Front. Math. China 2017, 12(3): 559–582
DOI 10.1007/s11464-017-0625-4

RESEARCH ARTICLE

An occupation time related potential measure for diffusion processes

Ye CHEN[1], **Yingqiu LI**[2], **Xiaowen ZHOU**[3]

1 Hunan Province Cooperative Innovation Center for the Construction and Development of Dongting Lake Ecological Economic Zone and College of Mathematics and Computational Science, Hunan University of Arts and Science, Changde 415000, China
2 School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, China
3 Department of Mathematics and Statistics, Concordia University, Montreal, Quebec H3G 1M8, Canada

Abstract In this paper, for homogeneous diffusion processes, the approach of Y. Li and X. Zhou [Statist. Probab. Lett., 2014, 94: 48–55] is adopted to find expressions of potential measures that are discounted by their joint occupation times over semi-infinite intervals $(-\infty, a)$ and (a, ∞). The results are expressed in terms of solutions to the differential equations associated with the diffusions generator. Applying these results, we obtain more explicit expressions for Brownian motion with drift, skew Brownian motion, and Brownian motion with two-valued drift, respectively.

Keywords Laplace transform, occupation time, potential measure, exit time, time-homogeneous diffusion, Brownian motion with two-valued drift, skew Brownian motion
MSC 60J60, 60G17

1 Introduction

The goal of this paper is to study a joint Laplace transform of occupation times over disjoint intervals for one-dimensional time homogeneous diffusion processes. In the following, we first give a brief review on the related previous results and their approaches.

Using the classical approach of solving the associated differential equation, Linetsky [26] and Davydov et al. [7] studied the Laplace transform of occupation time over semi-infinite intervals $(-\infty, 0)$ or $(0, \infty)$ and over finite

Received October 21, 2015; accepted December 26, 2016
Corresponding author: Ye CHEN, E-mail: chenyexfw@163.com

interval (a,b) of a geometric Brownian motion model, respectively. Cai et al. [5] generalized the previous results to the jump-diffusion process with double exponential jumps. They also derived a closed-form Laplace transform of the joint distribution of the occupation time and the terminal value of the jump-diffusion process. Applying excursion theory, Pitman and Yor [30,31] also obtained similar Laplace transforms for one-dimensional diffusions. In particular, they found a formula for the joint Laplace transform of the occupation times spent by the process either above or below a level up to a suitable random time.

Landriault et al. [18] proposed a perturbation approach to study the Laplace transform of occupation time for spectrally negative Lévy processes (SNLP). In [18], the occupation time is, respectively, over and under estimated by the approximating occupation times whose Laplace transforms can be computed via solutions to the exit problems for SNLP. The desired Laplace transform of the occupation time then follows by taking a limit. With the strategy of Landriault et al. [18] and for diffusion processes, Li and Zhou [22] studied the joint Laplace transforms of occupation times up to an independent exponential time. The results were expressed in terms of solutions to the differential equation associated to the diffusion generator. More recently, using the same approach, Li et al. [23] found expressions of double Laplace transform for diffusion processes. They also obtained the explicit Laplace transforms of the corresponding occupation time and the occupation density for Brownian motion with two-valued drift and that of occupation time for skew Ornstein-Uhlenbeck process.

In [27], the occupation time Laplace transforms were first considered for SNLP with sample paths of bounded variation, and then for general SNLP that can be approximated by SNLP of bounded variation. As a result, the Laplace transform of occupation time over a finite interval before certain first passage time was obtained. The above approximation method is similar to one that was used previously by Kyprianou et al. [17] to study the occupation time of the so called spectrally negative refracted Lévy processes. Similarly, Yin et al. [34] determined the joint laws for some occupation time related quantities for SNLP which are useful in risk theory. Recently, for SNLP and using the approximation scheme of [27], Guérin and Renaud [14] identified the expression of a quantity which determines the joint distribution of the occupation time over finite interval (a,b) up to time t and the value of the process at time t. Renaud [33] derived identities for the distribution of occupation times of refracted Lévy processes. More recently, Pérez and Yamazaki [28] studied the occupation times of the so called refracted-reflected SNLP, which was used in [29] to consider the occupation times of dual model.

In order to get around the approximation arguments in the aforementioned work, Li and Zhou [24] first studied the Laplace transforms of pre-exit joint occupation times for SNLP, where they proposed an alternative approach that identified the joint Laplace transform of the occupation time with the probability that two independent sequences of Poisson arrival times avoids their respective regions. Using formulas of potential measures, solutions to the exit problems, and identities on scale functions for SNLP, they found expressions for the desired probabilities. Li et al. [25] adopted the Poisson approach in [24] to obtain expressions of potential measures that are discounted by their joint occupation times over semi-infinite intervals $(-\infty, 0)$ and $(0, \infty)$. Recently, with the Poisson approach, Li and Palmowski [21] obtained fluctuations identities for Omega-killed SNLP.

The Poisson approach also works well for diffusion processes. Chen et al. [6] used this approach to express the joint Laplace transform for pre-exit diffusion of occupation times in terms of solutions to the associated differential equation. The direct approach of [24] allows us to handle more complex quantities involving Laplace transforms of occupation times. It thus has some advantages over the previous approaches.

Similar to [24], these expressions in [25] are in terms of the associated scale functions and the inverse functions of Laplace exponents. The problems considered in [25] and [14] are along the same line. But the methods of [25] and [14] are different. More fluctuation identities on SNLP observed at Poisson arrive times have been studied in Albrecher et al. [1].

Occupation times have be found many applications in mathematical finance and risk theory for insurance. In mathematical finance, the occupation times can be used to define and price options such as step options (e.g., Linetsky [26], Guérin et al. [14]) and corridor options (e.g., Fusai [13]). In risk theory, the occupation times can be utilized as a useful tool to manage insurable risks. Egidio dos Reis [10], Dickson and Egidio dos Reis [8], and Kolkovska et al. [16] studied different Laplace transform of occupation times of the classical compound Poisson risk model. Occupation times for other risk models were also studied, such as the SNLP risk model (e.g., Landriault et al. [18], Loeffen et al. [27], Yin et al. [34]), the Erlang-2 risk model (e.g., Dickson and Li [9]), the Markov additive risk model (e.g., Albrecher et al. [1]), the Markovian arrival process (e.g., Landriault et al. [19]), and the Markov-modulated Brownian motion (e.g., Breuer [4]).

Many explicit results on Laplace transforms for occupation times have been obtained for some well-known examples of diffusion processes; see, e.g., Borodin and Salminen [3] for a collection of such results. Potential measures are also interesting topics for stochastic processes. But for general diffusion processes, the potential measure discounted by their joint occupation times over disjoint intervals (a, ∞) and $(-\infty, a)$ have not been studied systematically.

In this paper, we adopt the Poisson approach of Li and Zhou [24] to consider the two-sided discounted potential measures for diffusion processes, the expressions are in terms of solutions to the associated differential equations. Moreover, the results can be applied to find more explicit Laplace transforms of the occupation times up to an independent exponential time for skew Brownian motion and for Brownian motion with two-valued drift. To our best knowledge, such results have not been known before. In addition, our results can also be applied to study option pricing.

The rest of this paper is arranged as follows. In Section 2, we review the basic facts we need for time-homogeneous diffusion processes. In Section 3, we first show a lemma which we need later. Then the desired results on two-sided discounted potential measures for diffusion processes are obtained. In Section 4, we find explicit expressions on two-sided discounted potential measures for several examples of diffusion processes, such as Brownian motion with drift, skew Brownian motion, and Brownian motion with two-valued drift.

2 Preliminaries

In this paper, we consider a one-dimensional diffusion process $X = (X_t)_{t\geqslant 0}$ defined on a filtered probability space $(\Omega, \mathbb{P}, (\mathscr{F}_t)_{t\geqslant 0})$. The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$, and we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. Process X takes values in interval I with end points $-\infty \leqslant l_1 < l_2 \leqslant \infty$, which is specified by the following stochastic differential equation:

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{1}$$

where $X_0 = x_0$ is the initial value and $W = \{W_t,\, t \geqslant 0\}$ is a one-dimensional standard Brownian motion. Throughout the paper, we assume that equation (1) allows a unique weak solution, which is guaranteed if there exists a constant $K > 0$ such that, for all $x, y \in I$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leqslant K|x - y|, \quad \mu^2(x) + \sigma^2(x) \leqslant K^2(1 + x^2); \tag{2}$$

see Evans [11].

