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RESEARCH ARTICLE

Tensor products of tilting modules

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Abstract We consider whether the tilting properties of a tilting A-module T and a tilting B-module T' can convey to their tensor product $T \otimes T'$. The main result is that $T \otimes T'$ turns out to be an $(n+m)$-tilting $A \otimes B$-module, where T is an m-tilting A-module and T' is an n-tilting B-module.

Keywords Tensor product, tilting module, n-tilting module, endomorphism algebra
MSC 15A09, 16D90, 16G20

1 Introduction

The classical notion of tilting modules was first considered by Brenner and Bulter [4] and by Happel and Ringel [10] in the 1980s. Since then it has played a very important role in the development of representation theory of finite dimensional algebras. First, given a tilting module T_A, the endomorphism algebra $B = \mathrm{End}_A(T)$ (it is called as a tilted algebra, if A is a finite-dimensional hereditary algebra) provides an interesting example of algebras in the representation theory, which is close to that of the original ones. Second, each tilting module T_A gives to a torsion theory in mod-A (the category of finitely generated right A-modules). The notion of tilting was further generalized by Bazzoni [3], Miyashita [12], Happel [9], and Hügel and Coelho [11]. Happel showed that, for finite-dimensional algebras, generalized tilting induces a derived equivalence between the corresponding module categories which inspired Rickard [13] to develop his Morita theory for derived categories.

Several authors have been interested in the following question: which properties of an A-module M and a B-module N are conveyed to the $A \otimes_k B$-module $M \otimes_k N$?" (see, e.g., [6–8,15,16]). Chen et al. have studied the tensor products of indecomposable, projective, injective, and flat modules, respectively

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(see, e.g., [6]). Recently, Yang [15] studied strongly graded vertex algebras and their strong graded modules, obtained that a tensor product of strongly graded irreducible modules for a tensor product of strongly graded vertex algebras is irreducible. The aim of this note is to investigate what will be with respect to the tensor product of an m-tilting A-module and an n-tilting B-module. In fact, we first find the tensor product of 1-tilting A-module and 1-tilting B-module is not 1-tilting again, then it is natural to consider what it is? Our main result is the following theorem.

Theorem 1.1 *Let T be an n-tilting A-module, and let T' be an m-tilting B-module. Then $T \otimes T'$ is an $(n+m)$-tilting $A \otimes B$-module.*

Throughout this note, unless otherwise stated, all algebras considered are finite-dimensional algebras (with an identity) over a field k. For a k-algebra A, every A-module can be regarded as a k-module, a left A-module M is denoted by ${}_AM$, and the right A-module N is denoted by N_A. The tensor product $\otimes_k$ will be denoted by $\otimes$ briefly. $\operatorname{add} T$ denotes the class of modules isomorphic to direct summands of finite direct sums of copies of T.

2 Preliminaries

In order to prove the main theorem, we collect some definitions and some basic properties of tensor products of modules in this section. First, We recall the following definition of an n-tilting module.

Definition 2.1 [3] An A-module T is said to be n-tilting provided it satisfies the following three conditions:

(T_1) $\mathrm{Pd}_A T \leqslant n$;

(T_2) $\mathrm{Ext}_A^i(T, T^{(\lambda)}) = 0$ for every $0 \leqslant i \leqslant n$ and every cardinal λ;

(T_3) there exists a long exact sequence

$$0 \to A \to T_0 \to T_1 \to \cdots \to T_n \to 0,$$

with $T_i \in \operatorname{add} T$ for every $0 \leqslant i \leqslant n$.

If $n = 1$, it coincides with the tilting module introduced by Happel and Ringel [10]. Here, we call it as a 1-tilting module. It is clear that any progenerator P is tilting (the trivial case $n = 0$). Miyashita [12] also called the n-tilting module as a tilting module of projective dimension $\leqslant n$.

For completeness, we also recall the concept of projective dimension of an A-module M.

Definition 2.2 [14, p. 233] If M is an A-module, then $\mathrm{Pd}_A(M) \leqslant m$ if there is a projective resolution

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

If no such finite resolution exists, define $\mathrm{Pd}_A M = \infty$; otherwise, the least such an integer n is called the projective dimension of M and denoted by $\mathrm{Pd}_A M$, sometimes abbreviate it as $\mathrm{Pd}\, M$. And the complex

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to 0$$

is called as the deleted projective resolution of M (see, e.g., [1]).

