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RESEARCH ARTICLE

Finite groups with permutable Hall subgroups

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Abstract Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$, and let G be a finite group. A set $\mathscr{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member $\neq 1$ of $\mathscr{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathscr{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. In this paper, we study the structure of G under the assuming that some subgroups of G permutes with all members of $\mathscr{H}$.

Keywords Finite group, Hall subgroup, complete Hall σ-set, permutable subgroup, supersoluble group
MSC 20D10, 20D15

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use $\pi(G)$ to denote the set of all primes dividing $|G|$. A subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. In this case, we say also that the subgroups A and B are *permutable*.

Following [14], we use σ to denote some partition of $\mathbb{P}$. Thus, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

A set $\mathscr{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G [7,15] if every nonidentity member of $\mathscr{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathscr{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$. If every two members of $\mathscr{H}$ are permutable, then $\mathscr{H}$ is said to be a *σ-basis* of G [16]. In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, a complete Hall σ-set $\mathscr{H}$ of G is also called *a complete set of Sylow subgroups* of G.

We use $\mathfrak{H}_\sigma$ to denote the class of all soluble groups G such that every complete Hall σ-set of G forms a σ-basis of G.

A large number of publications are connected with study the situation when

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some subgroups of G permute with all members of some fixed complete set of Sylow subgroups of G. For example, the classical Hall's result states that G is soluble if and only if it has a Sylow basis, that is, a complete set of pairwise permutable Sylow subgroups. Huppert [9] (see also [10, VI, § 3]) proved that G is a soluble group in which every complete set of Sylow subgroups forms a Sylow basis if and only if the automorphism group induced by G on every its chief p-factor H/K is of the order $p^a q^b$ for some q that depends only on H/K. Huppert [8] proved that if G is soluble and it has a complete set $\mathscr{S}$ of Sylow subgroups such that every maximal subgroup of every subgroup in $\mathscr{S}$ permutes with all other members of $\mathscr{S}$, then G is supersoluble.

The above-mentioned results in [8–10] and many other related results make natural to ask the following questions.

Question (I) Suppose that G has a complete Hall σ-set $\mathscr{H}$ such that every maximal subgroup of any subgroup in $\mathscr{H}$ permutes with all other members of $\mathscr{H}$. What we can say then about the structure of G? In particular, does it true then that G is supersoluble in the case when every member of $\mathscr{H}$ is supersoluble?

Question (II) Suppose that G possesses a complete Hall σ-set. What we can say then about the structure of G provided every complete Hall σ-set of G forms a σ-basis in G?

Our first observation is the following result concerning Question (I).

Theorem A *Suppose that G possesses a complete Hall σ-set $\mathscr{H}$ all whose members are supersoluble. If every maximal subgroup of every non-cyclic subgroup in $\mathscr{H}$ permutes with all other members of $\mathscr{H}$, then G is supersoluble.*

In the classical case when $\sigma = \{\{2\}, \{3\}, \dots\}$, we get from Theorem A the following two known results.

Corollary 1.1 [1] *If G has a complete set $\mathscr{S}$ of Sylow subgroups such that every maximal subgroup of every subgroup in $\mathscr{S}$ permutes with all other members of $\mathscr{S}$, then G is supersoluble.*

Note that Corollary 1.1 was proved in [1] on the base of the classification of all simple non-abelian groups. The proof of Theorem A does not use such a classification.

Corollary 1.2 [10, VI, Theorem 10.3] *If every Sylow subgroup of G is cyclic, then G is supersoluble.*

Recall that a *formation* $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and subdirect products. $\mathfrak{F}$ is said to be *saturated* if for any group G, $G/\Phi(G) \in \mathfrak{F}$ would imply that $G \in \mathfrak{F}$. $\mathfrak{F}$ is said to be *hereditary* provided $G \in \mathfrak{F}$ whenever $G \leqslant A \in \mathfrak{F}$.

Now, let $p > q > r$ be primes such that qr divides $p - 1$. Let P be a group of order p and $QR \leqslant \mathrm{Aut}(P)$, where Q and R are groups with orders q and r, respectively. Let $G = P \rtimes (QR)$. Then, in view of the above-mentioned Hupper's result in [9], G is not a group such that every complete set of Sylow subgroups forms a Sylow basis of G. But it is easy to see that every complete Hall σ-set of G, where $\sigma = \{\{2,3\},\{7\},\{2,3,7\}'\}$, is a σ-basis of G. This elementary example is a motivation for our next result, which gives the answer to Question (II) in the universe of all soluble groups.