Two basic characteristics of diffusion processes X, the *speed measure* m and the *scale function* s, are given by

$$m(\mathrm{d}x) = m(x)\mathrm{d}x := \frac{2\mathrm{e}^{B(x)}}{\sigma^2(x)}\,\mathrm{d}x, \quad s(x) := \int^x \mathrm{e}^{-B(y)}\mathrm{d}y, \quad l_1 < x < l_2,$$

respectively, where

$$B(x) := \int^x \frac{2\mu(y)}{\sigma^2(y)}\,\mathrm{d}y.$$

Let $p(\cdot\,;\cdot,\cdot)$ be the transition density of X with respect to the speed measure for diffusion processes, i.e.,

$$\mathbb{P}_x\{X_t \in \mathrm{d}y\} = p(t; x, y)m(\mathrm{d}y).$$

For $q > 0$, let $g_{-,q}(\cdot)$ and $g_{+,q}(\cdot)$ be two independent positive solutions to the (generalized) differential equation associated to the generator of X,

$$\frac{1}{2}\,\sigma^2(x)g''(x) + \mu(x)g'(x) = qg(x), \tag{3}$$

with $g_{-,q}(\cdot)$ decreasing and $g_{+,q}(\cdot)$ increasing, respectively. For some examples of diffusion processes, equation (3) yields explicit expressions for $g_{-,q}(\cdot)$ and $g_{+,q}(\cdot)$; see Borodin and Salminen [3]. Here, a solution $g(x)$ to equation (3) satisfies

$$q\int_{[a,b)} g(x)m(\mathrm{d}x) = g^-(b) - g^-(a), \tag{4}$$

where

$$g^-(x) := \lim_{h\to 0^+} \frac{g(x)-g(x-h)}{s(x)-s(x-h)}.$$

The *Green function* for X is

$$G_q(x,y) := \int_0^\infty \mathrm{e}^{-qt} p(t;x,y)\mathrm{d}t.$$

Then

$$G_q(x,y) = \begin{cases} \omega_q^{-1} g_{+,q}(x)g_{-,q}(y), & x \leqslant y, \\ \omega_q^{-1} g_{+,q}(y)g_{-,q}(x), & x \geqslant y, \end{cases}$$

where

$$\omega_q := g^+_{+,q}(x)g_{-,q}(x) - g_{+,q}(x)g^+_{-,q}(x) = g^-_{+,q}(x)g_{-,q}(x) - g_{+,q}(x)g^-_{-,q}(x)$$

is the so-called *Wronskian* with

$$g^+(x) := \lim_{h\to 0^+} \frac{g(x+h)-g(x)}{s(x+h)-s(x)}.$$

It is known that ω_q is independent of x.

We refer to [3, Chapter II] for the above facts and more details about diffusion processes.

Furthermore, for $q > 0$, define

$$f_q(y,z) := g_{-,q}(y)g_{+,q}(z) - g_{-,q}(z)g_{+,q}(y).$$

We have the following well-known solutions to the exit problems. Let

$$\tau_x := \inf\{t \geqslant 0 \colon X_t = x\}$$

be the first passage time of X at level x with the convention $\inf\phi = \infty$. For any $x \in (a,b)$ and $q \geqslant 0$,

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a}; \tau_a < \tau_b] = \frac{f_q(x,b)}{f_q(a,b)} \tag{5}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_b}; \tau_b < \tau_a] = \frac{f_q(a,x)}{f_q(a,b)}; \tag{6}$$

see, e.g., Borodin and Salminen [3] and Feller [12]. Moreover, for $q > 0$, we have

$$\lim_{x\to-\infty} g_{-,q}(x) = \lim_{x\to\infty} g_{+,q}(x) = \infty$$

and

$$\lim_{x\to\infty} g_{-,q}(x) = \lim_{x\to-\infty} g_{+,q}(x) = 0;$$

see, e.g., Li and Zhou [22]. Therefore, let $b \to \infty$ in (5) and $a \to -\infty$ in (6), for $x \in (a, b)$, we have

$$\mathbb{E}_x \mathrm{e}^{-q\tau_a} = \frac{g_{-,q}(x)}{g_{-,q}(a)}, \quad \mathbb{E}_x \mathrm{e}^{-q\tau_b} = \frac{g_{+,q}(x)}{g_{+,q}(b)}.$$

3 Main results

Throughout this section, we always take e_q to be an independent exponential random variable with rate q, and write $X(e_q)$ for X_{e_q}. We first prove a lemma which will be used later.

Lemma 1 *For any $y \in \mathbb{R}$,*

$$\int_0^\infty \mathbb{P}_y\{t < \tau_a,\, X_t \in \mathrm{d}x\}\mathrm{e}^{-qt}\mathrm{d}t$$
$$= \begin{cases} \Big[G_q(y,x) - \dfrac{g_{-,q}(y)}{g_{-,q}(a)}\, G_q(a,x)\Big] m(\mathrm{d}x), & y \in (a,\infty), \\ \Big[G_q(y,x) - \dfrac{g_{+,q}(y)}{g_{+,q}(a)}\, G_q(a,x)\Big] m(\mathrm{d}x), & y \in (-\infty,a). \end{cases}$$

The proof of Lemma 1 is given in Appendix.

We now consider the potential measure denoted by

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x)$$
$$:= \int_0^\infty \mathrm{e}^{-qt}\mathbb{E}_y[\mathrm{e}^{-\lambda_- \int_0^t 1_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^t 1_{(a,\infty)}(X_s)\mathrm{d}s}; X(t) \in \mathrm{d}x]\mathrm{d}t.$$

Theorem 1 *For any $\lambda_-, \lambda_+ > 0$, with $\lambda_- \neq \lambda_+$, we have*

for $x < a < y$,

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x) = \frac{g_{-,q+\lambda_+}(y)A(a,\mathrm{d}x)}{qg_{-,q+\lambda_+}(a)B(a)}; \tag{7}$$

for $x, y < a$,

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x)$$
$$= \Big[G_{q+\lambda_-}(y,x) - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\, G_{q+\lambda_-}(a,x)\Big] m(\mathrm{d}x) + \frac{g_{+,q+\lambda_-}(y)A(a,\mathrm{d}x)}{qg_{+,q+\lambda_-}(a)B(a)}; \tag{8}$$

for $x, y > a$,

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x)$$
$$= \Big[G_{q+\lambda_+}(y,x) - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\, G_{q+\lambda_+}(a,x)\Big] m(\mathrm{d}x) + \frac{g_{-,q+\lambda_+}(y)A'(a,\mathrm{d}x)}{q g_{-,q+\lambda_+}(a)B(a)}; \quad (9)$$

for $y < a < x$,

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x) = \frac{g_{+,q+\lambda_-}(y)A'(a,\mathrm{d}x)}{q g_{+,q+\lambda_-}(a)B(a)}, \quad (10)$$

where

$$A(a,\mathrm{d}x) := qG_{q+\lambda_-+\lambda_+}(a,x)m(\mathrm{d}x) + \lambda_+ q\int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,z)$$
$$\times \Big[G_{q+\lambda_-}(z,x)m(\mathrm{d}x) - \frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\, G_{q+\lambda_-}(a,x)m(\mathrm{d}x)\Big] m(\mathrm{d}z),$$

$$A'(a,\mathrm{d}x) := qG_{q+\lambda_-+\lambda_+}(a,x)m(\mathrm{d}x) + \lambda_- q\int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z)$$
$$\times \Big[G_{q+\lambda_+}(z,x)m(\mathrm{d}x) - \frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)}\, G_{q+\lambda_+}(a,x)m(\mathrm{d}x)\Big] m(\mathrm{d}z),$$

$$B(a) := 1 - \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z)\frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)}\, m(\mathrm{d}z)$$
$$- \lambda_+ \int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,z)\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\, m(\mathrm{d}z).$$

Proof For $x, y \in \mathbb{R}$, because

$$\mathscr{I}^y_{(-\infty,a)\cup(a,\infty)}(\mathrm{d}x) = \frac{1}{q}\,\mathbb{E}_y[\mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(a,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x],$$

let

$$f(y) := \mathbb{E}_y[\mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(a,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x].$$

Write

$$0 < T_1^- < T_2^- < \cdots, \quad 0 < T_1^+ < T_2^+ < \cdots,$$

for the arrived times of two independent Poisson processes with rate λ_- and λ_+, respectively. In particular, we denote

$$T_- = T_1^-, \quad T_+ = T_1^+.$$

We also assume that these Poisson processes are independent of process X. Using a property of Poisson process that

$$\mathbb{P}\{\{T_i^-\} \cap B = \emptyset\} = \mathrm{e}^{-\lambda_- m(B)}$$

for any Borel set $B \subset [0,\infty)$ and Lebesgue measure m on $\mathbb{R}$, by conditioning on X and e_q, we observe that

$$\begin{aligned} &f(y)\\ &\ = \mathbb{P}_y\{D, X(e_q) \in \mathrm{d}x\}\\ &:= \mathbb{P}_y\{\{T_i^-\} \cap \{s < e_q, X_s < a\} = \emptyset = \{T_i^+\} \cap \{s < e_q, X_s > a\}, X(e_q) \in \mathrm{d}x\}. \end{aligned}$$

For $x, y < a$,

$$\begin{aligned} f(y) &= \mathbb{E}_y[\mathrm{e}^{-\lambda_- e_q}; \tau_a > e_q, X(e_q) \in \mathrm{d}x] + \mathbb{E}_y[\mathrm{e}^{-\lambda_- \tau_a}; \tau_a < e_q] f(a)\\ &= q\int_0^\infty \mathbb{P}_y\{t < \tau_a, X(t) \in \mathrm{d}x\}\mathrm{e}^{-(q+\lambda_-)t}\mathrm{d}t + \mathbb{E}_y \mathrm{e}^{-(q+\lambda_-)\tau_a} f(a)\\ &= q\Big[G_{q+\lambda_-}(y,x) - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} G_{q+\lambda_-}(a,x)\Big] m(\mathrm{d}x) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} f(a), \end{aligned}$$

where in the third equation, we used Lemma 1.

