Front. Math. China 2016, 11(5): 1275–1290
DOI 10.1007/s11464-016-0578-z

SURVEY ARTICLE

Curvature notions on graphs

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Abstract We survey some geometric and analytic results under the assumptions of combinatorial curvature bounds for planar/semiplanar graphs and curvature dimension conditions for general graphs.

Keywords Planar graphs, combinatorial curvature, curvature dimension conditions, harmonic functions
MSC 31B05, 39A12

1 Introduction and setting

In Riemannian geometry, various curvature bounds, e.g., for sectional curvature, Ricci curvature, or scalar curvature, lead to many interesting geometric and analytic consequences. The research field, where people use geometric ideas to study discrete spaces, in particular, graphs, is called discrete geometric analysis, see, e.g., [48] and references therein.

In order to adopt geometric ideas, it is a key step to introduce some meaningful definitions of curvatures on graphs which mimic geometric properties for some curvatures in the Riemannian setting. Due to the discrete nature, it is hard to find a perfect notion of curvature which resembles many Riemannian properties. The typical strategy is to introduce a proper notion of discrete curvature and then to derive corresponding geometric and analytic results under the curvature assumptions.

In this survey, we will discuss two types of curvature definitions. One is called combinatorial curvature for a planar graph. The idea is to properly embed a planar graph into a metric surface which might be singular. By the embedding, one defines the combinatorial curvature notion of the graph by using the convexity of the surface, i.e., its Gaussian curvature, introduced by [18,27,46]. In this way, one can derive many global geometric properties for

Received May 3, 2016; accepted July 30, 2016
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the graph from those of the surface. The other one is the Ricci curvature of a graph defined via the Bochner formula using the so-called Γ-calculus, see, e.g., [1,22,39]. It turns out that many geometric and analytic properties follow provided the Bochner type inequalities. For other results not covered here, the readers may refer to, e.g., [2,3,16,28–30,38,40,43] and many others.

In the rest of the section, we recall the setting of weighted graphs. Let (V,E) be an undirected (finite or infinite) graph with the set of vertices V and the set of edges E, i.e., 2-element subsets of V. Two vertices x,y are called neighbours if $\{x,y\}\in E$, for simplicity, denoted by $x\sim y$. We do allow self-loops at vertices, e.g., $x\sim x$. A graph (V,E) is called *locally finite* if

$$\sharp\{y\in V \mid y\sim x\}<\infty,\quad \forall\, x\in V.$$

It is called connected if for any $x,y\in V$, there is a finite path connecting x and y, i.e., $\{x_i\}_{i=0}^n\subset V$ such that

$$x=x_0\sim x_1\sim\cdots\sim x_n=y.$$

In this paper, we only consider locally finite connected graphs. The graph (V,E) is endowed with a natural combinatorial metric d, given by the smallest number of edges among all paths connecting two vertices. For any $x\in V$, $R>0$, we denote by

$$B_R(x):=\{y\in V\colon d(y,x)\leqslant R\}$$

the ball of radius R centered at x.

We assign weights on the set of vertices V and set of edges E, respectively:

$$\begin{gathered}\mu\colon E\to(0,\infty),\\ E\ni\{x,y\}\mapsto\mu_{xy}=\mu_{yx},\end{gathered}$$

is a weight function on edges, and $m\colon V\to(0,\infty)$ is a weight function on vertices. We call the quadruple $G=(V,E,m,\mu)$ a *weighted graph*. We denote by $V(x,R):=m(B_R(x))$ the m-measure of $B_R(x)$.

For the convenience, we extend the function μ on E to $V\times V$, $\mu\colon V\times V\to[0,\infty)$, by setting $\mu_{xy}=0$ for any $x\not\sim y$. For any vertex x, the vertex degree of x is given by

$$\deg(x):=\sum_{y\in V}\mu_{xy}.$$

We denote by $C(V)$ the set of all functions defined on vertices, and by

$$C_0(V):=\{f\in C(V)\mid \sharp\{x\in V\mid f(x)\neq 0\}<\infty\}$$

those of finitely support. On the discrete measure space (V,m), ℓ^p-summable spaces, $p\in[1,\infty]$, are defined routinely: for any $f\in C(V)$,

$$\|f\|_{\ell^p(V,m)}:=\bigg(\sum_{x\in V}|f(x)|^p m(x)\bigg)^{1/p},\quad p\in[1,\infty),$$