Before giving the proof of main theorem, we need some preparations.

Lemma 2.3 [6, Proposition 4] *Let M_A and ${}_BN$ be projective (resp., injective, flat). Then $M \otimes N$ is projective (resp., injective, flat) (as a $A \otimes B$-module).*

Definition 2.4 [14] Let $(X^\cdot, \partial)$ be a complex of right A-modules, and let $(Y^\cdot, \delta)$ be a complex of left B-modules, that is,

$$X^\cdot\colon \cdots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \cdots, \quad Y^\cdot\colon \cdots \to Y_n \xrightarrow{\delta_n} Y_{n-1} \to \cdots.$$

Then the total complex $\mathbf{Hom}(X^\cdot, Y^\cdot)$ is a complex defined by

$$\mathbf{Hom}(X^\cdot, Y^\cdot)_n = \prod_{p\in\mathbb{Z}} \mathbf{Hom}(X_p, Y_{n+p}),$$

and

$$\begin{aligned} \tau_n\colon \mathbf{Hom}(X^\cdot, Y^\cdot)_n &\to \mathbf{Hom}(X^\cdot, Y^\cdot)_{n-1}, \\ \{f_{p,n+p}\} &\mapsto \{(-1)^{n+1} f_{p,n+p}\partial_{p+1} + \delta_{n+p+1} f_{p+1,n+p+1}, p \in \mathbb{Z}\}; \end{aligned}$$

and the total complex $X^\cdot \otimes Y^\cdot$ is usually called the tensor product of the complexes $X^\cdot$ and $Y^\cdot$, that is,

$$(X^\cdot \otimes Y^\cdot)_n = \bigoplus_{p+q=n} X_p \otimes Y_q,$$

and

$$\begin{aligned} d_n\colon (X^\cdot \otimes Y^\cdot)_n &\to (X^\cdot \otimes Y^\cdot)_{n-1}, \\ \{m_{pq} \in X_p \otimes Y_q\}_{p+q=n} &\mapsto \{(\partial_p \otimes \varepsilon_q)(m_{pq}) + (-1)^{p-1}(\varepsilon_{p-1} \otimes \delta_{q+1})(m_{p-1,q+1})\}. \end{aligned}$$

Lemma 2.5 ([17, p. 303], Künneth Theorem for $\otimes$) *Let $(X^\cdot, \partial)$ be a complex of right A-modules, and let $(Y^\cdot, \delta)$ be a complex of left B-modules with all $\mathrm{Im}\,\partial_p$ and $\ker \partial_p$ are flat. Then, for any n, there is a natural exact sequence*

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} H_p(X^\cdot) \otimes H_q(Y^\cdot) &\to H_n(X^\cdot \otimes Y^\cdot) \\ \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X^\cdot), H_q(Y^\cdot)) &\to 0. \end{aligned}$$

Similar to [5, Corollary 2.7], we have the following result.

Proposition 2.6 *If* $(P^{\cdot},\partial)$, $(Q^{\cdot},\delta)$ *are deleted projective resolutions of* M_A *and* ${}_BN$, *respectively, then* $(P^{\cdot}\otimes Q^{\cdot},d)$ *is the deleted projective resolution of* $M\otimes N$.

Proof First, we denote projective resolutions of M and N as

$$X^{\cdot}:\ \cdots\to P_n\xrightarrow{\partial_n}P_{n-1}\to\cdots\to P_0\to M\to 0,$$
$$Y^{\cdot}:\ \cdots\to Q_n\xrightarrow{\delta_n}Q_{n-1}\to\cdots\to Q_0\to N\to 0,$$

where P_i, Q_j are projective. Then we consider their deleted complexes:

$$P^{\cdot}:\ \cdots\to P_n\xrightarrow{\partial^n}P_{n-1}\to\cdots\to P_0\to 0,\quad H_p(P^{\cdot})=\begin{cases}M, & p=0,\\ 0, & p\neq 0,\end{cases}$$
$$Q^{\cdot}:\ \cdots\to Q_n\xrightarrow{\delta_n}Q_{n-1}\to\cdots\to Q_0\to 0,\quad H_q(Q^{\cdot})=\begin{cases}N, & q=0,\\ 0, & q\neq 0.\end{cases}$$

Consequently, by Definition 2.4, we have the total complex

$$(P^{\cdot}\otimes Q^{\cdot})_l=\bigoplus_{p+q=l}P_p\otimes Q_q,\quad l=0,1,2,\ldots,$$

here, it follows from Lemma 2.3 that $P_p\otimes Q_q$ is a projective $A\otimes B$-module, and thus, $(P^{\cdot}\otimes Q^{\cdot})_l$ is projective as well.

