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RESEARCH ARTICLE

Ideal counting function in cubic fields

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Abstract For a cubic algebraic extension K of $\mathbb{Q}$, the behavior of the ideal counting function is considered in this paper. More precisely, let $a_K(n)$ be the number of integral ideals of the field K with norm n, we prove an asymptotic formula for the sum $\sum_{n_1^2+n_2^2\leqslant x} a_K(n_1^2+n_2^2)$.

Keywords Non-normal extension, ideal counting function, Rankin-Selberg convolution
MSC 11F30, 11F66, 11N45, 11R42

1 Introduction

Many deep arithmetic properties of a number field are embedded into the associated Dedekind zeta function. Let K be an algebraic extension of $\mathbb{Q}$ with degree d. Its associated Dedekind zeta function is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \mathfrak{N}(\mathfrak{a})^{-s}, \quad \operatorname{Re} s > 1,$$

where the sum runs over all integral ideals in $\mathscr{O}_K$, and $\mathfrak{N}(\mathfrak{a})$ is the norm of the integral ideal $\mathfrak{a}$. Since the norm of an integral ideal is a positive rational integer, the Dedekind zeta function can be rewritten as an ordinary Dirichlet series

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_K(n) n^{-s}, \quad \operatorname{Re} s > 1, \tag{1}$$

where $a_K(n)$ is the so-called ideal counting function, which counts the number of integral ideals $\mathfrak{a}$ with norm n in K.

It is known that the ideal counting function $a_K(n)$ is a multiplicative function, and it has the upper bound

$$a_K(n) \ll \tau^d(n),$$

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where $\tau(n)$ is the divisor function, see [1]. However, in many applications, the upper bound is not enough and it is more interesting to study the asymptotic behavior of the ideal counting function in various sequences. Landau [12] proved the following asymptotic formula:

$$\sum_{n\leqslant x} a_K(n) = cx + O(x^{\frac{d-1}{d+1}+\varepsilon})$$

for arbitrary algebraic number field of degree $d \geqslant 2$. Many authors have studied this problem and for general algebraic number field of degree $d \geqslant 3$, the best result hitherto is due to Nowak [16]. Recently, for the Galois extension over $\mathbb{Q}$, by using the decomposition of prime p in $\mathscr{O}_K$ and the analytic properties of L-functions, Lü and Wang [14] considered the average behavior of moments of the ideal counting function

$$\sum_{n\leqslant x} a_K^l(n), \quad l = 1, 2, \ldots,$$

and gave a sharper estimate for $l = 1$ in the Galois extension over $\mathbb{Q}$, while later Lü and Yang [15] gave an asymptotic formula for the sum

$$\sum_{n\leqslant x} a_K^l(n^2), \quad l = 1, 2, \ldots,$$

in the Galois extension over $\mathbb{Q}$.

It is more difficult to study the ideal counting function for a non-normal extension K of $\mathbb{Q}$. However, by applying the so-called strong Artin conjecture, Fomenko [4] studied the sum

$$\sum_{n\leqslant x} a_K^l(n), \quad l = 2,\ 3,$$

when K is a non-normal cubic field extension. Later, Lü [13] improved the error term.

In this paper, we will be interested in the estimation of the following sum:

$$\sum_{n_1^2+n_2^2\leqslant x} a_K(n_1^2+n_2^2), \tag{2}$$

where K is the cubic algebraic extension of $\mathbb{Q}$.

Let $r(n)$ be the number of representation of an integer n as sums of two square integers. i.e.,

$$r(n) = \#\{n \in \mathbb{Z} \mid n = n_1^2 + n_2^2\}.$$

Then, we can rewrite formula (2) as

$$\sum_{n_1^2+n_2^2\leqslant x} a_K(n_1^2+n_2^2) = \sum_{n\leqslant x} a_K(n) \sum_{n=n_1^2+n_2^2} 1 = \sum_{n\leqslant x} a_K(n) r(n). \tag{3}$$

It is known that $r(n)$ is the ideal counting function of the Gaussian number field $\mathbb{Q}(\sqrt{-1})$ and we have

$$r(n) = 4\sum_{d|n}\chi'(d),$$

where χ' is the real primitive Dirichlet character modulo 4.