For $x < a < y$,

$$f(y) = \mathbb{E}_y[\mathrm{e}^{-\lambda_+ \tau_a}; \tau_a < e_q] f(a) = \mathbb{E}_y[\mathrm{e}^{-(q+\lambda_+)\tau_a}] f(a) = \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} f(a).$$

It follows that for $x < a$,

$$\begin{aligned} f(a) &= \mathbb{P}_a\{e_q < T_- \wedge T_+, X(e_q) \in \mathrm{d}x\} + \mathbb{P}_a\{T_- < e_q \wedge T_+, D, X(e_q) \in \mathrm{d}x\}\\ &\quad + \mathbb{P}_a\{T_+ < e_q \wedge T_-, D, X(e_q) \in \mathrm{d}x\}\\ &= \mathbb{E}_a[\mathrm{e}^{-(\lambda_- + \lambda_+)e_q}; X(e_q) \in \mathrm{d}x] + \int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z]\\ &\quad \times \mathbb{E}_z[f(a); \tau_a < e_q \wedge T_+] + \int_{-\infty}^a \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+}; X_{T_+} \in \mathrm{d}z]\\ &\quad \times (\mathbb{P}_z\{e_q < \tau_a \wedge T_-; X(e_q) \in \mathrm{d}x\} + \mathbb{P}_z\{\tau_a < e_q \wedge T_-\} f(a))\\ &= \mathbb{E}_a[\mathrm{e}^{-(\lambda_- + \lambda_+)e_q}; X(e_q) \in \mathrm{d}x] + \int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z]\\ &\quad \times \mathbb{E}_z[\mathrm{e}^{-(q+\lambda_+)\tau_a} f(a)] + \int_{-\infty}^a \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+}; X_{T_+} \in \mathrm{d}z]\\ &\quad \times (\mathbb{P}_z\{e_q < \tau_a \wedge T_-; X(e_q) \in \mathrm{d}x\} + \mathbb{P}_z\{\tau_a < e_q \wedge T_-\} f(a)). \end{aligned}$$

With some calculations, we have

$$\begin{aligned} &\int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z]\mathbb{E}_z[\mathrm{e}^{-(q+\lambda_+)\tau_a} f(a)]\\ &\quad = \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) \frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)} m(\mathrm{d}z) f(a) \end{aligned}$$

and

$$\begin{aligned}&\int_{-\infty}^{a}\mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+};X_{T_+}\in\mathrm{d}z]\mathbb{P}_z\{\tau_a<e_q\wedge T_-\}f(a)\\&=\lambda_+\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)m(\mathrm{d}z)\mathbb{E}_z[\mathrm{e}^{-(q+\lambda_-)\tau_a}]f(a)\\&=\lambda_+\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\,m(\mathrm{d}z)f(a).\end{aligned}$$

By Lemma 1, the following result can be deduced:

$$\begin{aligned}&\int_{-\infty}^{a}\mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+};X_{T_+}\in\mathrm{d}z]\mathbb{P}_z\{e_q<\tau_a\wedge T_-;X(e_q)\in\mathrm{d}x\}\\&=\lambda_+\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)m(\mathrm{d}z)\int_0^\infty q\mathrm{e}^{-qt}\mathbb{P}_z\{t<\tau_a\wedge T_-;X(t)\in\mathrm{d}x\}\mathrm{d}t\\&=\lambda_+\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)m(\mathrm{d}z)\int_0^\infty q\mathrm{e}^{-(q+\lambda_-)t}\mathbb{P}_z\{t<\tau_a;X(t)\in\mathrm{d}x\}\mathrm{d}t\\&=\lambda_+q\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)\Big[G_{q+\lambda_-}(z,x)m(\mathrm{d}x)\\&\quad-\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\,G_{q+\lambda_-}(a,x)m(\mathrm{d}x)\Big]m(\mathrm{d}z).\end{aligned}$$

So we can get

$$\begin{aligned}f(a)&=qG_{q+\lambda_-+\lambda_+}(a,x)m(\mathrm{d}x)+\lambda_-\int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z)\frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)}m(\mathrm{d}z)f(a)\\&\quad+\lambda_+q\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)\Big[G_{q+\lambda_-}(z,x)m(\mathrm{d}x)\\&\quad-\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\,G_{q+\lambda_-}(a,x)m(\mathrm{d}x)\Big]m(\mathrm{d}z)\\&\quad+\lambda_+\int_{-\infty}^{a}G_{q+\lambda_-+\lambda_+}(a,z)\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)}\,m(\mathrm{d}z)f(a).\end{aligned}$$

After some calculations, we have

$$f(a)=\frac{A(a,\mathrm{d}x)}{B(a)}.$$

Similarly, for $x>a$,

$$\begin{aligned}f(a)&=\mathbb{P}_a\{e_q<T_-\wedge T_+,X(e_q)\in\mathrm{d}x\}+\mathbb{P}_a\{T_-<e_q\wedge T_+,D,X(e_q)\in\mathrm{d}x\}\\&\quad+\mathbb{P}_a\{T_+<e_q\wedge T_-,D,X(e_q)\in\mathrm{d}x\}\\&=\mathbb{E}_a[\mathrm{e}^{-(\lambda_-+\lambda_+)e_q};X(e_q)\in\mathrm{d}x]+\int_{-\infty}^{a}\mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+};X_{T_+}\in\mathrm{d}z]\end{aligned}$$

$$
\begin{aligned}
&\quad\times \mathbb{P}_z\{\tau_a < e_q \wedge T_-\} f(a) + \int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z] \\
&\quad\times (\mathbb{P}_z\{e_q < \tau_a \wedge T_+; X(e_q) \in \mathrm{d}x\} + \mathbb{E}_z[f(a); \tau_a < e_q \wedge T_+]) \\
&= \mathbb{E}_a[\mathrm{e}^{-(\lambda_- + \lambda_+)e_q}; X(e_q) \in \mathrm{d}x] \\
&\quad + \int_{-\infty}^a \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_-)T_+}; X_{T_+} \in \mathrm{d}z] \mathbb{P}_z\{\tau_a < e_q \wedge T_-\} f(a) \\
&\quad + \int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z] \mathbb{P}_z\{e_q < \tau_a \wedge T_+; X(e_q) \in \mathrm{d}x\} \\
&\quad + \int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z] \mathbb{E}_z[\mathrm{e}^{-(q+\lambda_+)\tau_a}] f(a),
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_a^\infty \mathbb{E}_a[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}z] \mathbb{P}_z\{e_q < \tau_a \wedge T_+; X(e_q) \in \mathrm{d}x\} \\
&\quad = \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) m(\mathrm{d}z) \int_0^\infty q\mathrm{e}^{-qt} \mathbb{P}_z\{t < \tau_a \wedge T_+; X(t) \in \mathrm{d}x\} \mathrm{d}t \\
&\quad = \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) m(\mathrm{d}z) \int_0^\infty q\mathrm{e}^{-(q+\lambda_+)t} \mathbb{P}_z\{t < \tau_a; X(t) \in \mathrm{d}x\} \mathrm{d}t \\
&\quad = \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) \Big[q G_{q+\lambda_+}(z,x) m(\mathrm{d}x) \\
&\qquad - q \frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)} G_{q+\lambda_+}(a,x) m(\mathrm{d}x) \Big] m(\mathrm{d}z).
\end{aligned}
$$

So we can get

$$
\begin{aligned}
f(a) &= q G_{q+\lambda_-+\lambda_+}(a,x) m(\mathrm{d}x) + \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) \frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)} m(\mathrm{d}z) f(a) \\
&\quad + \lambda_- \int_a^\infty G_{q+\lambda_-+\lambda_+}(a,z) \Big[q G_{q+\lambda_+}(z,x) m(\mathrm{d}x) \\
&\quad - q \frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(a)} G_{q+\lambda_+}(a,x) m(\mathrm{d}x) \Big] m(\mathrm{d}z) \\
&\quad + \lambda_+ \int_{-\infty}^a G_{q+\lambda_-+\lambda_+}(a,z) \frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(a)} m(\mathrm{d}z) f(a).
\end{aligned}
$$

After some calculations, we have

$$
f(a) = \frac{A'(a, \mathrm{d}x)}{B(a)}.
$$

Now, we prove expressions (7)–(10).

Given $x < a$, for $y > a$,

$$
f(y) = \mathbb{E}_y[\mathrm{e}^{-\lambda_+ \tau_a}; \tau_a < e_q] f(a) = \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} f(a) = \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} \frac{A(a, \mathrm{d}x)}{B(a)};
$$

for $y < a$,

$$
\begin{aligned}
&f(y)\\
&= \mathbb{E}_y[\mathrm{e}^{-\lambda_- e_q}; \tau_a > e_q, X(e_q) \in \mathrm{d}x] + \mathbb{E}_y[\mathrm{e}^{-\lambda_- \tau_a}; \tau_a < e_q] f(a)\\
&= q\int_0^\infty \mathbb{P}_y\{t < \tau_a, X_t \in \mathrm{d}x\}\mathrm{e}^{-(q+\lambda_-)t}\mathrm{d}t + \mathbb{E}_y \mathrm{e}^{-(q+\lambda_-)\tau_a} f(a)\\
&= qG_{q+\lambda_-}(y,x)m(\mathrm{d}x) - q\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}G_{q+\lambda_-}(a,x)m(\mathrm{d}x) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} f(a)\\
&= q\Big[G_{q+\lambda_-}(y,x) - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}G_{q+\lambda_-}(a,x)\Big]m(\mathrm{d}x) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\frac{A(a,\mathrm{d}x)}{B(a)}.
\end{aligned}
$$

