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RESEARCH ARTICLE

Differential Structure of Kähler Manifold with Almost Nonnegative Orthogonal Bisectional Curvature

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Abstract In this short note, we suggest a definition of almost nonnegativity for orthogonal bisectional curvature and quadratic orthogonal bisectional curvature. Moreover we obtain the differential structure of universal covering of a compact Kähler manifold with almost nonnegative orthogonal bisectional curvature, which implies one of Fang's conjecture under an additional scalar curvature upper bound condition.

Keywords Orthogonal bisectional curvature, almost nonnegative

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1 Introduction

In the past several decades, a great deal of mathematical effort has been devoted to the study of differential and holomorphic structures of manifolds under certain curvature conditions. Our story dates back to the classic uniformization theorem of Riemannian surfaces. During recent years, a lot of progress has been made to generalizations of uniformization theorem in higher dimensions. There are two major lines. The first is Mori's work about the Hartshorne conjecture. And later on, Demailly–Peternell–Schneider [5] gave the structure of a compact Kähler manifold with nef tangent bundle. The second line is Siu and Yau's solution to the Frankel conjecture, and then some generalized results such as Mok's famous work [11]. In [7], Gu and Zhang used Kähler–Ricci flow combining Chen's result [4] to study the holomorphic structure of closed Kähler manifolds with nonnegative holomorphic orthogonal bisectional curvature, which extended Mok's result. Recall the notion of nonnegative holomorphic orthogonal bisectional curvature, which means that at any point

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$p \in X$, for any unitary frame $\{e_\alpha\}_{\alpha=1}^n$ of $T_p^{1,0}X$,

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0 \quad \text{for any } \alpha \neq \beta.$$

Furthermore, Chau and Tam studied the complex structure of compact Kähler manifolds with nonnegative quadratic orthogonal bisectional curvature (denoted by $QB \geq 0$) in [2]. The definition of $QB \geq 0$ is that for any unitary frame $\{e_\alpha\}_{\alpha=1}^n$ of $T_p^{1,0}X$, and any real numbers $\{\xi_\alpha\}_{\alpha=1}^n$,

$$\sum_{\alpha\neq\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}}(\xi_\alpha - \xi_\beta)^2 \geq 0.$$

Obviously, this is a weaker condition than nonnegative orthogonal bisectional curvature. For Kähler surfaces, the condition of nonnegative quadratic orthogonal bisectional curvature is equivalent to nonnegative orthogonal bisectional curvature. For $n > 2$, in [10], the authors constructed an example of compact Kähler manifold with nonnegative quadratic orthogonal bisectional curvature which does not admit any Kähler metrics with nonnegative orthogonal bisectional curvature.

For almost nonnegative curvatures, there also are some structure results. In [6], Fang studied the structure of compact Kähler manifolds with almost nonnegative bisectional curvature. He conjectured that there exists a constant $\epsilon(n) > 0$ such that if a simply connected compact Kähler manifold has holomorphic bisectional curvature satisfying $B \cdot \mathrm{diam}(X^n)^2 \geq -\epsilon(n)$, then X is diffeomorphic to $X_1 \times \cdots \times X_k$, where each X_j $(1 \leq j \leq k)$ is either a complex projective space or an irreducible Kähler symmetric space of rank ≥ 2. Huang solved this conjecture under an additional condition that the sectional curvature is bounded from above [9], where he used the technique of Peterson and Tao [13].

Now we suggest the notion of almost nonnegativity for orthogonal bisectional curvature and quadratic orthogonal bisectional curvature.

Definition 1.1. We say (X, ω, J) is a Kähler manifold with quadratic orthogonal bisectional curvature almost nonnegative with normalized coefficients, if the diameter of the manifold is 1, for any sufficiently small $\epsilon > 0$, at any $p \in X$, for any unitary frame $\{e_\alpha\}_{\alpha=1}^n$ of $T_p^{1,0}X$, and any real numbers $\{\xi_\alpha\}_{\alpha=1}^n$ satisfying $\sum_{\alpha\neq\beta}(\xi_\alpha - \xi_\beta)^2 = 1$,

$$\sum_{\alpha\neq\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}}(\xi_\alpha - \xi_\beta)^2 \geq -\epsilon.$$

If $R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq -\epsilon$ for all $\alpha \neq \beta$, then we say X has almost nonnegative orthogonal bisectional curvature.