In general, for a quadratic number field L with discriminant D', the ideal counting function of the field L is

$$a_L(n) = \sum_{d|n}\chi'(d),$$

where χ' is a real primitive Dirichlet character modulo $|D'|$. It is an interesting question to consider the sum

$$\sum_{n\leqslant x} a_K(n)a_L(n).$$

Fomenko [3] considered this convolution sum when both K and L are quadratic fields. In this paper, we shall discuss a more general case. Assume that $q \geqslant 1$ is an integer and χ is a primitive character modulo q. Define the function

$$f_\chi(n) = \sum_{k|n}\chi(k).$$

Then we have the following results.

Theorem 1 *Let K be a cubic normal extension of $\mathbb{Q}$, let $q \geqslant 1$ be an integer, and let χ be a primitive Dirichlet character modulo q. Then we have*

$$\sum_{n\leqslant x} a_K(n)f_\chi(n) = xP_4(\log x) + O(x^{\frac{7}{12}+\varepsilon}), \tag{4}$$

where $P_4(t)$ is a polynomial in t with degree 3, and $\varepsilon > 0$ is an arbitrarily small constant.

Theorem 2 *Let K be a cubic non-normal extension of $\mathbb{Q}$, let $q \geqslant 1$ be an integer, and let χ be a primitive Dirichlet character modulo q. Then we have*

$$\sum_{n\leqslant x} a_K(n)f_\chi(n) = xP_3(\log x) + O(x^{\frac{3}{5}+\varepsilon}), \tag{5}$$

where $P_3(t)$ is a polynomial in t with degree 2, and $\varepsilon > 0$ is an arbitrarily small constant.

According to the theorems above, we obtain the following corollaries.

Corollary 1 *Let K be a cubic normal extension of $\mathbb{Q}$, and let $r(n)$ be the number of representation of an integer n as sums of two square integers. Then we have*

$$\sum_{n\leqslant x} a_K(n)r(n) = xP_4(\log x) + O(x^{\frac{7}{12}+\varepsilon}),$$

where $P_4(t)$ is a polynomial in t with degree 3, and $\varepsilon > 0$ is an arbitrarily small constant.

Corollary 2 *Let K be a cubic non-normal extension of $\mathbb{Q}$, and let $r(n)$ be the number of representation of an integer n as sums of two square integers. Then we have*

$$\sum_{n\leqslant x} a_K(n)r(n) = xP_3(\log x) + O(x^{\frac{3}{5}+\varepsilon}),$$

where $P_3(t)$ is a polynomial in t with degree 2, and $\varepsilon > 0$ is an arbitrarily small constant.

2 Preliminaries

Let K be a cubic algebraic extension of $\mathbb{Q}$, and let $D = df^2$ (d squarefree) its discriminant. The Dedekind zeta function of K is given in (1). It has the Euler product

$$\zeta_K(s) = \prod_p \Big(1 + \frac{a_K(p)}{p^s} + \frac{a_K(p^2)}{p^{2s}} + \cdots\Big). \tag{6}$$

We will give some results about Dedekind zeta function of cubic field K in the following.

Lemma 1 *K is a normal extension if and only if $D = f^2$. In this case,*

$$\zeta_K(s) = \zeta(s)L(s,\varphi)L(s,\overline{\varphi}),$$

where $\zeta(s)$ is the Riemann zeta function and $L(s,\varphi)$ is an ordinary Dirichlet series (over $\mathbb{Q}$) corresponding to a primitive character φ modulo f.

Proof See the lemma in [17]. □

By using Lemma 1, the Euler product of Riemann zeta function $\zeta(s)$, and the Dirichlet L-functions, we have the following result.

Lemma 2 *Assume that $a_K(n)$ is the ideal counting function of the cubic normal extension K over $\mathbb{Q}$. Then we get*

$$a_K(n) = \sum_{xy|n} \varphi(x)\overline{\varphi}(y),$$

where x and y are integers. In particular, when $n = p$ is a prime, we get

$$a_K(p) = 1 + \varphi(p) + \overline{\varphi}(p), \tag{7}$$

where φ is a primitive character modulo f.

Assume that K is a non-normal cubic extension over $\mathbb{Q}$, which is given by an irreducible polynomial

$$f(x) = x^3 + ax^2 + bx + c.$$

Let E denote the normal closure of K that is normal over $\mathbb{Q}$ with degree 6, and denoted the Galois group $\mathrm{Gal}(E/\mathbb{Q}) = S_3$. First, we will introduce some properties about S_3 (see [5, pp. 226, 227] for detailed arguments).