and

$$\|f\|_{\ell^\infty(V,m)} := \sup_{x\in V} |f(x)|.$$

Moreover,

$$\ell^p(V,m) := \{f \in C(V) \mid \|f\|_{\ell^p(V,m)} < \infty\}.$$

Given a weighted graph (V, E, m, μ), it associates with a Dirichlet form with respect to the Hilbert space $\ell^2(V,m)$:

$$\begin{aligned} &Q\colon D(Q) \to \mathbb{R},\\ &\qquad f \mapsto Q(f) := \frac{1}{2}\sum_{x,y\in V} \mu_{xy}(f(y) - f(x))^2, \end{aligned}$$

where $D(Q)$, called the form domain, is the completion of $C_0(V)$ under the norm $\|\cdot\|_Q$, which is defined by

$$\|f\|_Q = \sqrt{\|f\|^2_{\ell^2(V,m)} + Q(f)}\,, \quad \forall\, f \in C(V),$$

see Keller-Lenz [32]. For the Dirichlet form Q, it associates with a C_0-semigroup denoted by

$$\mathrm{e}^{t\Delta}\colon \ell^2(V,m) \to \ell^2(V,m),$$

and an (infinitesimal) generator Δ, called Laplacian, given by

$$\Delta f := \lim_{t\to 0} \frac{\mathrm{e}^{t\Delta} f - f}{t}, \quad \forall\, f \in D(\Delta),$$

where

$$D(\Delta) := \Big\{ f \in D(Q)\colon \lim_{t\to 0} \frac{\mathrm{e}^{t\Delta} f - f}{t} \text{ exists in } \ell^2(V,m)\Big\}.$$

This Dirichlet form (or Laplacian) arises from imposing the trivial Dirichlet boundary condition at infinity on a graph.

For locally finite graphs, the generator acts as

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y\in X} \mu_{xy}(f(y) - f(x)), \quad \forall\, f \in C_0(V).$$

It is easy to see that the Laplacian is a bounded operator on $\ell^2(V,m)$ if and only if

$$D_m := \sup_{x\in V} \frac{\deg(x)}{m(x)} < \infty. \tag{1.1}$$

In many cases, the edge measure μ is fixed and the measure m varies, which of course induces different Laplacians. The typical choices of m are

(a) for $m \equiv \deg$ on V, the associated Laplacian is called the normalized Laplacian;

(b) for $m \equiv 1$ on V, the Laplacian is called physical (or combinatorial) Laplacian.

2 Combinatorial curvature on semiplanar graphs

Let (V, E) be a (possibly infinite) simple, locally finite, undirected combinatorial graph without self-loops, i.e., a 1-dimensional simplicial complex. In this section, we restrict to unweighted graphs, i.e., $\mu \equiv 1$ on E and $m \equiv \deg$ on V.

A graph is called *semiplanar* if it can be topologically embedded into a (possibly un-orientable) surface S without boundary. In the literature, planar graphs are those embeddable into plane, i.e., $S = \mathbb{R}^2$, which justifies the name of semiplanar graphs in the general case. For a semiplanar graph, we fix an embedding. A face of the graph is defined as a connected component of the complement of the image of the graph under the embedding map. Note that this structure depends on the choice of the embedding. To avoid pathological examples, we always assume the following conditions:

(1) every face is homeomorphic to a disk whose boundary consists of finitely many edges of the graph;

(2) every edge is contained in exactly two different faces;

(3) for any two faces whose closures have non-empty intersection, the intersection is either a vertex or an edge.

The embedding satisfying above conditions is called a tessellation of S, see, e.g., [29]. For simplicity, in this paper, we mean by a semiplanar graph a tessellation of a surface, denoted by the triple $G = (V, E, F)$ with the set of vertices V, edges E, and faces F, respectively. By the combinatorial structure, one may define the incidence relation: any two objects from V, E, or F are called incident if one is a proper subset of the closure of the other. Given a face σ, its degree $\deg(\sigma)$ is given by the number of edges incident to σ or edges in the boundary of σ. The tessellation assumptions (1)–(3) yield $3 \leqslant \deg(x) < \infty$ for any $x \in V$ and $2 \leqslant \deg(\sigma) < \infty$ for any $\sigma \in F$.