Second, since k is a field, every k-module is flat, and then $\mathrm{Tor}_1(\text{-},\text{-})=0$. By Lemma 2.5, it has

$$\begin{aligned}H_n(P^{\cdot}\otimes Q^{\cdot})&\cong\bigoplus_{p+q=n}(H_p(P^{\cdot})\otimes H_q(Q^{\cdot}))\\ &=\begin{cases}0, & n\neq 0,\\ H_0(P^{\cdot})\otimes H_0(Q^{\cdot})\cong M\otimes N, & n=0.\end{cases}\end{aligned}$$

Consequently, $P^{\cdot}\otimes Q^{\cdot}$ is the deleted projective resolution of $M\otimes N$. This completes the proof. □

By Definition 2.2 and Proposition 2.6, we are led to the conclusion as follows.

Corollary 2.7 *For a right* A*-module* M *and a left* B*-module* N, *the projective dimension of* $M\otimes N$ *as an* $A\otimes B$*-module is at most* $\mathrm{Pd}_AM+\mathrm{Pd}_BN$.

Proposition 2.8 *Let*

$$0\to C_0\to C_1\to\cdots\to C_n\to 0$$

be an exact sequence of right A*-modules, and let*

$$0\to D_0\to D_1\to\cdots\to D_m\to 0$$

be an exact sequence of left B-modules. Then

$$0 \to C_0 \otimes D_0 \to C_0 \otimes D_1 \oplus C_1 \otimes D_0 \to C_2 \otimes D_0 \oplus C_1 \otimes D_1 \oplus C_0 \otimes D_2$$
$$\to \cdots \to \bigoplus_{p+q=l} C_p \oplus D_q \to \cdots$$

is an exact sequence of $A \otimes B$-modules.

Proof Denote

$$C^{\cdot} : 0 \to C_0 \to C_1 \to \cdots \to C_n \to 0,$$
$$D^{\cdot} : 0 \to D_0 \to D_1 \to \cdots \to D_m \to 0.$$

The following tensor product of complexes will make $C^{\cdot} \otimes D^{\cdot}$ clear:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_0 \otimes D_0 \arrow[r] \arrow[d] & C_1 \otimes D_0 \arrow[r] \arrow[d] & C_2 \otimes D_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_0 \arrow[r] \arrow[d] & C_n \otimes D_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0 \otimes D_1 \arrow[r] \arrow[d] & C_1 \otimes D_1 \arrow[r] \arrow[d] & C_2 \otimes D_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_1 \arrow[r] \arrow[d] & C_n \otimes D_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0 \otimes D_2 \arrow[r] \arrow[d] & C_1 \otimes D_2 \arrow[r] \arrow[d] & C_2 \otimes D_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_2 \arrow[r] \arrow[d] & C_n \otimes D_2 \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_0 \otimes D_{m-2} \arrow[r] \arrow[d] & C_1 \otimes D_{m-2} \arrow[r] \arrow[d] & C_2 \otimes D_{m-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_{m-2} \arrow[r] \arrow[d] & C_n \otimes D_{m-2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0 \otimes D_{m-1} \arrow[r] \arrow[d] & C_1 \otimes D_{m-1} \arrow[r] \arrow[d] & C_2 \otimes D_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_{m-1} \arrow[r] \arrow[d] & C_n \otimes D_{m-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0 \otimes D_m \arrow[r] \arrow[d] & C_1 \otimes D_m \arrow[r] \arrow[d] & C_2 \otimes D_m \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n-1} \otimes D_m \arrow[r] \arrow[d] & C_n \otimes D_m \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

For every $0 \leqslant i \leqslant n+m$, $(C^{\cdot} \otimes D^{\cdot})_i$ could be regarded as the direct sum of modules on the i-th diagonal dotted line, for example,

$$(C^{\cdot} \otimes D^{\cdot})_0 = C_0 \otimes D_0, \quad (C^{\cdot} \otimes D^{\cdot})_1 = C_0 \otimes D_1 \oplus C_1 \otimes D_0, \quad \ldots,$$
$$(X^{\cdot} \otimes Y^{\cdot})_i = \bigoplus_{p+q=i} X_p \otimes Y_q.$$