The elements of S_3 fall into three conjugacy classes:

$C_1\colon (1)$;

$C_2\colon (1,2,3),\ (3,2,1)$;

$C_3\colon (1,2),\ (2,3),\ (1,3)$,

with the following three simple characters: the one dimensional characters ψ_1 (the principal character) and ψ_2 (the other character determined by the subgroup $C_1 \cup C_2$), and the two dimensional character ψ_3.

Let D be the discriminant of $f(x) = x^3 + ax^2 + bx + c$ and $K_2 = \mathbb{Q}(\sqrt{D}\,)$. The fields K_2 and K are the intermediate extensions fixed under the subgroups A_3 and $\{(1),(1,2)\}$, respectively. The extensions $K_2/\mathbb{Q}$, E/K_2, and E/K are abelian. The Dedekind zeta function satisfy the relations

$$\zeta_E(s) = L_{\psi_1} L_{\psi_2} L_{\psi_3}^2, \quad \zeta_{K_2}(s) = L_{\psi_1} L_{\psi_2}, \quad \zeta_K(s) = L_{\psi_1} L_{\psi_3}, \quad \zeta(s) = L_{\psi_1},$$

where

$$L_{\psi_2} = L(s, \psi_2, E/\mathbb{Q}), \quad L_{\psi_3} = L(s, \psi_3, E/\mathbb{Q}),$$

which are Artin L-functions.

Kim [11] proved that the strong Artin conjecture holds true for the group S_3. By using the strong Artin conjecture, the function L_{ψ_3} also can be interpreted in another way [2]. Let $\rho\colon S_3 \to GL_2(\mathbb{C})$ be the irreducible two-dimensional representation. Then ρ gives rise to a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Let

$$L(s,\pi) = \sum_{n=1}^{\infty} M(n) n^{-s}.$$

Below, we assume that ρ is odd, i.e. $D < 0$. Then $L(s,\pi) = L(s,f)$, where f is a holomorphic cusp form of weight 1 with respect to the congruence group $\Gamma_0(|D|)$:

$$f(z) = \sum_{n=1}^{\infty} M(n) \mathrm{e}^{2\pi i n z}.$$

Here, as usual, $L(s,\pi)$ denotes the L-function of the representation π, and $L(s,f)$ denotes the Hecke L-function of cusp form f. Thus $L_{\psi_3} = L(s,f)$ and

$$\zeta_K(s) = \zeta(s) L(s,f). \tag{8}$$

Formula (8) implies the following result.

Lemma 3 *The symbols are defined as above. We have*

$$a_K(n) = \sum_{d|n} M(d).$$

In particular,

$$a_K(p) = 1 + M(p),$$

where p is a prime integer.

To prove the theorems, we also need some well-known estimates of the relative L-functions. For subconvexity bounds, we have the following well-known estimates.

Lemma 4 *For any $\varepsilon > 0$, we have*

$$\zeta(\sigma + \mathrm{i}t) \ll_\varepsilon (1 + |t|)^{\frac{1}{3}(1-\sigma)+\varepsilon} \tag{9}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1$ and $|t| \geqslant 1$.

Proof See [18, Theorem II 3.6]. □

For the Dirichlet L-series, by using the Phragmen-Lindelöf principle for a strip [9] and the estimates given by Heath-Brown [7], we have the similar results:

$$L(\sigma + \mathrm{i}t, \chi) \ll_\varepsilon (1 + |t|)^{\frac{1}{3}(1-\sigma)+\varepsilon} \tag{10}$$

uniformly for $1/2 \leqslant \sigma \leqslant 1$ and $|t| \geqslant 1$, where χ is a Dirichlet character modulo q, and q is an integer.

For the mean values of the relative L-functions on the critical line, we have the following result.

Lemma 5 *For any $\varepsilon > 0$, let $L(s)$ be the Riemann zeta function $\zeta(s)$, or the Dirichlet L-function $L(s; \chi, q)$ with respect to the Dirichlet character χ modulo a fixed $q \geqslant 1$. Then we have*

$$\int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^A \ll_\varepsilon T^{1+\varepsilon} \tag{11}$$

uniformly for $T \geqslant 1$, where $A = 2$, 4.