For a semiplanar graph G, the combinatorial curvature at each vertex $x \in V$ is defined as

$$\Phi(x) = 1 - \frac{\deg(x)}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)},$$

where the sum is taken over all the faces incident to x. We introduce several collections of semiplanar graphs with positive or nonnegative curvature as follows:

- $\mathscr{PC}_+ := \{G\colon \inf_{x \in V} \Phi(x) > 0\}$ is the class of semiplanar graphs whose curvature is bounded below uniformly by a positive constant;
- $\mathscr{PC}_{>0} := \{G\colon \Phi(x) > 0, \forall\, x \in V\}$ is the class of semiplanar graphs with positive curvature everywhere;

- $\mathscr{PC}_{\geqslant 0} := \{G\colon \Phi(x) \geqslant 0,\, \forall\, x \in V\}$ is the class of semiplanar graphs with nonnegative curvature everywhere.

Similarly, one defines $\mathscr{PC}_-$, $\mathscr{PC}_{<0}$, and $\mathscr{PC}_{\leqslant 0}$, respectively. By definitions,

$$\mathscr{PC}_+ \subset \mathscr{PC}_{>0} \subset \mathscr{PC}_{\geqslant 0}, \quad \mathscr{PC}_- \subset \mathscr{PC}_{<0} \subset \mathscr{PC}_{\leqslant 0}.$$

At the moment, all discussion are based on combinatorial and topological structures of graphs and surfaces. In order to understand the geometric meaning of the combinatorial curvature, we will associate a metric structure to a semiplanar graph $G = (V, E, F)$ which is embedded into a surface S. For this purpose, we further assume that $\deg(\sigma) \geqslant 3$ for any $\sigma \in F$, which is not quite restrictive. We assign every $\sigma \in F$ a regular polygon in $\mathbb{R}^2$ of the same face degree and of unit edge length. This induces a unique metric structure on S and hence yields a metric space, denoted by $S(G)$, which is piecewise flat. We call $S(G)$ the regular polygonal surface associated to the semiplanar graph G. The metric is generally non-smooth near a vertex, while it is isometric to a flat domain in $\mathbb{R}^2$ near any interior point of an edge or a face. As a non-smooth metric surface $S(G)$, the Gaussian curvature K vanishes everywhere except at vertices and hence can be defined as a measure concentrated on vertices. One can show that the mass of the Gaussian curvature at each vertex x is given by

$$K(x) = 2\pi - \Sigma_x,$$

where $\Sigma_x := \sum_{\sigma \ni x} \angle_x \sigma$ is the total angle of $S(G)$ at x. Moreover, one can prove that

$$K(x) = 2\pi\Phi(x)$$

since it is straightforward to compute the total angle at the vertex by the combinatorial data of the graph. So that the geometric meaning of the combinatorial curvature at each vertex is nothing but a scalar multiple of the Gaussian curvature at that point.

Let $\chi(S)$ denote the Euler characteristic of the surface S. For finite semiplanar graphs, the Gauss-Bonnet formula is well-known, see, e.g., [15],

$$\sum_{x \in V} \Phi(x) = \chi(S).$$

The following Myer's type theorem is proved by [46].

Theorem 2.1 (Stone) *Any semiplanar graph $G \in \mathscr{PC}_+$ is a finite graph.*

Higuchi conjectured that it is a finite graph even if $G \in \mathscr{PC}_{>0}$, see [20, Conjecture 3.2]. This is certainly wrong in Riemannian geometry since there are many noncompact convex hypersurfaces in Euclidean spaces which have positive curvature everywhere. However, in the graph setting, by the combinatorial approach, [47] proved the conjecture for cubic graphs, i.e., satisfying $\deg(x) = 3$, $\forall\, x \in V$. Later, [15] verified this conjecture in full generality by proving the