Theorem B *The class $\mathfrak{H}_\sigma$ is a hereditary formation and it is saturated if and only if $|\sigma| \leqslant 2$. Moreover, $G \in \mathfrak{H}_\sigma$ if and only if G is soluble and the automorphism group induced by G on every its chief factor of order divisible by p is either a σ_i-group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different σ_i and σ_j such that $p \in \sigma_i$.*

In the case when $\sigma = \{\{2\},\{3\},\ldots\}$, we get from Theorem B the following result.

Corollary 1.3 [9] *Every complete set of Sylow subgroups of a soluble group G forms a Sylow basis of G if and only if the automorphism group induced by G on every its chief factor H/K has order divisible by at most one different from p prime, where $p \in \pi(H/K)$.*

2 Proof of Theorem A

Lemma 2.1 [12] *Let H, K and N be pairwise permutable subgroups of G and H is a Hall subgroup of G. Then*

$$N \cap HK = (N \cap H)(N \cap K).$$

Proof of Theorem A Assume that this theorem is false and let G be a counterexample of minimal order. Let $\mathscr{H} = \{H_1, H_2, \ldots, H_t\}$. We can assume, without loss of generality, that the smallest prime divisor p of $|G|$ belongs to $\pi(H_1)$. Let P be a Sylow p-subgroup of H_1.

(1) If R is a minimal normal subgroup of G, then G/R is supersoluble. Hence, R is the unique minimal normal subgroup of G, R is not cyclic, and $R \nleq \Phi(G)$.

We show that the hypothesis holds for G/R. First, note that

$$\mathscr{H}_0 = \{H_1R/R, H_2R/R, \ldots, H_tR/R\}$$

is a complete Hall σ-set of G/R, where

$$H_iR/R \simeq H_i/H_i \cap R$$

is supersoluble since H_i is supersoluble by hypothesis for all $i = 1, 2, \ldots, t$.

Now, let V/R be a maximal subgroup of H_iR/R. Then $V = R(V \cap H_i)$ and

$$|H_i : V \cap H_i| = |H_iV : V| = |H_iR : V|$$

is a prime. Thus, $V \cap H_i$ is maximal in H_i. Assume that H_iR/R is not cyclic. Then H_i is not cyclic, so

$$(V \cap H_i)H_j = H_j(V \cap H_i)$$

for all $j \neq i$ by hypothesis. Hence,

$$\begin{aligned}(V/R)(H_jR/R) &= (R(V \cap H_i)/R)(H_jR/R) \\ &= (H_jR/R)((V \cap H_i)R/R) \\ &= (H_jR/R)(V/R).\end{aligned}$$

Consequently, the hypothesis holds for G/R, and so G/R is supersoluble by the choice of G. Moreover, it is well known that the class of all supersoluble groups is a saturated formation (see [10, VI] or [6, Chapter 3, Example 4, Theorem 3.1.11]). Hence, the choice of G implies that R is the unique minimal normal subgroup of G, R is not cyclic, and $R \nleq \Phi(G)$.

(2) G is not soluble. Hence, R is not abelian and $2 \in \pi(R)$.

Assume that this is false. Then R is an abelian q-group for some prime q. Let $q \in \sigma_k$. Since R is non-cyclic by Claim (1) and $R \leqslant H_k$, H_k is non-cyclic. Hence, every member of $\mathscr{H}$ permutes with each maximal subgroup of H_k. Since

$$R \nleq \Phi(G), \quad R \nleq \Phi(H_k),$$

there exists a maximal subgroup V of H_k such that $RV = H_k$. Hence, $E = R \cap V \neq 1$ since $|R| > q$ and H_k is supersoluble. Clearly, E is normal in H_k. Now, assume that $i \neq k$. Then V permutes with H_i by hypothesis, so VH_i is a subgroup of G and

$$R \cap VH_i = (R \cap V)(R \cap H_i) = R \cap V = E$$

by Lemma 2.1, and so $H_i \leqslant N_G(E)$. Therefore, $H_i \leqslant N_G(E)$ for all $i = 1, 2, \ldots, t$. This implies that E is normal in G, which contradicts the minimality of R. Hence, we have (2).

(3) If R has a Hall $\{2, q\}$-subgroup for each q dividing $|R|$, then a Sylow 2-subgroup R_2 of R is non-abelian.