For Hecke L-functions defined in (8), we have the following result.

Lemma 6 *For any $\varepsilon > 0$, we have*

$$\int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, f\Big)\Big|^2 \mathrm{d}t \sim CT \log T, \tag{12}$$

$$\int_1^T \Big|L\Big(\frac{1}{2} + \mathrm{i}t, f\Big)\Big|^6 \mathrm{d}t \ll T^{2+\varepsilon}, \tag{13}$$

uniformly for $T \geqslant 1$, *and the subconvexity bound*

$$L(\sigma + \mathrm{i}t,\ f) \ll_{t,\ \varepsilon} (1+|t|)^{\max\{\frac{2}{3}(1-\sigma),\ 0\}+\varepsilon}$$

uniformly for $1/2 \leqslant \sigma \leqslant 2$ *and* $|t| \geqslant 1$.

Proof The first and third results due to Good [6], and the second was proved by Jutila [10]. □

By using Lemma 6 and Hölder's inequality, we have

$$\begin{aligned}
&\int_1^T \Big|L\Big(\frac{1}{2}+\mathrm{i}t, f\Big)\Big|^4 \mathrm{d}t \\
&\ll \bigg(\int_1^T \Big|L\Big(\frac{1}{2}+\mathrm{i}t, f\Big)\Big|^2 \mathrm{d}t\bigg)^{1/2} \bigg(\int_1^T \Big|L\Big(\frac{1}{2}+\mathrm{i}t, f\Big)\Big|^6 \mathrm{d}t\bigg)^{1/2} \\
&\ll T^{\frac{3}{2}+\varepsilon}. \qquad (14)
\end{aligned}$$

For the mean values of the Riemann zeta function in the critical strip $1/2 < \sigma < 1$, define $m(\sigma)$ $(\geqslant 4)$ as the supremum of all numbers m $(\geqslant 4)$ such that

$$\int_1^T |\zeta(\sigma+\mathrm{i}t)|^m \mathrm{d}t \ll T^{1+\varepsilon}, \quad \forall\, \varepsilon > 0. \qquad (15)$$

Ivić [8] proved that for $1/2 < \sigma < 5/8$,

$$m(\sigma) \geqslant \frac{4}{3-4\sigma}.$$

Let $\sigma = 7/12$. Then we can get

$$\int_1^T \Big|\zeta\Big(\frac{7}{12}+\mathrm{i}t\Big)\Big|^6 \mathrm{d}t \ll T^{1+\varepsilon}, \quad \forall\, \varepsilon > 0. \qquad (16)$$

Similarly, as the proof of the mean values of Riemann zeta function, for Dirichlet L-function $L(s;\chi,q)$ with respect to the Dirichlet character χ modulo a fixed $q \geqslant 1$, we have

$$\int_1^T \Big|L\Big(\frac{7}{12}+\mathrm{i}t;\chi,q\Big)\Big|^6 \mathrm{d}t \ll T^{1+\varepsilon}. \qquad (17)$$

3 Proofs of theorems

Assume that K is a cubic extension of $\mathbb{Q}$. The Dedekind zeta function of K is given in (1). Its Euler product is (6) with $\operatorname{Re} s > 1$.

Let q be an integer, and let χ be a primitive Dirichlet character modulo q. Define the function

$$f_\chi(n) = \sum_{k|n} \chi(k). \qquad (18)$$

It is easy to check that

$$f_\chi(mn)=f_\chi(m)f_\chi(n),\quad (m,n)=1.$$

Since $a_K(n)\ll n^\varepsilon$, so does $a_K(n)f_\chi(n)$. We can define an L-function associated to the function $a_K(n)f_\chi(n)$ in the half-plane $\operatorname{Re}s>1$,

$$L_{K,f_\chi}(s)=\sum_{n=1}^{\infty}a_K(n)f_\chi(n)n^{-s},\tag{19}$$

which is absolutely convergent in this region. Both $a_K(n)$ and $f_\chi(n)$ are multiplicative, so for $\operatorname{Re}s>1$, the function $L_{K,f_\chi}(s)$ can be expressed by the Euler product

$$L_{K,f_\chi}(s)=\prod_p\Big(1+\frac{a_K(p)f_\chi(p)}{p^s}+\frac{a_K(p^2)f_\chi(p^2)}{p^{2s}}+\cdots\Big),$$

where the product runs over all primes.