Gauss-Bonnet formula, see also [7]: let G be an infinite semiplanar graph with regular polygonal surface $S(G)$,

$$\sum_{x\in V}\Phi(x)\leqslant\chi(S(G)),\tag{2.1}$$

whenever $\Sigma_{x\in V}\min\{\Phi(x),0\}$ converges. As a corollary of this conjecture, one has

$$\mathscr{PC}_{+}=\mathscr{PC}_{>0}.$$

By the Gauss-Bonnet formula, any $G\in\mathscr{PC}_{>0}$ can be embedded into either the sphere S^2 or the real projective plane $\mathbb{R}\mathrm{P}^2$. There are four typical subclasses of $\mathscr{PC}_{>0}$, called prisms, anti-prisms in S^2 and their counterparts in $\mathbb{R}\mathrm{P}^2$. Besides them, the other graphs in $\mathscr{PC}_{>0}$ are called positive combinatorial curvature (PCC) graphs. It was proved by [15] that for any PCC graph, the number of vertices is always bounded above by 3444, and hence, there are only finitely many PCC graphs. We define

$$C_{S^2}:=\max_{G\hookrightarrow S^2}\sharp V,\quad C_{\mathbb{R}\mathrm{P}^2}:=\max_{G\hookrightarrow\mathbb{R}\mathrm{P}^2}\sharp V,$$

where the maxima are taken over PCC graphs embedded into S^2 and $\mathbb{R}\mathrm{P}^2$, respectively. A question raised by [15] is the following.

Problem 2.1 What are C_{S^2} and $C_{\mathbb{R}\mathrm{P}^2}$? Which PCC graphs attain the maxima in either cases?

These problems have been studied by [7,15,42,44,54] and the best known results, due to [42,54], are

$$208\leqslant C_{S^2}<580,\quad 104\leqslant C_{\mathbb{R}\mathrm{P}^2}<290.$$

Next, we turn to the class of semiplanar graphs with nonnegative curvature, i.e., $\mathscr{PC}_{\geqslant0}$, which includes many infinite graphs, in particular, all planar tilings using regular polygons. In this class, [6,7] obtained an interesting result, which was a conjecture in [7].

Theorem 2.2 (B. Chen and G. Chen) *For any $G\in\mathscr{PC}_{\geqslant0}$, there are only finitely many vertices with non-vanishing curvature.*

In the proof of Theorem 2.2, the following lemmas are useful, see [7, Lemmas 2.5, 2.7].

Lemma 2.1 *If $0<\Phi(x)<1/1722$, then x is incident to a face σ with* $\deg(\sigma)\geqslant43$.

Lemma 2.2 *If there is a face σ such that* $\deg(\sigma)\geqslant43$ *and* $\Phi(x)\geqslant0$ *for any vertex x incident to σ, then*

$$\sum_{x\in V,\,x\in\sigma}\Phi(x)\geqslant1.$$

In fact, any graph $G \in \mathscr{PC}_{\geqslant 0}$ with $\sup_{\sigma\in F}\deg(\sigma) \geqslant 43$ has rather special structure, whose regular polygonal surface is isometric to a half flat-cylinder in $\mathbb{R}^3$, see [25, Lemma 2.9, Theorem 2.10].

Lemma 2.3 *For any infinite $G \in \mathscr{PC}_{\geqslant 0}$ with $\sup_{\sigma\in F}\deg(\sigma) \geqslant 43$, it has vanishing curvature outside the unique face attaining the maximum facial degree.*

Given a semiplanar graph G, we denote by

$$N(G) := \sharp\{x \in V : \Phi(x) \neq 0\}$$

the number of vertices with non-vanishing curvature. By Lemma 2.3, one has

$$N(G) = \sup_{\sigma\in F}\deg(\sigma)$$

for any infinite graph $G \in \mathscr{PC}_{\geqslant 0}$ satisfying $\sup_{\sigma\in F}\deg(\sigma) \geqslant 43$. Let

$$N := \sup_G N(G),$$

where the supremum is taken over all infinite graphs G in $\mathscr{PC}_{\geqslant 0}$ with the maximum face degree less than 43. We propose the following questions.

Problem 2.2 How large is N? What kind of graphs attains the supremum in the definition of N?

By above lemmas, one can get a rough upper bound estimate.

Proposition 2.1 $N \leqslant 1722$.