Since $C^{\cdot}$ and $D^{\cdot}$ are exact sequences, namely, they are acyclic, by [14, p. 170], $H_i(C^{\cdot}) = 0$ and $H_j(D^{\cdot}) = 0$ for every $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$. From the fact that every module over a field is flat, it follows from Lemma 2.5 that

$$H_i(C^{\cdot} \otimes D^{\cdot}) \cong \bigoplus_{p+q=i} H_p(C^{\cdot}) \otimes H_q(D^{\cdot}) = 0.$$

Then we obtain the desired result. □

Proposition 2.9 *For module pairs* $(M_A,\ {}_BN;\ X_A,\ {}_BY)$, *there is a natural isomorphism*

$$\mathrm{Ext}^n_{A\otimes B}(M \otimes_k N, X \otimes_k Y) \cong \prod_{p+q=n} \mathrm{Ext}^p_A(M, X) \otimes_k \mathrm{Ext}^q_B(N, Y).$$

Proof First, for module pairs $(M_A,\ {}_BN;\ X_A,\ {}_BY)$, from [14, Lemma 3.83] and [14, Theorem 2.11, Adjoint Isomorphism], we have

$$\begin{aligned}\mathrm{Hom}_A(M, X) \otimes_k \mathrm{Hom}_B(N, Y) &\cong \mathrm{Hom}_A(M, X \otimes_k \mathrm{Hom}_B(N, Y))\\ &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, X \otimes_k Y))\\ &\cong \mathrm{Hom}_{A\otimes B}(M \otimes_k N, X \otimes_k Y). \qquad (1)\end{aligned}$$

Let $P^{\cdot}$ and $Q^{\cdot}$ be projective resolutions of M and N, respectively. Then

$$\mathrm{Hom}^{\cdot}_{A\otimes B}(P^{\cdot} \otimes_k Q^{\cdot}, X \otimes_k Y) \cong \mathrm{Hom}_A(P^{\cdot}, X) \otimes_k \mathrm{Hom}_B(Q^{\cdot}, Y).$$

By Lemma 2.5, we have

$$\begin{aligned}&H_n(\mathrm{Hom}_A(P^{\cdot}, X) \otimes \mathrm{Hom}_B(Q^{\cdot}, Y))\\ &\quad\cong \bigoplus_{p+q=n} H_p(\mathrm{Hom}_A(P^{\cdot}, X)) \otimes H_q(\mathrm{Hom}_B(Q^{\cdot}, Y)),\end{aligned}$$

that is,

$$\begin{aligned}&H_n(\mathrm{Hom}^{\cdot}_{A\otimes B}(P^{\cdot} \otimes_k Q^{\cdot}, X \otimes_k Y))\\ &\quad\cong \bigoplus_{p+q=n} H_p(\mathrm{Hom}_A(P^{\cdot}, X)) \otimes H_q(\mathrm{Hom}_B(Q^{\cdot}, Y)).\end{aligned}$$

Consequently, in view of [14, Theorem 10.85, Künneth Formula for Comology], it yields

$$\mathrm{Ext}^n_{A\otimes B}(M \otimes_k N, X \otimes_k Y) \cong \prod_{p+q=n} \mathrm{Ext}^p_A(M, X) \otimes_k \mathrm{Ext}^q_B(N, Y). \qquad □$$

Remark 2.10 If $i = 0$, then

$$\mathrm{Ext}^0_{A\otimes B}(M \otimes_k N, X \otimes_k Y) \cong \mathrm{Ext}^0_A(M, X) \otimes_k \mathrm{Ext}^0_B(N, Y),$$

that is,

$$\mathrm{Hom}_{A\otimes B}(M \otimes_k N, X \otimes_k Y) \cong \mathrm{Hom}_A(M, X) \otimes_k \mathrm{Hom}_B(N, Y),$$

which just was proved in the proof process of Proposition 2.9. It has also been shown in the proof of [6, Proposition 7].

Proposition 2.11 $\operatorname{add}({}_AT) \otimes \operatorname{add}({}_BT') \subseteq \operatorname{add}(T \otimes T')$.