Given $x > a$, for $y > a$,

$$
\begin{aligned}
&f(y)\\
&= \mathbb{E}_y[\mathrm{e}^{-\lambda_+ e_q}; \tau_a > e_q, X(e_q) \in \mathrm{d}x] + \mathbb{E}_y[\mathrm{e}^{-\lambda_+ \tau_a}; \tau_a < e_q] f(a)\\
&= q\int_0^\infty \mathbb{P}_y\{t < \tau_a, X_t \in \mathrm{d}x\}\mathrm{e}^{-(q+\lambda_+)t}\mathrm{d}t + \mathbb{E}_y \mathrm{e}^{-(q+\lambda_+)\tau_a} f(a)\\
&= qG_{q+\lambda_+}(y,x)m(\mathrm{d}x) - q\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}G_{q+\lambda_+}(a,x)m(\mathrm{d}x) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)} f(a)\\
&= q\Big[G_{q+\lambda_+}(y,x) - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}G_{q+\lambda_+}(a,x)\Big]m(\mathrm{d}x) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(a)}\frac{A'(a,\mathrm{d}x)}{B(a)};
\end{aligned}
$$

for $y < a$,

$$
f(y) = \mathbb{E}_y[\mathrm{e}^{-\lambda_- \tau_a}; \tau_a < e_q] f(a) = \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)} f(a) = \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(a)}\frac{A'(a,\mathrm{d}x)}{B(a)}. \quad \square
$$

Remark 1 From the proof of Theorem 1, we have

$$
\begin{aligned}
&\mathbb{E}_a[\mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(a,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x]\\
&\quad = \frac{1}{B(a)}\begin{cases} A(a,\mathrm{d}x), & x < a,\\ A'(a,\mathrm{d}x), & x > a. \end{cases}
\end{aligned}
$$

Some corollaries then follow from Theorem 1.

Corollary 1 *For $\lambda_-, \lambda_+ > 0$, we have*

$$
\mathbb{E}\mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s} = \frac{A_1}{B_1},
$$

where

$$
\begin{aligned}
A_1 := &\frac{q}{q+\lambda_- + \lambda_+} + \frac{q\lambda_-}{q+\lambda_+}\int_0^\infty G_{q+\lambda_- + \lambda_+}(0,y)\Big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\Big)m(\mathrm{d}y)\\
&+ \frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^0 G_{q+\lambda_- + \lambda_+}(0,y)\Big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\Big)m(\mathrm{d}y),
\end{aligned}
$$

$$B_1 := 1 - \lambda_- \int_0^\infty G_{q+\lambda_-+\lambda_+}(0,y)\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\, m(\mathrm{d}y) \\ - \lambda_+ \int_{-\infty}^0 G_{q+\lambda_-+\lambda_+}(0,y)\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\, m(\mathrm{d}y).$$

In general,

$$\mathbb{E}_y \mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s} \\ = \begin{cases} \dfrac{q}{q+\lambda_-}\Big(1 - \dfrac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\Big) + \dfrac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\,\dfrac{A_1}{B_1}, & y<0, \\ \dfrac{q}{q+\lambda_+}\Big(1 - \dfrac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\Big) + \dfrac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\,\dfrac{A_1}{B_1}, & y>0. \end{cases}$$

Proof For $y \in \mathbb{R}$, let

$$L_y := \mathbb{E}_y \mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}.$$

Similar to the proof of Theorem 1, we denote $0 < T_1^- < T_2^- < \cdots$ and $0 < T_1^+ < T_2^+ < \cdots$ for the arrived times of two independent Poisson processes with rate λ_- and λ_+, respectively, and they are both independent of process X. We have

$$L_y = \mathbb{P}_y\{D\} = \mathbb{P}_y\{\{T_i^-\}\cap\{s<e_q, X_s<0\} = \emptyset = \{T_i^+\}\cap\{s<e_q, X_s>0\}\}.$$

For $y<0$,

$$\begin{aligned} L_y &= \mathbb{E}_y[\mathrm{e}^{\lambda_- e_q}; e_q < \tau_0] + \mathbb{E}_y[\mathrm{e}^{\lambda_- \tau_0}; \tau_0 < e_q] L_0 \\ &= \mathbb{E}_y \mathrm{e}^{\lambda_- e_q} - \mathbb{E}_y[\mathrm{e}^{\lambda_- e_q}; \tau_0 < e_q] + \mathbb{E}_y \mathrm{e}^{-(q+\lambda_-)\tau_0} L_0 \\ &= \frac{q}{q+\lambda_-} - \frac{q}{q+\lambda_-}\mathbb{E}_y \mathrm{e}^{-(q+\lambda_-)\tau_0} + \mathbb{E}_y \mathrm{e}^{-(q+\lambda_-)\tau_0} L_0 \\ &= \frac{q}{q+\lambda_-}\Big(1 - \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\Big) + \frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)} L_0, \end{aligned} \tag{11}$$

and for $y>0$,

$$\begin{aligned} L_y &= \mathbb{E}_y[\mathrm{e}^{\lambda_+ e_q}; e_q < \tau_0] + \mathbb{E}_y[\mathrm{e}^{\lambda_+ \tau_0}; \tau_0 < e_q] L_0 \\ &= \frac{q}{q+\lambda_+}\Big(1 - \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\Big) + \frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)} L_0. \end{aligned} \tag{12}$$

Now, we consider L_0.

$$\begin{aligned} L_0 &= \mathbb{P}\{D\} \\ &= \mathbb{P}\{e_q < T_- \wedge T_+\} + \mathbb{P}\{T_- < e_q \wedge T_+, D\} + \mathbb{P}\{T_+ < e_q \wedge T_-, D\} \\ &= \mathbb{E}[\mathrm{e}^{-(\lambda_-+\lambda_+)e_q}] + \int_0^\infty \mathbb{E}[\mathrm{e}^{-(q+\lambda_+)T_-}; X_{T_-} \in \mathrm{d}y] L_y \end{aligned}$$

$$+\int_{-\infty}^{0}\mathbb{E}[\mathrm{e}^{-(q+\lambda_-)T_+};X_{T_+}\in\mathrm{d}y]L_y$$

$$=\frac{q}{q+\lambda_-+\lambda_+}+\lambda_-\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,y)m(\mathrm{d}y)L_y$$

$$+\lambda_+\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,y)m(\mathrm{d}y)L_y. \tag{13}$$

Combining (11)–(13), we have

$$\begin{aligned}L_0=&\frac{q}{q+\lambda_-+\lambda_+}+\lambda_-\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,y)m(\mathrm{d}y)\\&\times\Big[\frac{q}{q+\lambda_+}\Big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\Big)+\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}L_0\Big]\\&+\lambda_+\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,y)m(\mathrm{d}y)\\&\times\Big[\frac{q}{q+\lambda_-}\Big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\Big)+\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}L_0\Big]\\=&\frac{q}{q+\lambda_-+\lambda_+}+\frac{q\lambda_-}{q+\lambda_+}\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,y)\Big(1-\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}\Big)m(\mathrm{d}y)\\&+\lambda_-\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,y)\frac{g_{-,q+\lambda_+}(y)}{g_{-,q+\lambda_+}(0)}m(\mathrm{d}y)L_0\\&+\frac{q\lambda_+}{q+\lambda_-}\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,y)\Big(1-\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}\Big)m(\mathrm{d}y)\\&+\lambda_+\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,y)\frac{g_{+,q+\lambda_-}(y)}{g_{+,q+\lambda_-}(0)}m(\mathrm{d}y)L_0.\end{aligned}$$

After some algebras, we can obtain $B_1L_0=A_1$ with A_1 and B_1 as desired. □

Corollary 2 *For any* λ, *we have*

$$\mathbb{E}_y[\mathrm{e}^{-\lambda\int_0^{e_q}1_{(0,\infty)}(X_s)\mathrm{d}s};e_q<\tau_0,X(e_q)\in\mathrm{d}x]$$
$$=q\Big[G_{q+\lambda}(y,x)-\frac{g_{-,q+\lambda}(y)}{g_{-,q+\lambda}(0)}G_{q+\lambda}(0,x)\Big]m(\mathrm{d}x),\quad x,y>0,$$

$$\mathbb{E}_y[\mathrm{e}^{-\lambda\int_0^{e_q}1_{(-\infty,0)}(X_s)\mathrm{d}s};e_q<\tau_0,X(e_q)\in\mathrm{d}x]$$
$$=q\Big[G_{q+\lambda}(y,x)-\frac{g_{+,q+\lambda}(y)}{g_{+,q+\lambda}(0)}G_{q+\lambda}(0,x)\Big]m(\mathrm{d}x),\quad x,y<0.$$

Proof Given $x,y>0$, we have

$$\begin{aligned}&\mathbb{E}_y[\mathrm{e}^{-\lambda\int_0^{e_q}1_{(0,\infty)}(X_s)\mathrm{d}s};e_q<\tau_0,X(e_q)\in\mathrm{d}x]\\&=\mathbb{P}_y\{e_q<e_\lambda\wedge\tau_0,X(e_q)\in\mathrm{d}x\}\\&=q\int_0^\infty\mathbb{P}_y\{t<e_\lambda\wedge\tau_0,X(t)\in\mathrm{d}x\}\mathrm{e}^{-qt}\mathrm{d}t\end{aligned}$$

$$= q\int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{P}_y\{t<\tau_0, X(t)\in \mathrm{d}x\}\mathrm{e}^{-qt}\mathrm{d}t$$

$$= q\Big[G_{q+\lambda}(y,x) - \frac{g_{-,q+\lambda}(y)}{g_{-,q+\lambda}(0)}\, G_{q+\lambda}(0,x)\Big]m(\mathrm{d}x).$$

With similar arguments, we can get the result of case $x, y<0$. □

Letting $\lambda_- \to 0_+$ and $\lambda_+ \to 0_+$, respectively, in Corollary 1, we obtain the following corollary.