Proof For any infinite graph $G \in \mathscr{PC}_{\geqslant 0}$ with $\sup_{\sigma\in F}\deg(\sigma) < 43$, Lemma 2.1 yields that $\Phi(x) \geqslant 1/1722$ for any $x \in V$ with $\Phi(x) > 0$. Since G is an infinite graph, $\chi(S(G)) \leqslant 1$. Hence, (2.1) implies that

$$\sum_{x\in V}\Phi(x) \leqslant 1,$$

which yields the result. □

It is known that for any infinite $G \in \mathscr{PC}_{\geqslant 0}$, the total curvature of G satisfies

$$0 \leqslant \sum_{x\in V}\Phi(x) \leqslant 1.$$

Réti[1)] asked how small the total curvature of a graph $G \in \mathscr{PC}_{\geqslant 0}$ could differ from zero? Precisely, we define

$$TC := \inf_G \sum_{x\in V}\Phi(x),$$

1) Personal communication from Tamás Réti

where the infimum is taken over all $G \in \mathscr{PC}_{\geqslant 0}$ with positive total curvature, which is the gap of the total curvature from zero in the class $\mathscr{PC}_{\geqslant 0}$. His conjecture is the following.

Conjecture 2.1 (Réti)

$$TC \geqslant \frac{1}{6}$$

and the infimum is attained by the planar graph consisting of a pentagon and infinitely many hexagons.

By a simple argument as in the proof of Proposition 2.1, Lemmas 2.1 and 2.2 yield that

$$TC \geqslant \frac{1}{1722}.$$

This is obviously far from optimal.

By the geometric meaning of combinatorial curvature, one yields

$$\Phi(x) \geqslant 0, \quad \forall\, x \in V,$$

if and only if

$$K \geqslant 0,$$

i.e., $S(G)$ is a convex surface (non-negatively curved in the sense of Alexandrov). obtained the topological classification of regular polygonal surfaces of infinite semiplanar graphs with nonnegative curvature: $\mathbb{R}^2$, the cylinder without boundary, and the projective plane minus one point.

In other direction, one can study the classes of $\mathscr{PC}_{-}$, $\mathscr{PC}_{<0}$, and $\mathscr{PC}_{\leqslant 0}$. The following observation in [20] is unexpected.

Proposition 2.2 *Let G be a semiplanar graph. If $\Phi(x) < 0$ for some $x \in V$, then $\Phi(x) \leqslant -\varepsilon_0$ with $\varepsilon_0 := 1/1806$.*

This yields that

$$\mathscr{PC}_{-} = \mathscr{PC}_{<0}.$$

As strong applications of this proposition, many authors, [19–21,29,33,35,49,55], proved the positivity of the isoperimetric constant, or called Cheeger constant in the literature, for the semiplanar graphs $G \in \mathscr{PC}_{<0}$. This yields the exponential volume growth of the graph and the positivity of the bottom of the ℓ^2 spectrum.

At the end of this section, we turn to the cases of non-negatively curved graphs. First, we introduce the definitions of the volume doubling property and the Poincaré inequality on weighted graphs.

Definition 2.1 (DV) A graph $G = (V, E, m, \mu)$ satisfies the *volume doubling* property $DV(C)$ for constant $C > 0$, if for all $x \in V$ and all $r > 0$,

$$V(x, 2r) \leqslant CV(x, r).$$

(P) A graph G satisfies the *Poincaré inequality* $P(C)$ for a constant $C > 0$ if

$$\sum_{x\in B(x_0,r)} m(x)|f(x)-f_B|^2 \leqslant Cr^2 \sum_{x,y\in B(x_0,2r)} \mu_{xy}(f(y)-f(x))^2$$

for all functions f, all $x_0 \in V$, and all $r > 0$, where

$$f_B := \frac{1}{V(x_0,r)} \sum_{x\in B(x_0,r)} m(x)f(x).$$

[25] proved these properties for the class $\mathscr{PC}_{\geqslant 0}$.

Theorem 2.3 [25] *For any $G \in \mathscr{PC}_{\geqslant 0}$, the volume doubling property $DV(C_1)$ and the Poincaré inequality $P(C_2)$ hold for some $C_1, C_2 > 0$.*

The general principle hidden in this result dates back to [12], in which they showed that the volume doubling property and the Poincaré inequality are both quasi-isometric invariant, see [45,50] for definitions. Since the semiplanar graph G is properly embedded into the regular polygonal surface $S(G)$, they are in fact quasi-isometric to each other. For convex surfaces, even more general Alexandrov spaces with nonnegative curvature, the volume doubling property follows from the Bishop-Gromov volume comparison [5] and the Poincaré inequality was obtained by [23,34].