Obviously, almost nonnegative bisectional curvature defined in [6] implies almost nonnegative orthogonal bisectional curvature, which further indicates quadratic orthogonal bisectional curvature is almost nonnegative. When $n = 2$, the three conditions are equivalent to each other. Some curvature inequalities are developed in Section 2.

In this short note, we are going to use Chau–Tam's result and Gu–Zhang's classification results to study the differential structure of closed Kähler manifolds with almost nonnegative orthogonal bisectional curvature. Compare to Huang's paper, besides the bisectional curvature bound condition is weaken, we only need the upper bound of scalar curvature instead of sectional curvature. Our main result is

Theorem 1.1. *For any $\Lambda, D > 0, v > 0$ there exists a constant $\epsilon(n, \Lambda, D, v) > 0$, such that if X is an n-dim simply connected closed Kähler manifold with scalar curvature bounded from above by Λ, diameter bounded from above by D, volume bounded from below by v, and the orthogonal bisectional curvature is larger than $-\epsilon$. If $n = 2$, an additional condition of bounded holomorphic sectional curvature is needed. Then the universal covering of X is diffeomorphic to the product $\mathbb{R}^{2l} \times \mathbb{CP}^{k_1} \times \cdots \times \mathbb{CP}^{k_j} \times X_1 \times \cdots \times X_k$ $(l \geq 0)$, where $\{X_p\}_{p=1}^k$ are irreducible compact Hermitian symmetric spaces of rank ≥ 2.*

Remark 1.1. Under the additional condition of scalar curvature upper bound and lower volume bound, Fang's conjecture [6] of the structure of compact manifolds with almost nonnegative bisectional curvature follows immediately. It should be noted that this conjecture can be obtained by a recent work of Bamler, Cabezas-Rivas and Wilking (cf. [1]).

2 Curvature Bounds

In this section, we derive some curvature inequalities for Kähler manifolds with quadratic orthogonal bisectional curvature lower bounded.

Proposition 2.1. *Let (X, ω, J) be a Kähler manifold of complex dimension $n \geq 3$. Assume the quadratic orthogonal bisectional curvature at some point $p \in X$ is bounded below by $-\epsilon$. Then there exists a constant $C(n)$ only depending on the dimension, such that*

$$|\mathrm{Rm}|(p) \leq C(n)(R + \epsilon),$$

here R is the scalar curvature.

Proof. Assume $\{e_\alpha\}_{\alpha=1}^n$ is a unitary frame of $T_p^{1,0}X$. For any $\alpha \neq \beta$, we know that $\frac{e_\alpha + e_\beta}{\sqrt{2}}$ and $\frac{e_\alpha - e_\beta}{\sqrt{2}}$ are also orthogonal, so are $\frac{e_\alpha + \sqrt{-1}e_\beta}{\sqrt{2}}$ and $\frac{e_\alpha - \sqrt{-1}e_\beta}{\sqrt{2}}$.

Fix $\alpha \neq \beta$, let $\xi_\alpha = 0, \xi_\beta = \frac{2}{\sqrt{4n}}$, and let $\xi_\gamma = \frac{1}{\sqrt{4n}}$ for $\gamma \neq \alpha, \beta$, then almost nonnegative quadratic orthogonal bisectional curvature suggests

$$4R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\gamma \neq \alpha, \beta} (R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} + R_{\beta\bar{\beta}\gamma\bar{\gamma}}) \geq -4n\epsilon.$$

Similar to Niu did in [12], we can calculate the curvatures bounds as follows. We consider

$$\begin{aligned}
&4R\Big(\frac{e_\alpha+e_\beta}{\sqrt{2}},\frac{\bar e_\alpha+\bar e_\beta}{\sqrt{2}},\frac{e_\alpha-e_\beta}{\sqrt{2}},\frac{\bar e_\alpha-\bar e_\beta}{\sqrt{2}}\Big)\\
&\quad+\sum_{\gamma\neq\alpha,\beta}\Big[R\Big(\frac{e_\alpha+e_\beta}{\sqrt{2}},\frac{\bar e_\alpha+\bar e_\beta}{\sqrt{2}},\gamma,\bar\gamma\Big)+R\Big(\frac{e_\alpha-e_\beta}{\sqrt{2}},\frac{\bar e_\alpha-\bar e_\beta}{\sqrt{2}},\gamma,\bar\gamma\Big)\Big]\\
&\quad=(R_{\alpha\bar\alpha\alpha\bar\alpha}+R_{\beta\bar\beta\beta\bar\beta}-R_{\alpha\bar\beta\alpha\bar\beta}-R_{\beta\bar\alpha\beta\bar\alpha})+\sum_{\gamma\neq\alpha,\beta}(R_{\alpha\bar\alpha\gamma\bar\gamma}+R_{\beta\bar\beta\gamma\bar\gamma})\\
&\quad\ge-4n\epsilon,
\end{aligned}$$