Corollary 3 *When* $\lambda_- \to 0_+$, $\lambda_+ = \lambda$, *we have*

$$\mathbb{E}\mathrm{e}^{-\lambda\int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s} = 1 - \frac{\frac{\lambda}{q+\lambda} - \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}\, m(\mathrm{d}y)},$$

$$\mathbb{E}_y\mathrm{e}^{-\lambda\int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}$$

$$= \begin{cases} 1 - \dfrac{g_{+,q}(y)}{g_{+,q}(0)}\cdot \dfrac{\frac{\lambda}{q+\lambda} - \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}\, m(\mathrm{d}y)}, & y<0,\\ \dfrac{q}{q+\lambda} + \dfrac{g_{-,q+\lambda}(y)}{g_{-,q+\lambda}(0)}\left(\dfrac{\lambda}{q+\lambda} - \dfrac{\frac{\lambda}{q+\lambda} - \lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_{-\infty}^0 G_{q+\lambda}(0,y)\frac{g_{+,q}(y)}{g_{+,q}(0)}\, m(\mathrm{d}y)}\right), & y>0; \end{cases}$$

when $\lambda_+ \to 0_+$, $\lambda_- = \lambda$, *we have*

$$\mathbb{E}\mathrm{e}^{-\lambda\int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s} = 1 - \frac{\frac{\lambda}{q+\lambda} - \lambda\int_0^\infty G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_0^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(0)}\, m(\mathrm{d}y)},$$

$$\mathbb{E}_y\mathrm{e}^{-\lambda\int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s}$$

$$= \begin{cases} \dfrac{q}{q+\lambda} + \dfrac{g_{+,q+\lambda}(y)}{g_{+,q+\lambda}(0)}\left(\dfrac{\lambda}{q+\lambda} - \dfrac{\frac{\lambda}{q+\lambda} - \lambda\int_0^\infty G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_0^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(0)}\, m(\mathrm{d}y)}\right), & y<0,\\ 1 - \dfrac{g_{-,q}(y)}{g_{-,q}(0)}\cdot \dfrac{\frac{\lambda}{q+\lambda} - \lambda\int_0^\infty G_{q+\lambda}(0,y)m(\mathrm{d}y)}{1-\lambda\int_0^\infty G_{q+\lambda}(0,y)\frac{g_{-,q}(y)}{g_{-,q}(0)}\, m(\mathrm{d}y)}, & y>0. \end{cases}$$

4 Examples

In this section, we apply the results in Section 3 to some examples to find more explicit expressions.

4.1 Brownian motion with drift

Let $X_t = \mu t + W_t$ be a Brownian motion with drift. The corresponding differential equation (3) is

$$\frac{1}{2}\, g''(x) + \mu g'(x) = qg(x), \quad q>0,$$

with two independent solutions

$$g_{+,q}(x) = \mathrm{e}^{(-\mu+\sqrt{\mu^2+2q})x}, \quad g_{-,q}(x) = \mathrm{e}^{(-\mu-\sqrt{\mu^2+2q})x};$$

see [3, pp. 127, 128]. We also have

$$m(\mathrm{d}x) = 2\mathrm{e}^{2\mu x}\mathrm{d}x, \quad \omega_q = 2\sqrt{2q+\mu^2}, \quad G_q(x,y) = \omega_q^{-1}\mathrm{e}^{-\mu(x+y)}\mathrm{e}^{-\sqrt{\mu^2+2q}\,|y-x|}.$$

By Theorem 1, for $x < 0$, with some computing, we have

$$\begin{aligned}
B(0) &= 1 - \frac{\lambda_-}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}\int_0^\infty \mathrm{e}^{-(\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_+)})z}\mathrm{d}z \\
&\quad - \frac{\lambda_+}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}\int_{-\infty}^0 \mathrm{e}^{(\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_-)})z}\mathrm{d}z \\
&= \frac{\sqrt{\mu^2+2(q+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_-)}}{2\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}},
\end{aligned}$$

$$\begin{aligned}
&A(0,\mathrm{d}x) \\
&= \frac{q\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}} + \frac{q\lambda_+}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}\,\sqrt{\mu^2+2(q+\lambda_-)}} \\
&\quad \times \bigg[\int_{-\infty}^0 \mathrm{e}^{\mu x+z\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}\,\mathrm{e}^{-\sqrt{\mu^2+2(q+\lambda_-)}\,|x-z|}\mathrm{d}z \\
&\quad - \mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-)})x}\int_{-\infty}^0 \mathrm{e}^{(\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_-)})z}\mathrm{d}z\bigg]\mathrm{d}x \\
&= \frac{q\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}} + \frac{q}{2\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}\,\sqrt{\mu^2+2(q+\lambda_-)}} \\
&\quad \times [2\sqrt{\mu^2+2(q+\lambda_-)}\,\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-)})x} \\
&\quad - 2\sqrt{\mu^2+2(q+\lambda_-)}\,\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}]\mathrm{d}x \\
&= \frac{q\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}.
\end{aligned}$$

By Remark 1, we have

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\lambda_-\int_0^{e_q}1_{(-\infty,0)}(X_s)\mathrm{d}s-\lambda_+\int_0^{e_q}1_{(0,\infty)}(X_s)\mathrm{d}s}; X(e_q)\in\mathrm{d}x] \\
&\quad = \frac{A(0,\mathrm{d}x)}{B(0)} \\
&\quad = \frac{2q\mathrm{e}^{(\mu+\sqrt{\mu^2+2(q+\lambda_-)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_-)}}. \qquad (14)
\end{aligned}$$

For $x>0$, we have

$$
\begin{aligned}
&A'(0,\mathrm{d}x)\\
&=\frac{q\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}+\frac{q\lambda_-}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}\sqrt{\mu^2+2(q+\lambda_+)}}\\
&\quad\times\bigg[\int_0^\infty \mathrm{e}^{\mu x-z\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}\,\mathrm{e}^{-\sqrt{\mu^2+2(q+\lambda_+)}\,|x-z|}\mathrm{d}z\\
&\quad-\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_+)})x}\int_0^\infty \mathrm{e}^{-(\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_+)})z}\mathrm{d}z\bigg]\mathrm{d}x\\
&=\frac{q\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}}-\frac{q}{2\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}\sqrt{\mu^2+2(q+\lambda_+)}}\\
&\quad\times[2\sqrt{\mu^2+2(q+\lambda_+)}\,\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)})x}\\
&\quad-2\sqrt{\mu^2+2(q+\lambda_+)}\,\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_+)})\,x}]\mathrm{d}x\\
&=\frac{q\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_-+\lambda_+)}},
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&\mathbb{E}[\mathrm{e}^{-\lambda_-\int_0^{e_q}1_{(-\infty,0)}(X_s)\mathrm{d}s-\lambda_+\int_0^{e_q}1_{(0,\infty)}(X_s)\mathrm{d}s};X(e_q)\in\mathrm{d}x]\\
&\quad=\frac{A'(0,\mathrm{d}x)}{B(0)}\\
&\quad=\frac{2q\mathrm{e}^{(\mu-\sqrt{\mu^2+2(q+\lambda_+)})x}\mathrm{d}x}{\sqrt{\mu^2+2(q+\lambda_+)}+\sqrt{\mu^2+2(q+\lambda_-)}}.
\end{aligned}
\tag{15}
$$

Expressions (14) and (15) agree with [3, Expression 1.6.5, p. 260] for $r=0$.

4.2 Skew Brownian motion

Skew Brownian motion, proposed by Itô and McKean [15], is a natural generalization of the Brownian motion. We now briefly introduce the skew Brownian motion. Let X be a skew Brownian motion of parameter β with $\beta\in(0,1)$. Process X is specified by the following stochastic differential equation

$$\mathrm{d}X_t=\mathrm{d}W_t+(2\beta-1)\mathrm{d}L_t^0(X), \tag{16}$$

where W_t is a one-dimensional standard Brownian motion and $L_t^0(X)$ is the local time at 0 for X. Equation (16) has a unique strong solution; see Lejay [20]. In addition, from [3, p. 126], we have

$$
m(\mathrm{d}x)=\begin{cases}2\beta\mathrm{d}x, & x>0,\\ 2(1-\beta)\mathrm{d}x, & x<0,\end{cases}\qquad s(x)=\begin{cases}\dfrac{x}{\beta}, & x\geqslant 0,\\ \dfrac{x}{1-\beta}, & x<0,\end{cases}
$$

and

$$G_q(x,y)=\frac{\mathrm{e}^{-|x-y|\sqrt{2q}}-\mathrm{e}^{-(|x|+|y|)\sqrt{2q}}}{\sqrt{2q}\,(1+(2\beta-1)\mathrm{sgn}(x\wedge y))}+\frac{\mathrm{e}^{-(|x|+|y|)\sqrt{2q}}}{\sqrt{2q}}.$$

In addition, $\omega_q=\sqrt{2q}$. We refer to Borodin and Salminen [3] and Lejay [20] for more details about skew Brownian motion and Appuhamillage et al. [2] for an occupation time related results on skew Brownian motion.