A function $f \in C(V)$ is called harmonic on $\Omega \subset V$ if $\Delta f \equiv 0$ on Ω. For any $k > 0$, we denote by

$$H^k(G) := \{f \in C(V) \mid \Delta f \equiv 0,\ |f(x)| \leqslant C(1+d(x,p))^k,\ \text{for some}\, p \in V,\ C > 0\}$$

the space of harmonic functions on V of polynomial growth whose growth order are less than or equal to k. By Moser iteration, [13] proved the elliptic Harnack inequality on graphs under the assumptions of the volume doubling property and the Poincaré inequality. In particular, Theorem 2.3 implies the Harnack inequality that for $G \in \mathscr{PC}_{\geqslant 0}$ and any positive harmonic function f on $B_{2R}(p) \subset V$, we have

$$\max_{B_R(p)} f \leqslant C \inf_{B_R(p)} f,$$

where C is a universal constant. This further yields that

$$\dim H^d(G) = 1$$

for some $d \in (0,1)$. In fact, the combination of volume doubling property and the Poincaré inequality turns out to be equivalent to the parabolic Harnack inequalities [14], see Subsection 3.4 later.

A function f on G is called of sublinear growth if for some (hence all) $p \in V$ such that

$$\max_{B_R(p)} |f| = o(R) \quad (R \to \infty).$$

We conjecture that the following Cheng type Liouville theorem holds on semiplanar graphs with nonnegative curvature.

Conjecture 2.2 For any $G \in \mathscr{PC}_{\geqslant 0}$, any harmonic function on G of sublinear growth is constant.

Moreover, one can prove the finite-dimensional property of the space of harmonic functions of polynomial growth with growth rate bounded above, following Colding-Minicozzi and Li [8–11,36,37].

Theorem 2.4 [24,25] *For any* $G \in \mathscr{PC}_{\geqslant 0}$,

$$\dim H^d(G) \leqslant Cd, \quad \forall\, d \geqslant 1,$$

where C *is a universal constant.*

3 Curvature dimension conditions on graphs

3.1 Gamma calculus

We introduce the Gamma calculus and curvature dimension conditions on graphs, see, e.g., [1,39]. First, we define two natural bilinear forms associated to the Laplacian.

Definition 3.1 The gradient form Γ is defined by

$$\Gamma(f,g)(x) = \frac{1}{2}\,(\Delta(fg) - f\Delta g - g\Delta f)(x).$$

The iterated gradient form is defined by

$$\Gamma_2(f,g) = \frac{1}{2}\,(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)).$$

For simplicity, we write

$$\Gamma(f) = \Gamma(f,f), \quad \Gamma_2(f) = \Gamma_2(f,f).$$

Now, we are ready to define *curvature dimension conditions*, also called CD-inequalities, on graphs. In the following, let $n \in (0,\infty]$ and $K \in \mathbb{R}$, which will serve as the upper bound of the dimension and the lower bound of the Ricci curvature mimicking the Riemannian case, respectively.

Definition 3.2 The graph G satisfies the $CD(n,K)$ property if for any $f \in C(V)$,

$$\Gamma_2(f) \geqslant \frac{1}{n}\,(\Delta f)^2 + K\Gamma(f).$$

Moreover, [1] introduced two other conditions, CDE and CDE' which stand for *exponential curvature dimension conditions*, both of which we recall below.

Definition 3.3 We say that a graph G satisfies the $CDE(n,K)$ property if for any $x \in V$, and any positive function f satisfying $\Delta f(x) < 0$, we have

$$\widetilde{\Gamma}_2(f)(x) := \Gamma_2(f)(x) - \Gamma\Big(f, \frac{\Gamma(f)}{f}\Big)(x) \geqslant \frac{1}{n}\,((\Delta f)(x))^2 + K\Gamma(f)(x).$$

Definition 3.4 We say that a graph G satisfies the $CDE'(n,K)$ property, if for any $x \in V$ and any positive function f,

$$\widetilde{\Gamma_2}(f)(x) \geqslant \frac{1}{n}\,(f(x))^2((\Delta \log f)(x))^2 + K\Gamma(f)(x).$$

One can show that $CDE'(n,K)$ implies $CDE(n,K)$, see [22, Remark 1]. It turns out that $CDE'(n,K)$ yields $CD(n,K)$, see [41]. Cayley graphs of finitely generated Abelian groups, more generally, Ricci flat graphs, satisfy $CDE'(0,d)$ and $CDE(0,d')$ for some $d,d' > 0$.