Assume that this is false. Then by Claim (2) and [11, XI, Theorem 13.7], the composition factors of R are isomorphic to one of the following groups:

a) $PSL(2, 2^f)$;

b) $PSL(2, q)$, where 8 divides $q - 3$ or $q - 5$;

c) the Janko group J_1;

d) a Ree group.

But with respect to each of these groups, it is well known (see, for example, [17, Theorem 1]) that the group has no Hall $\{2, q\}$-subgroup for at least one odd prime q dividing its order. Hence, we have (3).

(4) If H_i or H_k is non-cyclic, then $H_iH_k = H_kH_i$.

This follows from the fact that every maximal subgroup of H_i permutes with H_k.

(5) $H = H_1$ is not cyclic.

This directly follows from Claim (2), [10, IV, 2.8], and the Feit-Thompson theorem.

In view of Claim (5), $\mathscr{H}$ contains non-cyclic subgroups. Without loss of generality, we may assume that $H_1, H_2, \dots, H_r$ are non-cyclic groups and all groups $H_{r+1}, H_{r+2}, \dots, H_t$ are cyclic.

(6) Let $E_{\{i,j\}} = H_iH_j$, where $i \leqslant r$. If r is the smallest prime dividing $|E_{\{i,j\}}|$, then $E_{\{i,j\}}$ is p-nilpotent, so it is soluble. Therefore, $E_{\{i,j\}} \neq G$.

Clearly, the hypothesis holds for $E_{\{i,j\}}$. Hence, if $E_{\{i,j\}} < G$, then this subgroup is supersoluble by the choice of G, and so it is p-nilpotent. Now, assume that $E_{\{i,j\}} = G$. Then $r = p = 2$ and

$$E_{\{i,j\}} = HH_j = H_jH.$$

Let $\{V_1, V_2, \dots, V_t\}$ be the set of all maximal subgroups of a Sylow 2-subgroup P of H. Since H is supersoluble, it has a normal 2-complement S. Then SV_i is a maximal subgroup of H, so $SV_iH_j = H_jSV_i$ is a subgroup of G by hypothesis. Moreover, this subgroup is normal in $G = E_{\{i,j\}}$ since $|G : H_jSV_i| = 2$. Now, let

$$E = SV_1H_j \cap SV_2H_j \cap \dots \cap SV_tH_j.$$

Then E is normal in G and clearly, $E \cap P \leqslant \Phi(P)$. Therefore, E is 2-nilpotent by Tate's theorem [10, IV, Satz 4.7], so the Feit-Thompson theorem implies that G has an abelian minimal normal subgroup, which contradicts Claim (2). Thus, (6) holds.

(7) $E_i = HH_i$ is supersoluble for all $i = 2, 3, \dots, t$.

Since the hypothesis holds for E_i and $E_i < G$ by Claim (6), this follows from the choice of G.

(8) $E = H_1H_2 \cdots H_r$ is soluble.

We argue by induction on r. For $r = 2$, it is true by Claim (6). Now, let $r > 2$ and assume that the assertion is true for $r - 1$. Then by Claim (4), E has at least three soluble subgroups E_1, E_2, E_3, whose indices $|E : E_1|$, $|E : E_2|$, $|E : E_3|$ are pairwise coprime. But then E is soluble by the Wielandt theorem [3, Chapter I, 3.4].

(9) R has a Hall $\{2, q\}$-subgroup for each q dividing $|R|$.

It is clear in the case when $q \in \pi(H)$. Now, assume that $q \in \pi(H_i)$ for some $i > 1$. Then Claim (6) implies that $B = HH_i$ is a Hall soluble subgroup of G. Hence, B has a Hall $\{2, q\}$-subgroup V and so $V \cap R$ is a Hall $\{2, q\}$-subgroup of R.

(10) A Sylow 2-subgroup R_2 of R is non-abelian.

This follows from Claims (3) and (9).

(11) If $q \in \pi(H_k)$ for some $k > r$, then q does not divide $|R : N_R((R_2)')|$.

By Claim (7), $B = HH_k$ is supersoluble. Hence, there is a Sylow q-subgroup of Q of B such that PQ is a Hall $\{2, q\}$-subgroup of B. Then

$$U = PQ \cap R = (P \cap R)(Q \cap R) = R_2(Q \cap R)$$

is a Hall supersoluble subgroup of R with a cyclic Sylow q-subgroup $Q \cap R$. By [10, VI, 9.1], $Q \cap R$ is normal in U, and $U/C_U(Q \cap R)$ is an abelian group by [4, Ch. 5, 4.1]. Hence,

$$R_2 C_U(Q \cap R)/C_U(Q \cap R) \simeq R_2/R_2 \cap C_U(Q \cap R)$$

is abelian and so

$$(R_2)' \leqslant C_U(Q \cap R).$$

Consequently,

$$Q \cap R \leqslant N_R((R_2)'),$$

which yields that q does not divide $|R : N_R((R_2)')|$.