Proof Let

$$ {}_AT = \bigoplus_{i=1}^{n} T_i, \quad {}_BT' = \bigoplus_{j=1}^{m} T'_j. $$

Then

$$ T \otimes T' = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} T_i \otimes T'_j \quad (T_i \otimes T'_j \in \operatorname{add}(T \otimes T')), $$
$$ T_i^{(n)} \otimes T_j \cong (T_i \otimes T_j)^{(n)}, \quad T_i \otimes T_j^{(m)} \cong (T_i \otimes T_j)^{(m)}, $$

which gives the result. □

3 Proof of main theorem and an example

Now, let us complete the proof of Theorem 1.1.

Proof of Theorem 1.1 First, by Definition 2.1, we see $\operatorname{Pd}_A T \leqslant n$, $\operatorname{Pd}_B T' \leqslant m$. Then, from Corollary 2.7,

$$ \operatorname{Pd}_{A \otimes B}(T \otimes T') \leqslant n + m, $$

so (T_1) follows.

In view of Definition 2.1,

$$ \operatorname{Ext}_A^p(T, T) = 0, \quad \operatorname{Ext}_B^q(T', T') = 0, \quad \forall\, p, q > 0. $$

Combining with Proposition 2.9, we see

$$ \operatorname{Ext}_{A \otimes B}^i(T \otimes T', T \otimes T') = \bigoplus_{p+q=i} \operatorname{Ext}_A^p(T, T) \otimes \operatorname{Ext}_B^q(T', T') = 0, \quad \forall\, i > 0. $$

Thus, we get (T_2).

As for (T_3), since T is an n-tilting A-module and T' is an m-tilting B-module, there are exact sequences

$$ 0 \to {}_AA \to T_0 \to \cdots \to T_n \to 0, $$
$$ 0 \to {}_BB \to T'_0 \to \cdots \to T'_m \to 0, $$

where $T_i \in \operatorname{add}(T)$, $T'_j \in \operatorname{add}(T')$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$. From Proposition 2.8,

$$ 0 \to A \otimes B \to (X^{\cdot} \otimes Y^{\cdot})_0 \to \cdots \to (X^{\cdot} \otimes Y^{\cdot})_{m+n} \to 0 $$

is exact. And it follows from Proposition 2.11 that $(X^{\cdot} \otimes Y^{\cdot})_l \in \mathrm{add}(T \otimes T')$ for $l = 0, 1, \dots, n+m$.

Consequently, $X^{\cdot} \otimes Y^{\cdot}$ is an $(n+m)$-tilting $(A \otimes B)$-module. $\square$

At the end of this note, we illustrate our main theorem by the following example in details.

Example 3.1 Let $A = k\vec{Q}$, where $\vec{Q}\colon \circ \xrightarrow{\alpha_1} \circ \xrightarrow{\alpha_2} \circ$, $I = \langle \alpha_1\alpha_2 \rangle$, and let $B = k\vec{Q}'$, where $\vec{Q}'\colon\ \circ \longrightarrow \circ$. Then $A \otimes_k B = k(\vec{Q} \times \vec{Q}')/I$, where

$$\begin{tikzcd}
\circ \arrow[r] \arrow[d] & \circ \arrow[r] \arrow[d] & \circ \arrow[d] \\
\circ \arrow[r] & \circ \arrow[r] & \circ
\end{tikzcd}$$

and the dotted lines mean the zero relations.

First, all tilting A-modules are

$$\begin{aligned}
P_1 \oplus P_2 \oplus P_3 &= 0\,1\,1 \oplus 1\,1\,0 \oplus 0\,0\,1 \quad (0\text{-tilting}),\\
P_1 \oplus P_2 \oplus S_2 &= 0\,1\,1 \oplus 1\,1\,0 \oplus 0\,1\,0 \quad (1\text{-tilting}),\\
P_1 \oplus P_2 \oplus I_1 &= 0\,1\,1 \oplus 1\,1\,0 \oplus 1\,0\,0 \quad (2\text{-tilting}).
\end{aligned}$$

Second, all tilting B-modules are

$$P_2' \oplus P_1' = 1\,1 \oplus 0\,1 \ (0\text{-tilting}), \quad P_2' \oplus I_2' = 1\,1 \oplus 1\,0 \ (1\text{-tilting}).$$

Now, we consider their tensor products.

(i) If

$${}_AT = 0\,1\,1 \oplus 1\,1\,0 \oplus 0\,0\,1 = {}_AA, \quad {}_BT' = 0\,1 \oplus 1\,1 = {}_BB,$$

then $T \otimes T' = A \otimes B$, which is trivial—a 0-tilting module.