3.2 Stochastic completeness on graphs

A graph is called *stochastically complete* (or *conservative*) if the continuous time heat kernel $p_t(\cdot,\cdot)$, which is the kernel of the semigroup $\mathrm{e}^{t\Delta}$, satisfies for some (hence all) $x \in V$, $t > 0$,

$$\sum_y p_t(x,y)m(y) = 1.$$

The stochastic completeness of graphs, in particular for unbounded Laplacians, has been studied by many authors, e.g., [17,31,32,51–53].

We say that a graph $G = (V,E,m,\mu)$ is *complete* if there exists a non-decreasing sequence of $\{\eta_k\}_{k=1}^{\infty} \subset C_0(V)$ such that

$$\lim_{k\to\infty} \eta_k = \mathbb{1} \quad \text{and} \quad \Gamma(\eta_k) \leqslant \frac{1}{k},$$

where $\mathbb{1}$ is the constant function 1 on V. Under the curvature conditions, we proved the following gradient bounds for heat semigroups.

Theorem 3.1 [26, Theorem 1.1] *Let $G = (V,E,m,\mu)$ be a complete graph, and let m be non-degenerate, i.e., $\inf_{x\in V} m(x) > 0$. Then the following are equivalent*:

(a) *G satisfies $CD(\infty,K)$*;

(b) *for any $f \in C_0(V)$,*

$$\Gamma(\mathrm{e}^{t\Delta} f) \leqslant \mathrm{e}^{-2Kt}\mathrm{e}^{t\Delta}(\Gamma(f)).$$

Note that we need the non-degeneracy of the measure m which is conjecturally not necessary. The stochastic completeness follows directly from the above gradient bounds, see [26, Theorem 1.2].

Theorem 3.2 *Let $G = (V,E,\mu,m)$ be a complete graph satisfying the $CD(\infty,K)$ condition for some $K \in \mathbb{R}$. Suppose that the measure m is non-degenerate. Then G is stochastically complete.*

3.3 Li-Yau estimate on graphs

[1] proved the Li-Yau gradient estimate for positive solutions of heat equations under CDE conditions on graphs.

In the rest of the paper, we only consider weighted graphs $G=(V,E,m,\mu)$ satisfying

$$\inf_{x\sim y}\mu_{xy}>0,\quad D_\mu:=\sup_{x\sim y}\frac{\deg(x)}{\mu_{xy}}<\infty,$$

and (1.1) holds. Moreover, we restrict to finite-dimensional curvature dimension conditions, i.e., dimensional constants n are assumed to be finite.

For $\Omega\subset V$ and an interval $I\subset\mathbb{R}$, we say that a function $u\colon\mathbb{R}\times V\to\mathbb{R}$ satisfies the heat equation on $I\times\Omega$, if

$$\partial_t u=\Delta u\quad\text{on } I\times\Omega.$$

In case $I=\mathbb{R}$, we simply say that u satisfies the heat equation on Ω. Similarly, we say that a function $v\colon\mathbb{Z}\times V\to\mathbb{R}$ satisfies the discrete-time heat equation on $I\cap\mathbb{Z}\times\Omega$ if

$$v(k+1,x)-v(k,x)=\Delta v(k,x),\quad k\in I\cap\mathbb{Z},\ x\in\Omega.$$

Theorem 3.3 [1, Theorem 4.20] *Let G be a weighted graph satisfying $CDE(n,0)$. For any $R>0$, $x_0\in V$, let u be a positive solution to the heat equation on $B_{2R}(x_0)$. Then*

$$\frac{\Gamma(\sqrt{u}\,)}{u}-\frac{\partial_t\sqrt{u}}{\sqrt{u}}<\frac{n}{2t}+\frac{n(1+D_\mu)D_m}{R}\quad\text{on } B_R(x_0).$$

This gradient estimate leads to the Harnack inequality [1, Corollary 5.3] and hence the pointwise heat kernel bounds [1, Theorem 7.6] and [4, Theorem 1.2].

3.4 Volume doubling and Poincaré inequalities

[22] proved the volume doubling property and the Poincaré inequality for graphs satisfying CDE' conditions. In this section, we only consider normalized Laplacians, i.e., setting $m\equiv\deg$. The combination of these properties turns out to be equivalent to the Gaussian bounds for the heat kernel, or the parabolic Harnack inequality, see [14].