(12) The final contradiction.

In view of Claim (11),

$$R = (E \cap R) N_R((R_2)').$$

Hence,

$$((R_2)')^R = ((R_2)')^{(E \cap R) N_R((R_2)')} = ((R_2)')^{E \cap R} \leqslant E \cap R.$$

But by Claim (8), $E \cap R$ is soluble and so $((R_2)')^R$ is soluble. On the other hand, Claim (10) implies that $(R_2)' \neq 1$. But $((R_2)')^R$ is a normal subgroup of R and R is a direct product of isomorphic simple groups, so R is soluble, contrary to Claim (2).

The final contradiction completes the proof of Theorem A. □

3 Proof of Theorem B

The following lemma can be proved by the direct calculations on the base of well-known properties of Hall subgroups of soluble subgroups.

Lemma 3.1 *The class $\mathfrak{H}_\sigma$ is closed under taking homomorphic images, subgroups, and direct products.*

Proof Let $E \leqslant G \in \mathfrak{H}_\sigma$. Then G is soluble, so for any normal subgroup R of G, any complete Hall σ-set $\mathscr{H}_0$ of G/R is of the form

$$\mathscr{H}_0 = \{H_1R/R, H_2R/R, \ldots, H_tR/R\},$$

where $\mathscr{H} = \{H_1, H_2, \ldots, H_t\}$ is a complete Hall σ-set of G. But since $G \in \mathfrak{H}_\sigma$, $\mathscr{H}$ is a σ-basis of G. Hence, for all i, j, $H_iH_j = H_jH_i$, and so

$$(H_iR/R)(H_jR/R) = H_iH_jR/R = H_jH_iR/R = (H_jR/R)(H_iR/R).$$

Hence, $\mathscr{H}_0$ is a basis of G/R, and so $G/R \in \mathfrak{H}_\sigma$. On the other hand, for any complete Hall σ-set $\mathscr{E} = \{E_1, E_2, \dots, E_r\}$ of E, there is a complete Hall σ-set $\mathscr{H} = \{H_1, H_2, \dots, H_t\}$ of G such that $E_i = H_i \cap E$ for all $i = 1, 2, \dots, t$. Then

$$\langle E_i, E_j \rangle \leqslant E \cap H_i H_j \leqslant E_{i,j},$$

where $E_{i,j}$ is a Hall π-subgroup of E and

$$\pi = \pi(H_i) \cup \pi(H_j).$$

Hence, $E_{i,j} = E_i E_j$, so $\mathscr{E}$ is σ-basis of E. Thus, $E \in \mathfrak{H}_\sigma$.

Finally, we show that if $A, B \in \mathfrak{H}_\sigma$, then

$$G = A \times B \in \mathfrak{H}_\sigma.$$

First, note that $\mathscr{H}$ is a complete Hall σ-set of G. Then

$$\mathscr{H} = \{A_1 \times B_1, A_2 \times B_2, \dots, A_t \times B_t\},$$

where $\{A_1, A_2, \dots, A_t\}$ is a complete Hall σ-set of A and $\{B_1, B_2, \dots, B_t\}$ is a complete Hall σ-set of B. Then

$$(A_i B_i)(A_j B_j) = (A_j B_j)(A_i B_i), \quad \forall\, i, j,$$

since $A, B \in \mathfrak{H}_\sigma$ and $[A_k, B_l] = 1$ for all $k \neq l$. The lemma is proved. □

Proof of Theorem B First, from Lemma 3.1, $\mathfrak{H}_\sigma$ is a hereditary formation.

Now, we prove that $G \in \mathfrak{H}_\sigma$ if and only if G is soluble and the automorphism group induced by G on every its chief factor of order divisible by p is either a σ_i-group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different σ_i and σ_j such that $p \in \sigma_i$.

Necessity. Assume that this is false and let G be a counterexample of minimal order. Then G has a chief factor H/K of order divisible by p such that $A = G/C_G(H/K)$ is neither a σ_i-group, where $p \notin \sigma_i$, nor a $(\sigma_i \cup \sigma_j)$-group, where $\sigma_i \neq \sigma_j$ and $p \in \sigma_i$. Since

$$G/C_G(H/K) \simeq (G/K)/(C_G(H/K)/K) = (G/K)/C_{G/K}(H/K)$$

and the hypothesis hods for G/K by Lemma 3.1, the choice of G implies that $K = 1$.