$$\begin{aligned}
&R\Big(\frac{e_\alpha+\sqrt{-1}e_\beta}{\sqrt{2}},\frac{\bar e_\alpha-\sqrt{-1}\bar e_\beta}{\sqrt{2}},\frac{e_\alpha-\sqrt{-1}e_\beta}{\sqrt{2}},\frac{\bar e_\alpha+\sqrt{-1}\bar e_\beta}{\sqrt{2}}\Big)\\
&\quad+\sum_{\gamma\neq\alpha,\beta}\Big[R\Big(\frac{e_\alpha+\sqrt{-1}e_\beta}{\sqrt{2}},\frac{\bar e_\alpha-\sqrt{-1}\bar e_\beta}{\sqrt{2}},\gamma,\bar\gamma\Big)\\
&\quad+R\Big(\frac{e_\alpha-\sqrt{-1}e_\beta}{\sqrt{2}},\frac{\bar e_\alpha+\sqrt{-1}\bar e_\beta}{\sqrt{2}},\gamma,\bar\gamma\Big)\Big]\\
&\quad=(R_{\alpha\bar\alpha\alpha\bar\alpha}+R_{\beta\bar\beta\beta\bar\beta}+R_{\alpha\bar\beta\alpha\bar\beta}+R_{\beta\bar\alpha\beta\bar\alpha})+\sum_{\gamma\neq\alpha,\beta}(R_{\alpha\bar\alpha\gamma\bar\gamma}+R_{\beta\bar\beta\gamma\bar\gamma})\\
&\quad\ge-4n\epsilon.
\end{aligned}$$

From the above two inequalities we obtain

$$R_{\alpha\bar\alpha\alpha\bar\alpha}+R_{\beta\bar\beta\beta\bar\beta}+\sum_{\gamma\neq\alpha,\beta}\big(R_{\alpha\bar\alpha\gamma\bar\gamma}+R_{\beta\bar\beta\gamma\bar\gamma}\big)\ge-4n\epsilon.$$

And then

$$\begin{aligned}
R_{\alpha\bar\alpha}+R_{\beta\bar\beta}&=R_{\alpha\bar\alpha\alpha\bar\alpha}+R_{\beta\bar\beta\beta\bar\beta}+2R_{\alpha\bar\alpha\beta\bar\beta}+\sum_{\gamma\neq\alpha,\beta}(R_{\alpha\bar\alpha\gamma\bar\gamma}+R_{\beta\bar\beta\gamma\bar\gamma})\\
&\ge-4n\epsilon+2R_{\alpha\bar\alpha\beta\bar\beta}.
\end{aligned}$$

Now we fix α. Letting $\xi_\alpha=\frac{1}{\sqrt{n-1}}$ and $\xi_\beta=0$ for any $\beta\neq\alpha$, we know

$$\sum_{\beta\neq\alpha}R_{\alpha\bar\alpha\beta\bar\beta}\ge-(n-1)\epsilon.$$

Summing over $\beta\neq\alpha$ we get

$$\begin{aligned}
&(n-1)R_{\alpha\bar\alpha}+\sum_{\beta\neq\alpha}R_{\beta\bar\beta}=(n-2)R_{\alpha\bar\alpha}+R\ge-C(n)\epsilon,\\
&(n-2)R_{\alpha\bar\alpha}\ge-R-C(n)\epsilon.
\end{aligned}$$