We need the following definitions.

Definition 3.5 ($\mathscr{H}$) Fix $\eta\in(0,1)$, $0<\theta_1<\theta_2<\theta_3<\theta_4$, and $C>0$. G satisfies the *continuous time parabolic Harnack inequality* $\mathscr{H}(\eta,\theta_1,\theta_2,\theta_3,\theta_4,C)$, if for all $x_0\in V$, $s,R>0$, and every positive solution $u(t,x)$ to the heat equation on

$$Q=[s,s+\theta_4R^2]\times B(x_0,R),$$

we have

$$\sup_{Q^-}u(t,x)\leqslant C\inf_{Q^+}u(t,x),$$

where

$$Q^- = [s + \theta_1 R^2, s + \theta_2 R^2] \times B(x_0, \eta R),$$
$$Q^+ = [s + \theta_3 R^2, s + \theta_4 R^2] \times B(x_0, \eta R).$$

(H) Fix $\eta \in (0,1)$, $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$, and $C > 0$. G satisfies the *discrete-time parabolic Harnack inequality* $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C)$, if for all $x_0 \in V$, $s, R > 0$, and every positive solution $u(k,x)$ to the discrete heat equation on

$$Q = ([s, s + \theta_4 R^2] \cap \mathbb{Z}) \times B(x_0, R),$$

for any

$$(k^-, x^-) \in Q^-, \quad (k^+, x^+) \in Q^+, \quad d(x^-, x^+) \leqslant k^+ - k^-,$$

we have

$$u(k^-, x^-) \leqslant C u(k^+, x^+),$$

where

$$Q^- = ([s + \theta_1 R^2, s + \theta_2 R^2] \cap \mathbb{Z}) \times B(x_0, \eta R),$$
$$Q^+ = ([s + \theta_3 R^2, s + \theta_4 R^2] \cap \mathbb{Z}) \times B(x_0, \eta R).$$

(G) Fix positive constants $c_l, C_l, C_r, c_r > 0$. The graph G satisfies the *Gaussian estimate* $G(c_l, C_l, C_r, c_r)$ if, whenever $d(x,y) \leqslant k$,

$$\frac{c_l m(y)}{V(x, \sqrt{k}\,)} \,\mathrm{e}^{-C_l d(x,y)^2/k} \leqslant p_k(x,y) \leqslant \frac{C_r m(y)}{V(x, \sqrt{k}\,)} \,\mathrm{e}^{-c_r d(x,y)^2/k}.$$

Moreover, for any $\alpha > 0$, we say that the graph G satisfies the property $\Delta(\alpha)$ if the following conditions hold:

(1) $x \sim x$ for any $x \in V$, and

(2) $\mu_{xy} \geqslant \alpha \deg(x)$ for any $x \sim y$.

Now, we are ready to state [22, Theorem 2.2].

Theorem 3.4 *For any graph G satisfying $CDE'(n,0)$ and $\Delta(\alpha)$, the following four properties hold*:

1) *there exist $C_1, C_2 > 0$ such that $DV(C_1)$ and $P(C_2)$ hold*;

2) *there exist $c_l, C_l, C_r, c_r > 0$ such that $G(c_l, C_l, C_r, c_r)$ is true*;

3) *there exists C_H such that $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_H)$ is true*;

4) *there exists $C_{\mathscr{H}}$ such that $\mathscr{H}(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_{\mathscr{H}})$ is true.*

In particular, this yields that the space of polynomial growth harmonic functions $H^d(G)$ is of finite dimension for such a graph, see [22, Theorem 2.3]. We pose a conjecture on the optimal dimension estimate for $H^d(G)$.

Conjecture 3.1 Let a graph G satisfy $CDE'(n,0)$ condition. Then there is a constant C such that

$$\dim H^d(G) \leqslant C d^{n-1}, \quad \forall\, d \geqslant 1.$$

Acknowledgements B. H. was supported by the National Natural Science Foundation of China (Grant No. 11401106); Y. L. was supported by the National Natural Science Foundation of China (Grant No. 11271011), the Fundamental Research Funds for the Central Universities, and the Research Funds of Renmin University of China (11XNI004).

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