Then for $a<0<b$, the corresponding differential equation (4) of skew Brownian motion is

$$2q(1-\beta)\int_{[a,0)}g(x)\mathrm{d}x+2q\beta\int_{[0,b)}g(x)\mathrm{d}x=\beta g'_-(b)-(1-\beta)g'_-(a),$$

where g'_- denotes the usual left derivative, and it has two independent positive solutions

$$g_{-,q}(x)=\Big[\frac{1-2\beta}{1-\beta}\,\mathrm{sh}(x\sqrt{2q}\,)+\mathrm{e}^{-x\sqrt{2q}}\Big]\mathbf{1}_{(-\infty,0)}(x)+\mathrm{e}^{-x\sqrt{2q}}\mathbf{1}_{[0,\infty)}(x)$$

and

$$g_{+,q}(x)=\mathrm{e}^{x\sqrt{2q}}\mathbf{1}_{(-\infty,0]}(x)+\Big[\frac{1-2\beta}{\beta}\,\mathrm{sh}(x\sqrt{2q})+\mathrm{e}^{x\sqrt{2q}}\Big]\mathbf{1}_{(0,\infty)}(x).$$

For $x<0$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,z)G_{q+\lambda_-}(z,x)m(\mathrm{d}z)\\
&\quad=\int_{-\infty}^{x}\frac{\mathrm{e}^{z\sqrt{2(q+\lambda_-+\lambda_+)}}}{\sqrt{2(q+\lambda_-+\lambda_+)}}\,\frac{\mathrm{e}^{z\sqrt{2(q+\lambda_-)}}}{2(1-\beta)\sqrt{2(q+\lambda_-)}}\\
&\qquad\times\big[\mathrm{e}^{-x\sqrt{2(q+\lambda_-)}}+(1-2\beta)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\,\big]2(1-\beta)\mathrm{d}z\\
&\qquad+\int_{x}^{0}\frac{\mathrm{e}^{z\sqrt{2(q+\lambda_-+\lambda_+)}}}{\sqrt{2(q+\lambda_-+\lambda_+)}}\,\frac{\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}}{2(1-\beta)\sqrt{2(q+\lambda_-)}}\\
&\qquad\times\big[\mathrm{e}^{-z\sqrt{2(q+\lambda_-)}}+(1-2\beta)\mathrm{e}^{z\sqrt{2(q+\lambda_-)}}\,\big]2(1-\beta)\mathrm{d}z\\
&\quad=\frac{1}{2\lambda_+\sqrt{2(q+\lambda_-+\lambda_+)}\,\sqrt{2(q+\lambda_-)}}\,\big[-2\sqrt{2(q+\lambda_-)}\,\mathrm{e}^{x\sqrt{2(q+\lambda_-+\lambda_+)}}\\
&\qquad+(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_-)}\,)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\\
&\qquad+(1-2\beta)(\sqrt{2(q+\lambda_-+\lambda_+)}-\sqrt{2(q+\lambda_-)}\,)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\,\big]
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{0}G_{q+\lambda_-+\lambda_+}(0,z)\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(0)}\,m(\mathrm{d}z)\\
&\quad=\frac{2(1-\beta)}{\sqrt{2(q+\lambda_-+\lambda_+)}}\int_{-\infty}^{0}\mathrm{e}^{z(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_-)}\,)}\mathrm{d}z\\
&\quad=\frac{(1-\beta)(\sqrt{2(q+\lambda_-+\lambda_+)}-\sqrt{2(q+\lambda_-)}\,)}{\lambda_+\sqrt{2(q+\lambda_-+\lambda_+)}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}A(0,\mathrm{d}x)&=\frac{2q(1-\beta)\mathrm{e}^{x\sqrt{2(q+\lambda_-+\lambda_+)}}\,\mathrm{d}x}{\sqrt{2(q+\lambda_-+\lambda_+)}}+\frac{q(1-\beta)\mathrm{d}x}{\sqrt{2(q+\lambda_-+\lambda_+)}\,\sqrt{2(q+\lambda_-)}}\\&\quad\times\big[-2\sqrt{2(q+\lambda_-)}\,\mathrm{e}^{x\sqrt{2(q+\lambda_-+\lambda_+)}}+2\sqrt{2(q+\lambda_-)}\,\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\,\big]\\&=\frac{2q(1-\beta)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\,\mathrm{d}x}{\sqrt{2(q+\lambda_-+\lambda_+)}}.\end{aligned}$$

Similarly, for $x>0$, we have the following results through some algebras:

$$\begin{aligned}&\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,z)G_{q+\lambda_+}(z,x)m(\mathrm{d}z)\\&=\frac{1}{2\lambda_-\sqrt{2(q+\lambda_-+\lambda_+)}\,\sqrt{2(q+\lambda_+)}}\big[-2\sqrt{2(q+\lambda_+)}\mathrm{e}^{-x\sqrt{2(q+\lambda_-+\lambda_+)}}\\&\quad+(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_+)}\,)\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}\\&\quad-(2\beta-1)(\sqrt{2(q+\lambda_+)}-\sqrt{2(q+\lambda_-+\lambda_+)}\,)\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}\,\big],\end{aligned}$$

$$\begin{aligned}&\int_0^\infty G_{q+\lambda_-+\lambda_+}(0,z)\frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(0)}\,m(\mathrm{d}z)\\&=\frac{2\beta}{\sqrt{2(q+\lambda_-+\lambda_+)}}\int_0^\infty \mathrm{e}^{-z(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_+)}\,)}\mathrm{d}z\\&=\frac{\beta(\sqrt{2(q+\lambda_-+\lambda_+)}-\sqrt{2(q+\lambda_+)}\,)}{\lambda_-\sqrt{2(q+\lambda_-+\lambda_+)}}.\end{aligned}$$

So,

$$\begin{aligned}A'(0,\mathrm{d}x)&=\frac{2q\beta\mathrm{e}^{-x\sqrt{2(q+\lambda_-+\lambda_+)}}\,\mathrm{d}x}{\sqrt{2(q+\lambda_-+\lambda_+)}}+\frac{2q\beta\mathrm{d}x}{2\sqrt{2(q+\lambda_-+\lambda_+)}\,\sqrt{2(q+\lambda_+)}}\\&\quad\times\big[-2\sqrt{2(q+\lambda_+)}\,\mathrm{e}^{-x\sqrt{2(q+\lambda_-+\lambda_+)}}+2\sqrt{2(q+\lambda_+)}\,\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}\,\big]\\&=\frac{2q\beta\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}\,\mathrm{d}x}{\sqrt{2(q+\lambda_-+\lambda_+)}}.\end{aligned}$$

Then

$$\begin{aligned}B(0)&=1-\frac{2\lambda_-\beta}{\sqrt{2(q+\lambda_-+\lambda_+)}}\int_0^\infty \mathrm{e}^{-(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_+)}\,)z}\mathrm{d}z\\&\quad-\frac{2\lambda_+(1-\beta)}{\sqrt{2(q+\lambda_-+\lambda_+)}}\int_{-\infty}^0 \mathrm{e}^{(\sqrt{2(q+\lambda_-+\lambda_+)}+\sqrt{2(q+\lambda_-)}\,)z}\mathrm{d}z\\&=\frac{\beta\sqrt{2(q+\lambda_+)}+(1-\beta)\sqrt{2(q+\lambda_-)}}{\sqrt{2(q+\lambda_-+\lambda_+)}}.\end{aligned}$$

By Theorem 1 and Remark 1, we have

$$\mathbb{E}[\mathrm{e}^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x]$$
$$= \begin{cases} \dfrac{A(0,\mathrm{d}x)}{B(0)} = \dfrac{2q(1-\beta)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\,\mathrm{d}x, & x<0, \\ \dfrac{A'(0,\mathrm{d}x)}{B(0)} = \dfrac{2q\beta\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\,\mathrm{d}x, & x>0; \end{cases}$$

$$\mathscr{I}^y_{(-\infty,0)\cup(0,\infty)}(\mathrm{d}x)$$
$$= \begin{cases} \dfrac{2(1-\beta)\mathrm{e}^{x\sqrt{2(q+\lambda_-)}}\,\mathrm{e}^{-y\sqrt{2(q+\lambda_+)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\,\mathrm{d}x, & x<0<y, \\ \Bigg[\dfrac{\mathrm{e}^{-|x-y|\sqrt{2(q+\lambda_-)}} - \mathrm{e}^{(x+y)\sqrt{2(q+\lambda_-)}}}{\sqrt{2(q+\lambda_-)}} \\ \quad + \dfrac{2(1-\beta)\mathrm{e}^{(x+y)\sqrt{2(q+\lambda_-)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\Bigg]\mathrm{d}x, & x,y<0, \\ \Bigg[\dfrac{\mathrm{e}^{-|x-y|\sqrt{2(q+\lambda_+)}} - \mathrm{e}^{-(x+y)\sqrt{2(q+\lambda_+)}}}{\sqrt{2(q+\lambda_+)}} \\ \quad + \dfrac{2\beta\mathrm{e}^{-(x+y)\sqrt{2(q+\lambda_+)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\Bigg]\mathrm{d}x, & x,y>0, \\ \dfrac{2\beta\mathrm{e}^{-x\sqrt{2(q+\lambda_+)}}\,\mathrm{e}^{y\sqrt{2(q+\lambda_-)}}}{\beta\sqrt{2(q+\lambda_+)} + (1-\beta)\sqrt{2(q+\lambda_-)}}\,\mathrm{d}x, & y<0<x. \end{cases}$$

Letting $\beta = 1/2$, one can recover the well-known results for Brownian motion.