First, we show that $H \neq C_G(H)$. Indeed, assume that $H = C_G(H)$. By hypothesis, every complete Hall σ-set $\mathscr{W} = \{W_1, W_2, \dots, W_t\}$ of G forms a σ-basis of G. Without loss of generality, we can assume that $p \in \pi(W_1)$. It is cleat that $t > 2$. Since $H = C_G(H)$, H is the unique minimal normal subgroup of G and $H \nleq \Phi(G)$ by [3, Ch.A, 9.3(c)] since G is soluble. Hence,

$$H = O_p(G) = F(G)$$

by [3, Ch.A, 15.6]. Then, for some maximal subgroup M of G, we have

$$G = H \rtimes M.$$

Let $V = W_3$. We now show that $V^x \leqslant C_G(W_2)$ for all $x \in G$. First, note that $W_2V^x = V^xW_2$ is a Hall $(\sigma_2 \cup \sigma_3)$-subgroup of G. Since $|G : M|$ is a power of p, any Hall σ_0-subgroup of M, where $p \notin \pi_0$, is a Hall π_0-subgroup of G. Hence, we can assume, without loss of generality, that $W_2V^x \leqslant M$ since G is soluble. By hypothesis,

$$W_2(V^x)^y = (V^x)^yW_2, \quad \forall\, y \in G,$$

so

$$D = \langle (W_2)^{V^x} \rangle \cap \langle (V^x)^{W_2} \rangle$$

is subnormal in G by [2, 1.1.9(2)]. But

$$D \leqslant \langle W_2, V^x \rangle \leqslant M,$$

so

$$D^G = D^{HM} = D^M \leqslant M_G = 1$$

by [3, Ch. A, 14.3], which implies that $[W_2, V^x] = 1$. Thus, $V^x \leqslant C_G(W_2)$ for all $x \in G$. It follows that

$$H \leqslant (W_3)^G \leqslant N_G(W_2),$$

and therefore,

$$W_2 \leqslant C_G(H) = H,$$

a contradiction. Hence, $H \neq C_G(H)$.

Finally, let

$$D = G \times G, \quad A^* = \{(g, g) \mid g \in G\},$$
$$C = \{(c, c) \mid c \in C_G(H)\}, \quad R = \{(h, 1) \mid h \in H\}.$$

Then $C \leqslant C_D(R)$, R is a minimal normal subgroup of A^*R, and the factors $R/1$ and RC/C are (A^*R)-isomorphic. Moreover,

$$C_{A^*R}(R) = R(C_{A^*R}(R) \cap A^*) = RC,$$

so

$$A^*R/C = (RC/C) \rtimes (A^*/C),$$

where $A^*/C \simeq A$ and RC/C is a minimal normal subgroup of A^*R/C such that

$$C_{A^*R/C}(RC/C) = RC/C.$$

As $H < C_G(H)$, we see that

$$|A^*R/C| < |G|.$$

On the other hand, by Lemma 3.1, the hypothesis holds for A^*R/C, so the choice of G implies that $A \simeq A^*/C$ is either a σ_i-group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different σ_i and σ_j such that $p \in \sigma_i$. This contradiction completes the proof of the necessity.

Sufficiency. Assume that this is false and let G be a counterexample of minimal order. Then G has a complete Hall σ-set $\mathscr{W} = \{W_1, W_2, \dots, W_t\}$ such that for some i and j, we have $W_iW_j \neq W_jW_i$. Let R be a minimal normal subgroup of G.

(1) $G/R \in \mathfrak{H}_\sigma$, so R is a unique minimal normal subgroup of G.

It is clear that the hypothesis holds for G/R, so $G/R \in \mathfrak{H}_\sigma$ by the choice of G. If G has a minimal normal subgroup $L \neq R$, then we also have $G/L \in \mathfrak{H}_\sigma$. Hence, G is isomorphic to some subgroup of $(G/R) \times (G/L)$ by [10, I, 9.7]. It follows from Lemma 3.1 that $G \in \mathfrak{H}_\sigma$. This contradiction shows that we have Claim (1).

(2) The hypothesis holds for any subgroup E of G.