(ii) If ${}_AT = {}_AA$ and ${}_BT' = 1\,0 \oplus 1\,1$, then

$$T \otimes T' = \begin{smallmatrix} 0\,1\,1 \\ 0\,0\,0 \end{smallmatrix} \oplus \begin{smallmatrix} 0\,1\,1 \\ 0\,1\,1 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,1\,0 \\ 0\,0\,0 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,1\,0 \\ 1\,1\,0 \end{smallmatrix} \oplus \begin{smallmatrix} 0\,0\,1 \\ 0\,0\,0 \end{smallmatrix} \oplus \begin{smallmatrix} 0\,0\,1 \\ 0\,0\,1 \end{smallmatrix}.$$

(T_1)

$$\mathrm{Pd}\begin{smallmatrix} 0\,1\,1 \\ 0\,1\,1 \end{smallmatrix} = 0 = \mathrm{Pd}\begin{smallmatrix} 1\,1\,0 \\ 1\,1\,0 \end{smallmatrix} = \mathrm{Pd}\begin{smallmatrix} 0\,0\,1 \\ 0\,0\,1 \end{smallmatrix}, \quad \mathrm{Pd}\begin{smallmatrix} 0\,1\,1 \\ 0\,0\,0 \end{smallmatrix} = 1 = \mathrm{Pd}\begin{smallmatrix} 1\,1\,0 \\ 0\,0\,0 \end{smallmatrix} = \mathrm{Pd}\begin{smallmatrix} 0\,0\,1 \\ 0\,0\,0 \end{smallmatrix};$$

hence, $\mathrm{Pd}\, T \otimes T' = 1$.

(T_2) By the Auslander-Reiten formulas (see [2, IV.2, Theorem 2.13]), we see that

$$\begin{aligned}\operatorname{Ext}^1_{A\otimes B}(T\otimes T', T\otimes T') &= \operatorname{Ext}^1_{A\otimes B}\Big(\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}, \begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,1\end{matrix}\Big)\\
&= D\overline{\operatorname{Hom}}\Big(\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,1\end{matrix}, \tau\Big(\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\Big)\Big)\\
&= D\operatorname{Hom}\Big(\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,1\end{matrix}, \begin{matrix}0\,0\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,0\,0\\1\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,0\\0\,1\,1\end{matrix}\Big)\\
&= 0,\end{aligned}$$

where τ denotes the Auslander-Reiten transition.

(T$_3$)

$$\begin{aligned}0\longrightarrow A\otimes B\longrightarrow{}& \begin{matrix}1\,1\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,1\,1\\0\,1\,1\end{matrix}\oplus\begin{matrix}0\,1\,1\\0\,1\,1\end{matrix}\oplus\begin{matrix}1\,1\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,1\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,1\end{matrix}\\
&\longrightarrow \begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\longrightarrow 0;\end{aligned}$$

hence, ${}_AT\otimes_B T'$ is 1-tilting.

(iii) Similar to (ii), if ${}_AT = 0\,1\,1\oplus 1\,1\,0\oplus 0\,1\,0$ (1-tilting), ${}_BT' = {}_BB$, then

$$T\otimes T' = \begin{matrix}0\,0\,0\\0\,1\,1\end{matrix}\oplus\begin{matrix}0\,1\,1\\0\,1\,1\end{matrix}\oplus\begin{matrix}0\,0\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,0\,0\\0\,1\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}$$

is 1-tilting.

(iv) If ${}_AT = 0\,1\,1\oplus 1\,1\,0\oplus 0\,1\,0$ (1-tilting), ${}_BT' = 1\,0\oplus 1\,1$ (1-tilting), then

$$T\otimes T' = \begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,1\\0\,1\,1\end{matrix}\oplus\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}.$$

(T$_1$)

$$\operatorname{Pd}\begin{matrix}0\,1\,1\\0\,1\,1\end{matrix} = 0 = \operatorname{Pd}\begin{matrix}1\,1\,0\\1\,1\,0\end{matrix},\quad \operatorname{Pd}\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix} = 1 = \operatorname{Pd}\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix} = \operatorname{Pd}\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix},$$

so $\operatorname{Pd} T\otimes T' = 2$.