4.3 Brownian motion with two-valued drift

Let X be a Brownian motion with two-valued drift, specified by the following stochastic differential equation:

$$\mathrm{d}X_t = (\mu_L 1_{(-\infty,0)}(X_t) - \mu_R 1_{(0,\infty)}(X_t))\mathrm{d}t + \mathrm{d}W_t, \tag{17}$$

where $\mu_L, \mu_R \in \mathbb{R}$ and W_t is a standard one-dimensional Brownian motion. The Brownian motion with two-valued drift, referred as a refracted Brownian motion, is also interested to risk theory.

Although the Lipschitz assumption (2) for drift function

$$\mu(\cdot) = \mu_L 1_{(-\infty,0)}(\cdot) - \mu_R 1_{(0,\infty)}(\cdot)$$

fails, equation (17) still has a unique strong solution, see Prokhorov and Shiryaev [32].

In this two-valued drift model, for $q > 0$, two independent, positive, and convex solutions of the differential equation

$$\frac{1}{2}\, g''(x) + (\mu_L 1_{(-\infty,0)}(x) - \mu_R 1_{(0,\infty)}(x))g'(x) - qg(x) = 0,$$

with $g_{-,q}(\cdot)$ strictly decreasing and $g_{+,q}(\cdot)$ strictly increasing are given by

$$\begin{aligned} g_{-,q}(x) = {}& \mathrm{e}^{(\mu_R - \sqrt{\mu_R^2+2q})x} 1_{x>0} \\ & + \big[c_-^q \mathrm{e}^{(-\mu_L + \sqrt{\mu_L^2+2q})x} + (1-c_-^q)\mathrm{e}^{(-\mu_L - \sqrt{\mu_L^2+2q})x}\big] 1_{x<0} \end{aligned}$$

and

$$\begin{aligned} g_{+,q}(x) = {}& \big[c_+^q \mathrm{e}^{(\mu_R + \sqrt{\mu_R^2+2q})x} + (1-c_+^q)\mathrm{e}^{(\mu_R - \sqrt{\mu_R^2+2q})x}\big] 1_{x>0} \\ & + \mathrm{e}^{(-\mu_L + \sqrt{\mu_L^2+2q})x} 1_{x<0}, \end{aligned}$$

respectively, where

$$c_-^q = \frac{\mu_R - \sqrt{\mu_R^2+2q} + \mu_L + \sqrt{\mu_L^2+2q}}{2\sqrt{\mu_L^2+2q}},$$

$$c_+^q = \frac{-\mu_L + \sqrt{\mu_L^2+2q} - \mu_R + \sqrt{\mu_R^2+2q}}{2\sqrt{\mu_R^2+2q}};$$

see [22, Section 5.2]. In addition,

$$B(x) = -2\mu_R x 1_{x>0} + 2\mu_L x 1_{x<0},$$

$$m(x) = 2(\mathrm{e}^{-2\mu_R x} 1_{x>0} + \mathrm{e}^{2\mu_L x} 1_{x<0}),$$

$$s(x) = \frac{\mathrm{e}^{2\mu_R x} - 1}{2\mu_R}\, 1_{x>0} + \frac{1 - \mathrm{e}^{-2\mu_L x}}{2\mu_L}\, 1_{x<0},$$

$$\omega_q = -\mu_L - \mu_R + \sqrt{\mu_L^2+2q} + \sqrt{\mu_R^2+2q}.$$

We can also get the Green function of this process

$$G_q(x,y) = \begin{cases} \omega_q^{-1} \mathrm{e}^{(-\mu_L + \sqrt{\mu_L^2+2q})y} & \\ \quad \times \big[c_-^q \mathrm{e}^{(-\mu_L + \sqrt{\mu_L^2+2q})x} + (1-c_-^q)\mathrm{e}^{(-\mu_L - \sqrt{\mu_L^2+2q})x}\big], & 0 \geqslant x \geqslant y, \\ \omega_q^{-1} \mathrm{e}^{(-\mu_L + \sqrt{\mu_L^2+2q})y} \mathrm{e}^{(\mu_R - \sqrt{\mu_R^2+2q})x}, & x \geqslant 0 \geqslant y, \\ \omega_q^{-1} \mathrm{e}^{(\mu_R - \sqrt{\mu_R^2+2q})x} & \\ \quad \times \big[c_+^q \mathrm{e}^{(\mu_R + \sqrt{\mu_R^2+2q})y} + (1-c_+^q)\mathrm{e}^{(\mu_R - \sqrt{\mu_R^2+2q})y}\big], & x \geqslant y \geqslant 0. \end{cases}$$

For $x<0$,

$$\begin{aligned}
&\int_{-\infty}^{0} G_{q+\lambda_-+\lambda_+}(0,z)G_{q+\lambda_-}(z,x)m(\mathrm{d}z)\\
&\quad=\frac{2(1-c_-^{q+\lambda_-})\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)})x}+2c_-^{q+\lambda_-}\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x}}{\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_-}\big(\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}\big)}\\
&\qquad+\frac{2(1-c_-^{q+\lambda_-})\big(\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x}-\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)})x}\big)}{\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_-}\big(\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}-\sqrt{\mu_L^2+2(q+\lambda_-)}\big)}\\
&\quad=\frac{1}{\lambda_+\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_-}}\big[-2(1-c_-^{q+\lambda_-})\sqrt{\mu_L^2+2(q+\lambda_-)}\\
&\qquad\times\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)})x}+\big(\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}\\
&\qquad+\sqrt{\mu_L^2+2(q+\lambda_-)}-2c_-^{q+\lambda_-}\sqrt{\mu_L^2+2(q+\lambda_-)}\big)\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x}\big],
\end{aligned}$$

$$\begin{aligned}
&G_{q+\lambda_-}(0,x)\int_{-\infty}^{0} G_{q+\lambda_-+\lambda_+}(0,z)\frac{g_{+,q+\lambda_-}(z)}{g_{+,q+\lambda_-}(0)}m(\mathrm{d}z)\\
&\quad=\frac{\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}-\sqrt{\mu_L^2+2(q+\lambda_-)}}{\lambda_+\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_-}}\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x},
\end{aligned}$$

$$A(0,\mathrm{d}x)=\frac{q\mathrm{e}^{(\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x}\mathrm{d}x}{(1-c_-^{q+\lambda_-+\lambda_+})\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}}.$$

Similarly, for $x>0$,

$$\begin{aligned}
&\int_{0}^{\infty} G_{q+\lambda_-+\lambda_+}(0,z)G_{q+\lambda_-}(z,x)m(\mathrm{d}z)\\
&\quad=\frac{1}{\lambda_-\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_+}}\big[-2c_+^{q+\lambda_+}\sqrt{\mu_R^2+2(q+\lambda_+)}\\
&\qquad\times\mathrm{e}^{(\mu_R-\sqrt{\mu_R^2+2(q+\lambda_-+\lambda_+)})x}+\big(\sqrt{\mu_R^2+2(q+\lambda_-+\lambda_+)}\\
&\qquad-\sqrt{\mu_R^2+2(q+\lambda_+)}+2c_+^{q+\lambda_+}\sqrt{\mu_R^2+2(q+\lambda_+)}\big)\mathrm{e}^{(\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})x}\big],
\end{aligned}$$

$$\begin{aligned}
&G_{q+\lambda_+}(0,x)\int_{0}^{\infty} G_{q+\lambda_-+\lambda_+}(0,z)\frac{g_{-,q+\lambda_+}(z)}{g_{-,q+\lambda_+}(0)}m(\mathrm{d}z)\\
&\quad=\frac{\sqrt{\mu_R^2+2(q+\lambda_-+\lambda_+)}-\sqrt{\mu_R^2+2(q+\lambda_+)}}{\lambda_-\omega_{q+\lambda_-+\lambda_+}\omega_{q+\lambda_+}}\mathrm{e}^{(\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})x}.
\end{aligned}$$

So

$$A'(0,\mathrm{d}x)=\frac{q\mathrm{e}^{(-\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})x}\mathrm{d}x}{c_+^{q+\lambda_-+\lambda_+}\sqrt{\mu_R^2+2(q+\lambda_-+\lambda_+)}}.$$

Then

$$B(0)=1-\frac{2\lambda_-}{\omega_{q+\lambda_-+\lambda_+}}\int_0^\infty \mathrm{e}^{-(\sqrt{\mu_R^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu_R^2+2(q+\lambda_+)})z}\mathrm{d}z\\ -\frac{2\lambda_+}{\omega_{q+\lambda_-+\lambda_+}}\int_{-\infty}^0 \mathrm{e}^{(\sqrt{\mu_L^2+2(q+\lambda_-+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)})z}\mathrm{d}z\\ =\frac{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}{\omega_{q+\lambda_-+\lambda_+}}.$$