Summing over α, we have

$$(n-2)R = (n-2)\sum_{\alpha} R_{\alpha\bar{\alpha}} \geq -C(n)(R+\epsilon).$$

Then we obtain

$$\begin{aligned}
&R \geq -C(n)\epsilon,\\
&R_{\alpha\bar{\alpha}} = R - \sum_{\gamma\neq\alpha} R_{\gamma\bar{\gamma}} \leq C(n)(R+\epsilon),\\
&2R_{\alpha\bar{\alpha}\beta\bar{\beta}} = R_{\alpha\bar{\alpha}} + R_{\beta\bar{\beta}} - (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\beta\bar{\beta}\beta\bar{\beta}}) - \sum_{\gamma\neq\alpha,\beta}(R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} + R_{\beta\bar{\beta}\gamma\bar{\gamma}})\\
&\qquad\quad \leq C(n)(R+\epsilon),\\
&-\epsilon \leq R_{\alpha\bar{\alpha}\beta\bar{\beta}} + \sum_{\gamma\neq\alpha,\beta} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \leq R_{\alpha\bar{\alpha}\beta\bar{\beta}} + C(n)(R+\epsilon),\\
&R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq -C(n)(R+\epsilon).
\end{aligned}$$

From

$$R_{\alpha\bar{\alpha}} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + \sum_{\beta\neq\alpha} R_{\alpha\bar{\alpha}\beta\bar{\beta}},$$

we get

$$-C(n)(R+\epsilon) \leq R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq C(n)(R+\epsilon).$$

From the above calculation we know that there exists a constant $C(n)$ only depending on the dimension $n > 2$, such that

$$\begin{aligned}
&-C(n)\epsilon \leq R_{\alpha\bar{\alpha}\beta\bar{\beta}} \leq C(n)(R+\epsilon),\\
&-C(n)(R+\epsilon) \leq R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leq C(n)(R+\epsilon),\\
&-C(n)(R+\epsilon) \leq R_{\alpha\bar{\beta}\alpha\bar{\beta}} + R_{\beta\bar{\alpha}\beta\bar{\alpha}} \leq C(n)(R+\epsilon).
\end{aligned}$$

Assume $e_\alpha = \frac{1}{\sqrt{2}}(u_\alpha - \sqrt{-1}Ju_\alpha)$, then

$$\begin{aligned}
&R_{\alpha\bar{\alpha}\beta\bar{\beta}} = R(u_\alpha, Ju_\beta, Ju_\beta, u_\alpha) + R(u_\alpha, u_\beta, u_\beta, u_\alpha),\\
&R_{\alpha\bar{\beta}\alpha\bar{\beta}} + R_{\beta\bar{\alpha}\beta\bar{\alpha}} = R(u_\alpha, Ju_\beta, Ju_\beta, u_\alpha) - R(u_\alpha, u_\beta, u_\beta, u_\alpha).
\end{aligned}$$

Then we have

$$|R(u_\alpha, Ju_\beta, Ju_\beta, u_\alpha)| \leq C(n)(R+\epsilon), \quad |R(u_\alpha, u_\beta, u_\beta, u_\alpha)| \leq C(n)(R+\epsilon),$$

and

$$|R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}| = |R(u_\alpha, Ju_\alpha, Ju_\alpha, u_\alpha)| \leq C(n)(R+\epsilon).$$

It is easy to derive that the curvature tensor is bounded by $C(n)(R+\epsilon)$ from these inequalities. □

This estimate is a pointwise result. If we assume the scalar curvature is bounded from above, then the sectional curvature is bounded from both sides.

For $n=2$, the above estimates may not be true. One can refer Example 1.2 in [7], which implies that a manifold may have nonnegative orthogonal bisectional curvature and zero scalar curvature, yet sectional curvatures may be arbitrarily large. In order to bound sectional curvature, we need some additional condition. From the above we have

$$\begin{aligned} &-4\epsilon \le R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} \le R + 2\epsilon, \\ &-\epsilon \le R_{1\bar{1}2\bar{2}} \le \frac{1}{2}R + 2\epsilon. \end{aligned}$$

If we assume the scalar curvature and the holomorphic sectional curvature are bounded from above, then we can get the same conclusion as above of the $n>2$ case.

3 Proof of the Main Theorem

We prove our main theorem by a contradiction argument. Assume (X_i, ω_i, J_i) is a sequence of n-dim closed Kähler manifolds with diam $\le D$ and Vol $\ge v > 0$, scalar curvature is bounded from above by Λ, and the orthogonal bisectional curvature $OB \ge -\frac{1}{i}$. In addition, we assume all $\tilde{X}_i$ are not diffeomorphic to the product $\mathbb{R}^{2l} \times \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 \times X_1 \times \cdots \times X_k$ $(l \ge 0)$, as the theorem stated.