(T$_2$) By the Auslander-Reiten formulas,

$$\begin{aligned}&\operatorname{Ext}^1_{A\otimes B}(T\otimes T', T\otimes T')\\
&= \operatorname{Ext}^1_{A\otimes B}\Big(\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}, \begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\Big)\\
&= D\overline{\operatorname{Hom}}\Big(\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}, \tau\Big(\begin{matrix}0\,1\,1\\0\,0\,0\end{matrix}\oplus\begin{matrix}1\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}\Big)\Big)\\
&= D\operatorname{Hom}\Big(\begin{matrix}0\,1\,0\\0\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,0\\0\,1\,0\end{matrix}, \begin{matrix}0\,0\,0\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,0\,0\\1\,0\,0\end{matrix}\oplus\begin{matrix}0\,1\,1\\1\,1\,0\end{matrix}\oplus\begin{matrix}0\,0\,1\\0\,0\,0\end{matrix}\Big)\\
&= 0.\end{aligned}$$

In view of [9, p. 30], we obtain that

$$\begin{aligned}
&\mathrm{Ext}^2_{A\otimes B}(T\otimes T', T\otimes T')\\
&\quad = \mathrm{Ext}^2_{A\otimes B}\Big(\begin{matrix}011\\000\end{matrix}\oplus\begin{matrix}110\\000\end{matrix}\oplus\begin{matrix}010\\000\end{matrix}\oplus\begin{matrix}010\\010\end{matrix}, \begin{matrix}010\\010\end{matrix}\oplus\begin{matrix}010\\000\end{matrix}\Big)\\
&\quad = \mathrm{Hom}_{D^b(A\otimes B)}\Big(\begin{matrix}011\\000\end{matrix}\oplus\begin{matrix}110\\000\end{matrix}\oplus\begin{matrix}010\\000\end{matrix}\oplus\begin{matrix}010\\010\end{matrix}, \sum{}^2\Big(\begin{matrix}010\\010\end{matrix}\oplus\begin{matrix}010\\000\end{matrix}\Big)\Big)\\
&\quad = 0,
\end{aligned}$$

where $\sum$ is a shift functor.

(T_3)

$$\begin{aligned}
0 \longrightarrow A\otimes B \longrightarrow &\begin{matrix}110\\110\end{matrix}\oplus\begin{matrix}011\\011\end{matrix}\oplus\begin{matrix}011\\011\end{matrix}\oplus\begin{matrix}110\\110\end{matrix}\oplus\begin{matrix}011\\011\end{matrix}\oplus\begin{matrix}011\\011\end{matrix}\\
&\longrightarrow \begin{matrix}110\\000\end{matrix}\oplus\begin{matrix}011\\000\end{matrix}\oplus\begin{matrix}011\\000\end{matrix}\oplus\begin{matrix}010\\010\end{matrix}\oplus\begin{matrix}010\\010\end{matrix}\longrightarrow\begin{matrix}010\\000\end{matrix}\longrightarrow 0;
\end{aligned}$$

consequently, $T\otimes T'$ is 2-tilting.

(v) Similar to (iv), $T\otimes T'$ is 2-tilting, if ${}_AT = 011\oplus 110\oplus 100$ (2-tilting) and ${}_BT' = {}_BB$.

(vi) If ${}_AT = 011\oplus 110\oplus 100$ (2-tilting), ${}_BT' = 10\oplus 11$ (1-tilting), then

$$T\otimes T' = \begin{matrix}011\\000\end{matrix}\oplus\begin{matrix}011\\011\end{matrix}\oplus\begin{matrix}110\\000\end{matrix}\oplus\begin{matrix}110\\110\end{matrix}\oplus\begin{matrix}100\\000\end{matrix}\oplus\begin{matrix}100\\100\end{matrix} = D(A).$$

(T_1)

$$\mathrm{Pd}\begin{matrix}011\\011\end{matrix} = \mathrm{Pd}\begin{matrix}110\\110\end{matrix} = 0,\quad \mathrm{Pd}\begin{matrix}011\\000\end{matrix} = \mathrm{Pd}\begin{matrix}110\\000\end{matrix} = 1,\quad \mathrm{Pd}\begin{matrix}100\\100\end{matrix} = 2,\quad \mathrm{Pd}\begin{matrix}100\\000\end{matrix} = 3,$$