Let H/K be any chief factor of G of order divisible by p such that

$$H \cap E \neq K \cap E.$$

Then $G/C_G(H/K)$ is either a σ_i-group, where $p \notin \sigma_i$, or a $(\sigma_i \cup \sigma_j)$-group for some different σ_i and σ_j such that $p \in \sigma_i$. Let H_1/K_1 be a chief factor of E such that

$$K \cap E \leqslant K_1 < H_1 \leqslant H \cap E.$$

Then H_1/K_1 is a p-group and

$$EC_G(H/K)/C_G(H/K) \simeq E/(E \cap C_G(H/K))$$

is either a σ_i-group or a $(\sigma_i \cup \sigma_j)$-group. Since

$$C_G(H/K) \cap E \leqslant C_E(H \cap E/K \cap E) \leqslant C_E(H_1/K_1),$$

$E/C_E(H_1/K_1)$ is also either a σ_i-group or a $(\sigma_i \cup \sigma_j)$-group. Therefore, the hypothesis holds for every factor H_1/K_1 of some chief series of E. Now, applying the Jordan-Hölder Theorem for chief series, we get Claim (2).

(3) R is a Sylow p-subgroup of G.

Since $G/R \in \mathfrak{H}_\sigma$ by Claim (1),

$$(W_iR/R)(W_jR/R) = (W_jR/R)(W_iR/R),$$

so W_iW_jR is a subgroup of G. Assume that R is not a Sylow p-subgroup of G and let $B = W_iW_jR$. Then $B \neq G$. On the other hand, the hypothesis holds for B by Claim (2). The choice of G implies that $B \in \mathfrak{H}_\sigma$, so $W_iW_j = W_jW_i$, a contradiction. Hence, Claim (3) holds.

(4) Final contradiction for sufficiency.

In view of Claims (1) and (3), there is a maximal subgroup M of G such that

$$G = R \rtimes M, \quad M_G = 1.$$

Hence,

$$R = C_G(R) = O_p(G)$$

by [3, Ch.A, 15.6]. Since p does not divide $|G : R| = |G : C_G(R)|$ by Claim (3), the hypothesis implies that $M \simeq G/R$ is a Hall σ_k-group for some $\sigma_k \in \sigma$, so one of the subgroups W_i or W_j coincides with R. Thus,

$$G = W_i W_j = W_j W_i.$$

This contradiction completes the proof of the sufficiency.

Finally, we prove that $\mathfrak{H}_\sigma$ is saturated if and only if $|\sigma| \leqslant 2$. It is clear that $\mathfrak{H}_\sigma$ is a saturated formation for any σ with $|\sigma| \leqslant 2$. Now, we show that for any σ such that $|\sigma| > 2$, the formation $\mathfrak{H}_\sigma$ is not saturated.

Indeed, since $|\sigma| > 2$, there are primes $p < q < r$ such that for some distinct σ_i, σ_j, and σ_k in σ, we have $p \in \sigma_i$, $q \in \sigma_j$, and $r \in \sigma_k$. Let C_q and C_r be groups of order q and r, respectively. Let P_1 be a simple $\mathbb{F}_p C_q$-module which is faithful for C_q (see [3, Chapter B, Theorem 10.9] or [13, Lemma 2.6]), and let P_2 be a simple $\mathbb{F}_p C_r$-module which is faithful for C_r. Let $H = P_1 \rtimes C_q$ and Q be a simple $\mathbb{F}_q H$-module which is faithful for H. Let

$$E = (Q \rtimes H) \times (P_2 \rtimes C_r).$$

Let $A = A_p(E)$ be the p-Frattini module of E ([3, p. 853]), and let G be a non-splitting extension of A by E. In this case, $A \subseteq \Phi(G)$ and $G/A \simeq E$. Then $G/\Phi(G) \in \mathfrak{H}_\sigma$, where $\sigma = \{\sigma_i, \sigma_j, \sigma_k\}$. By [5, Corollary 1],

$$QP_1P_2 = O_{p',p}(E) = C_E(A/\mathrm{Rad}(A)),$$

where $\mathrm{Rad}(A)$ is the radical of A, that is, the intersection of all maximal submodules of A (see [6, p. 235]). Hence, for some normal subgroup N of G, we have

$$A/N \leqslant \Phi(G/N)$$

and

$$G/C_G(A/N) \simeq C_q \times C_r$$

is a $(\sigma_j \cup \sigma_k)$-group. But neither $p \notin \sigma_j$ nor $p \in \sigma_k$. Hence, $G \notin \mathfrak{H}_\sigma$ by the necessity. Theorem B is proved. □

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