Now we consider Ricci flows starting with (X_i, ω_i, J_i). From the discussion of Section 2 we know that the curvature tensor of (X_i, ω_i) are uniformly bounded, say by $K(n, \Lambda)$, then by the doubling time estimate of Ricci flow we know that for $0 \le t \le \frac{1}{16}K$, the curvature tensor $|\mathrm{Rm}| \le 2K$.

By

$$\frac{d}{dt}\mathrm{Vol}(g_i(t)) = -\int_{X_i} R(g_i) d\mu \ge -C(n)K\mathrm{Vol}(g_i(t)),$$

we know the volume of (X_i, g_i) is uniformly bounded from below by $e^{-C(n)K^2}v$ for $t \in [0, \frac{1}{16}K]$. Moreover, the curvature bound implies a uniform upper bound of diameters. According to Corollary 2.2 of Cheeger [3], one can get a uniform lower bound of injective radius for manifolds with uniformly bounded sectional curvature, bounded diameter and lower bounded volume. Then by Hamiltion's Cheeger–Gromov compactness theorem, after passing to a subsequence, which we still denote by $(X_i, g_i(t), J_i)$,

$$(X_i, g_i(t), J_i) \to (X_\infty, g_\infty(t), J_\infty),$$

in Cheeger–Gromov sense, where g_i is the Riemannian metric corresponding to ω_i. $(X_\infty, g_\infty(t), J_\infty)$ is a solution to Kähler–Ricci flow. The convergence is $C^{1,\alpha}$ at time 0 and C^∞ for $0 \le t \le \frac{1}{16}K$.

Since it is claimed by Cao and Hamilton in one unpublished work (one can also refer to Gu–Zhang's paper [7]) that the cone of nonnegative of orthogonal holomorphic bisectional curvature is invariant under the Ricci flow, thus the orthogonal bisectional curvature of $g_i(t)$ is bounded below by $-C(n)\frac{1}{i}\exp(Ct)$ due to the proof of Hamilton's Theorem 4.3 in [8]. Let P denote the fiber bundle with fibers orthogonal 2-vectors $\{\xi,\eta\}\subset T_p^{1,0}(X)$, and define a function u on P as

$$u(\{\xi,\eta\},t)=R(\xi,\bar{\xi},\eta,\bar{\eta}).$$

Then Gu and Zhang [7] calculated

$$\frac{\partial u}{\partial t}\geq Lu+c\min\left\{0,\inf_{\xi\in V,|\xi|=1}D^2u(\xi,\xi)\right\}-c\sup_{\xi\in V,|\xi|=1}Du(\xi)-cu$$

for some $c>0$, where L is the horizontal Laplacian on P and V is the vertical subspace of the bundle. Then by the proof of Hamilton's maximum principle (Theorem 4.3 in [8]) we obtain the lower bound for orthogonal bisectional curvature. The curvature bounds imply that the orthogonal bisectional curvature of $g_\infty(t)$ is nonnegative for $t>0$. Using Chau and Tam's classification of Kähler manifolds with almost nonnegative quadratic orthogonal bisectional curvature, we know that (X_∞,J_∞) is biholomorphic to one of the followings:

(i) $\mathbb{C}^l\times\mathbb{C}P^1\times\cdots\times\mathbb{C}P^1\times Y_1\times\cdots\times Y_k$ $(l\geq 0)$;

(ii) $\mathbb{B}^l\times\mathbb{C}P^1\times\cdots\times\mathbb{C}P^1\times Y_1\times\cdots\times Y_k$ $(l\geq 0)$.

Because the differential structure of both $\mathbb{B}^l$ and $\mathbb{C}^l$ are $\mathbb{R}^{2l}$, we can obtain that the universal covering of $X_\infty, g_\infty(t)$ is diffemorphic to $\mathbb{R}^{2l}\times\mathbb{CP}^1\times\cdots\times\mathbb{CP}^1\times Y_1\times\cdots\times Y_k$ $(l\geq 0)$, where $\{Y_p\}_{p=1}^k$ are simply connected compact Fano manifolds with nonnegative orthogonal bisectional curvature. Then by Gu and Zhang's result (cf. [7]), we know that $\{Y_p\}_{p=1}^k$ are either $\mathbb{C}P^k$ or symmetric Hermitian space of rank ≥ 2.

From Cheeger–Colding theory, we know that for i large enough, X_i is diffeomorphic to X_∞. Then we meet a contradiction.

Since the complex structure may vary when taking Cheeger–Gromov limit, we do not know whether X_∞ is biholomorphic to one of the above.

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