these are due to

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix}001\\001\end{matrix} \arrow[r] & \begin{matrix}011\\011\end{matrix} \arrow[rr] \arrow[dr] & & \begin{matrix}110\\110\end{matrix} \arrow[r] & \begin{matrix}100\\100\end{matrix} \arrow[r] & 0, \\
& & & \begin{matrix}010\\010\end{matrix} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix}000\\001\end{matrix} \arrow[r] & \begin{matrix}001\\001\end{matrix}\oplus\begin{matrix}000\\011\end{matrix} \arrow[rr] \arrow[dr] & & \begin{matrix}011\\011\end{matrix}\oplus\begin{matrix}000\\110\end{matrix} \arrow[rr] \arrow[dr] & & \begin{matrix}110\\110\end{matrix} \arrow[r] & \begin{matrix}100\\000\end{matrix} \arrow[r] & 0. \\
& & & \begin{matrix}001\\011\end{matrix} \arrow[ur] & & \begin{matrix}010\\110\end{matrix} \arrow[ur] & & &
\end{tikzcd}$$

Thus, $\operatorname{Pd} T \otimes T' = 3$.

(T$_2$)

$$\begin{aligned}\operatorname{Ext}^3_{A\otimes B}(T \otimes T', T \otimes T') &= \operatorname{Ext}^2_{A\otimes B}(T \otimes T', T \otimes T')\\ &= \operatorname{Ext}^1_{A\otimes B}(T \otimes T', T \otimes T')\\ &= 0.\end{aligned}$$

(T$_3$)

$$\begin{aligned} 0 \longrightarrow A \otimes B \longrightarrow & \begin{smallmatrix}1\,1\,0\\1\,1\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}1\,1\,0\\1\,1\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,1\,1\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,1\,1\end{smallmatrix}\\ \longrightarrow & \begin{smallmatrix}1\,1\,0\\1\,1\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\,1\,0\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\,1\,0\\1\,1\,0\end{smallmatrix} \oplus \begin{smallmatrix}0\,1\,1\\0\,0\,0\end{smallmatrix}\\ \longrightarrow & \begin{smallmatrix}1\,0\,0\\1\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\,1\,0\\0\,0\,0\end{smallmatrix} \oplus \begin{smallmatrix}1\,0\,0\\1\,0\,0\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\,0\,0\\0\,0\,0\end{smallmatrix} \longrightarrow 0;\end{aligned}$$

so $T \otimes T'$ is 3-tilting. □

By seeking into Example 3.1, we find that tensor products of indecomposable modules are also indecomposable, since it follows from (1) in Proposition 2.9 that

$$\operatorname{End}_{A\otimes B}(M \otimes N) = \operatorname{End}_A(M) \otimes \operatorname{End}_B(N)$$

(see [6, Proposition 7]). Conversely, it does not work, i.e., an indecomposable $A \otimes B$-module do not always be decomposed into the tensor product of an indecomposable A-module and an indecomposable B-module. From Example 3.1, there are 5 indecomposable A-modules, 3 indecomposable B-modules, the number of the tensor product of indecomposable A-modules and indecomposable B-modules should be 15. However, there are 20 indecomposable $A\otimes B$-modules, so there are still 5 indecomposable $A\otimes B$-modules, which could not presented as the tensor products of an indecomposable A-modules and an indecomposable B-module, they are

$$\begin{smallmatrix}0\,0\,1\\0\,1\,1\end{smallmatrix} \quad \begin{smallmatrix}0\,1\,1\\0\,1\,0\end{smallmatrix} \quad \begin{smallmatrix}0\,1\,1\\1\,1\,0\end{smallmatrix} \quad \begin{smallmatrix}0\,1\,0\\1\,1\,0\end{smallmatrix} \quad \begin{smallmatrix}1\,1\,0\\1\,0\,0\end{smallmatrix}$$

From our main result, for the classical (1-)tilting module ${}_AT$ and ${}_BT'$, $T\otimes T'$ turns out to be 2-tilting. In this case, if A is hereditary, then

$$C = \operatorname{End}({}_AT), \quad D = \operatorname{End}({}_BT),$$

are tilted algebras. But $\operatorname{End}_{A\otimes B}(M \otimes N) = \operatorname{End}_A(M) \otimes \operatorname{End}_B(N)$ may not be tilted, in which $T \otimes T'$ is 2-tilting. There is a natural question: when does the tensor products of tilted algebras will be tilted? So does the problem of the tensor products of torsion theories. We will discuss them in another paper.

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