By Theorem 1 and Remark 1, we have

$$\mathbb{E}[\mathrm{e}^{-\lambda_-\int_0^{e_q}1_{(-\infty,0)}(X_s)\mathrm{d}s-\lambda_+\int_0^{e_q}1_{(0,\infty)}(X_s)\mathrm{d}s};X(e_q)\in\mathrm{d}x]\\ =\begin{cases}\dfrac{A(0,\mathrm{d}x)}{B(0)}=\dfrac{2q\mathrm{e}^{(\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x\mathrm{d}x}}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & x<0,\\ \dfrac{A'(0,\mathrm{d}x)}{B(0)}=\dfrac{2q\mathrm{e}^{(-\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})x\mathrm{d}x}}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & x>0;\end{cases}$$

$$\mathscr{I}^y_{(-\infty,0)\cup(0,\infty)}(\mathrm{d}x)\\ =\begin{cases}\dfrac{2\mathrm{e}^{(\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})y}\mathrm{e}^{(\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})x}\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & x<0<y,\\ \dfrac{2(c_-^{q+\lambda_-}-1)\mathrm{e}^{\mu_L(x-y)}\big[\mathrm{e}^{\sqrt{\mu_L^2+2(q+\lambda_-)}\,(x+y)}-\mathrm{e}^{-\sqrt{\mu_L^2+2(q+\lambda_-)}\,|x-y|}\big]\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_-)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}\\ \quad+\dfrac{2\mathrm{e}^{\mu_L(x-y)}\mathrm{e}^{\sqrt{\mu_L^2+2(q+\lambda_-)}\,(x+y)}\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & x,y<0,\\ \dfrac{2c_+^{q+\lambda_+}\mathrm{e}^{\mu_R(y-x)}\big[\mathrm{e}^{-\sqrt{\mu_R^2+2(q+\lambda_+)}\,|x-y|}-\mathrm{e}^{-\sqrt{\mu_R^2+2(q+\lambda_+)}\,(x+y)}\big]\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_+)}}\\ \quad+\dfrac{2\mathrm{e}^{\mu_R(y-x)}\mathrm{e}^{-\sqrt{\mu_R^2+2(q+\lambda_+)}\,(x+y)}\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & x,y>0,\\ \dfrac{2\mathrm{e}^{(-\mu_R-\sqrt{\mu_R^2+2(q+\lambda_+)})x}\mathrm{e}^{(-\mu_L+\sqrt{\mu_L^2+2(q+\lambda_-)})y}\mathrm{d}x}{-\mu_L-\mu_R+\sqrt{\mu_R^2+2(q+\lambda_+)}+\sqrt{\mu_L^2+2(q+\lambda_-)}}, & y<0<x.\end{cases}$$

Letting $\mu_L=-\mu_R=\mu$, we can recover the results for Brownian motion with drift, see [3, Expression 1.6.5, p. 260] for $r=0$.

Acknowledgements The authors would like to thank the anonymous referees for helpful suggestions and comments. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11171044, 11571052, 11671132), the Natural Science Foundation of Hunan Province (Grant No. 2016JJ4061), the Scientific Research Project of Hunan University of Arts and Science (Grant No. 15ZD05), and the Project Sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry (2015).

Appendix Proof of Lemma 1

Proof of Lemma 1 Note that

$$\int_0^\infty \mathbb{P}_y\{t < \tau_a, X_t \in \mathrm{d}x\}\mathrm{e}^{-qt}\mathrm{d}t = \frac{1}{q}\,\mathbb{P}_y\{e_q < \tau_a, X(e_q) \in \mathrm{d}x\}.$$

Since

$$\begin{aligned}
&\mathbb{P}_y\{e_q < \tau_a, X(e_q) \in \mathrm{d}x\}\\
&\quad = \mathbb{P}_y\{X(e_q) \in \mathrm{d}x\} - \mathbb{P}_y\{e_q > \tau_a, X(e_q) \in \mathrm{d}x\}\\
&\quad = \mathbb{P}_y\{X(e_q) \in \mathrm{d}x\} - \mathbb{P}_y\{\tau_a < e_q\}\mathbb{P}_a\{X(e_q) \in \mathrm{d}x\}\\
&\quad = q\int_0^\infty \mathrm{e}^{-qt}p(t;y,x)m(\mathrm{d}x)\mathrm{d}t - q\mathbb{E}_y\mathrm{e}^{-q\tau_a}\int_0^\infty \mathrm{e}^{-qt}p(t;a,x)m(\mathrm{d}x)\mathrm{d}t\\
&\quad = qG_q(y,x)m(\mathrm{d}x) - q\mathbb{E}_y\mathrm{e}^{-q\tau_a}G_q(a,x)m(\mathrm{d}x)\\
&\quad = \begin{cases} q\Big[G_q(y,x) - \dfrac{g_{-,q}(y)}{g_{-,q}(a)}G_q(a,x)\Big]m(\mathrm{d}x), & y \in (a,\infty),\\ q\Big[G_q(y,x) - \dfrac{g_{+,q}(y)}{g_{+,q}(a)}G_q(a,x)\Big]m(\mathrm{d}x), & y \in (-\infty,a), \end{cases}
\end{aligned}$$

the result of Lemma 1 then follows. □

References

1. Albrecher H, Ivanovs J, Zhou X. Exit identities for Lévy processes observed at Poisson arrival times. Bernoulli (to appear)
2. Appuhamillage T, Bokil V, Thomann E, Waymire E, Wood B. Occupation and local times for skew Brownian motion with applications to dispersion across an interface. Ann Probab, 2011, 21: 183–214
3. Borodin A N, Salminen P. Handbook of Brownian Motion-Facts and Formulae. 2nd ed. Basel: Birkhäuser Verlag, 2002
4. Breuer L. Occupation times for Markov-modulated Brownian motion. J Appl Probab, 2012, 49: 549–565
5. Cai N, Chen N, Wan X. Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options. Math Oper Res, 2010, 35: 412–437
6. Chen Y, Yang X, Li Y, Zhou X. A joint Laplace transform for pre-exit diffusion of occupation times. Acta Math Sin (Engl Ser) (to appear)

7. Davydov A, Linetsky V. Structuring, pricing and hedging double-barrier step options. J Comput Finance, 2002, 5: 55–87
8. Dickson D C M, Egídio dos Reis A D. On the distribution of the duration of negative surplus. Scand Actuar J, 1996, 2: 148–164
9. Dickson D C M, Li S. The distributions of the time to reach a given level and the duration of negative surplus in the Erlang(2) risk model. Insurance Math Econom, 2013, 52: 490–497
10. Egídio dos Reis A D. How long is the surplus below zero? Insurance Math Econom, 1993, 12: 23–38
11. Evans L C. An Introduction to Stochastic Differential Equations. Berkeley: UC Berkeley, 2008
12. Feller W. Diffusion processes in one dimension. Trans Amer Math Soc, 1954, 77: 1–31
13. Fusai G. Corridor options and arc-sine law. Ann Appl Probab, 2000, 10: 634–663
14. Guérin H, Renaud J F. Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view. arXiv: 1406.3130
15. Itô K, McKean H. Diffusion and Their Sample Paths. New York: Springer-Verlag, 1974
16. Kolkovska E T, López-Mimbela J A, Morales J V. Occupation measure and local time of classical risk processes. Insurance Math Econom, 2005, 37: 573–584
17. Kyprianou A E, Pardo J C, Pérez J L. Occupation times of refracted Lévy processes. J Theoret Probab, 2014, 27: 1292–1315
18. Landriault D, Renaud J F, Zhou X. Occupation times of spectrally negative Lévy processes with applications. Stochastic Process Appl, 2011, 121: 2629–2641
19. Landriault D, Shi T X. Occupation times in the MAP risk model. Insurance Math Econom, 2015, 60: 75–82
20. Lejay A. On the constructions of the skew Brownian motion. Probab Surv, 2006, 3: 413–466
21. Li B, Palmowski Z. Fluctuations of Omega-killed spectrally negative Lévy processes. arXiv: 1603.07962
22. Li B, Zhou X. The joint Laplace transforms for diffusion occupation times. Adv Appl Probab, 2013, 45: 1–19
23. Li Y, Wang S, Zhou X, Zhu N. Diffusion occupation time before exiting. Front Math China, 2014, 9: 843–861
24. Li Y, Zhou X. On pre-exit joint occupation times for spectrally negative Lévy processes. Statist Probab Lett, 2014, 94: 48–55
25. Li Y, Zhou X, Zhu N. Two-sided discounted potential measures for spectrally negative Lévy processes. Statist Probab Lett, 2015, 100: 67–76
26. Linetsky V. Step options. Math Finance, 1999, 9: 55–96
27. Loeffen R L, Renaud J F, Zhou X. Occupation times of intervals until first passage times for spectrally negative Lévy processes. Stochastic Process Appl, 2014, 124: 1408–1435
28. Pérez J L, Yamazaki K. On the refracted-reflected spectrally negative Lévy processes. arXiv: 1511.06027
29. Pérez J L, Yamazaki K. Refraction-Reflection Strategies in the Dual Model. arXiv: 1511.07918
30. Pitman J, Yor M. Laplace transforms related to excursions of a one-dimensional diffusion. Bernoulli, 1999, 5: 249–255
31. Pitman J, Yor M. Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches. Bernoulli, 2003, 9: 1–24
32. Prokhorov Y V, Shiryaev A N. Probability Theory III. Stochastic Calculus. Berlin: Springer-Verlag, 1998
33. Renaud J F. On the time spent in the red by a refracted Lévy process. J Appl Prob, 2014, 51: 1171–1188
34. Yin C, Yuen C. Exact joint laws associated with spectrally negative Lévy processes and applications to insurance risk theory. Front Math China, 2014, 9: 1453–1471