Proof of Theorem 1 When K is a cubic normal extension, according to (7) and (18), we get the formula

$$a_K(p)f_\chi(p)=1+\varphi(p)+\overline{\varphi}(p)+\chi(p)+\varphi(p)\chi(p)+\overline{\varphi}(p)\chi(p)=:A(p),\tag{20}$$

where p is a prime.

For $\operatorname{Re}s>1$, we can write

$$M_{K,f_\chi}(s):=\zeta(s)L(s,\varphi)L(s,\overline{\varphi})L(s,\chi)L(s,\varphi\times\chi)L(s,\overline{\varphi}\times\chi)$$

as an Euler product of the form

$$\prod_p\Big(1+\frac{A(p)}{p^s}+\frac{A(p^2)}{p^{2s}}+\cdots\Big),$$

where the functions $L(s,\varphi\times\chi)$ and $L(s,\overline{\varphi}\times\chi)$ are the Rankin-Selberg convolution L-function of the Dirichlet L-functions $L(s,\varphi)$ and $L(s,\overline{\varphi})$ with the Dirichlet L-functions $L(s,\chi)$, respectively.

By comparing it with the Euler product of $L_{K,f_\chi}(s)$, and using (20), we obtain

$$L_{K,f_\chi}(s)=M_{K,f_\chi}(s)\cdot U_1(s),\tag{21}$$

where $U_1(s)$ denotes a Dirichlet series, which is absolutely convergent for $\operatorname{Re}s>1/2$, and uniformly convergent for $\operatorname{Re}s>\frac12+\varepsilon$. Therefore, the function $L_{K,f_\chi}(s)$ admits an analytic continuation into the half-plane $\sigma>1/2$, having as its only singularity a pole of order 4 at $s=1$.

By using the well-known inversion formula for Dirichlet series, we obtain

$$\sum_{n\leqslant x}a_K(n)f_\chi(n)=\frac{1}{2\pi \mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T}L_{K,f_\chi}(s)\frac{x^s}{s}\,\mathrm{d}s+O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where $b = 1 + \varepsilon$ and $1 \leqslant T \leqslant x$ is a parameter to be chosen later.

Shifting the path of integration to the line $\sigma = \frac{7}{12} + \varepsilon$. By using Cauchy's residue theorem, we have

$$\sum_{n\leqslant x} a_K(n) f_\chi(n) = \frac{1}{2\pi \mathrm{i}}\Big\{\int_{\frac{7}{12}+\varepsilon-\mathrm{i}T}^{\frac{7}{12}+\varepsilon+\mathrm{i}T} + \int_{\frac{7}{12}+\varepsilon+\mathrm{i}T}^{b+\mathrm{i}T} + \int_{b-\mathrm{i}T}^{\frac{7}{12}+\varepsilon-\mathrm{i}T}\Big\} L_{K,f_\chi}(s)\frac{x^s}{s}\,\mathrm{d}s$$
$$+ \operatorname*{Res}_{s=1} L_{K,f_\chi}(s)\frac{x^s}{s} + O\Big(\frac{x^{1+\varepsilon}}{T}\Big)$$
$$=: I_1 + I_2 + I_3 + xP_4(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{22}$$

where $P_4(t)$ is a polynomial in t with degree 3.

Using the lemmas in Section 2 about the bound for the Dirichlet series, we will estimate I_i, $i = 1, 2, 3$, in the following.

For I_1, we have

$$I_1 \ll x^{\frac{7}{12}+\varepsilon} + x^{\frac{7}{12}+\varepsilon}\int_1^T \Big|M_{K,f_\chi}\Big(\frac{7}{12} + \varepsilon + \mathrm{i}t\Big)\Big| t^{-1}\mathrm{d}t, \tag{23}$$

where we have used that $U_1(s)$ is absolutely convergent in the region $\operatorname{Re} s \geqslant \frac{1}{2}+\varepsilon$ and behaves as $O(1)$ there.

By Hölder's inequality, (16), and (17), we have

$$\int_1^T \Big|M_{K,f_\chi}\Big(\frac{7}{12} + \varepsilon + \mathrm{i}t\Big)\Big| t^{-1}\mathrm{d}t \ll \log T \sup_{1\leqslant T_1\leqslant T} T_1^{-1}\cdot (T_1^{\frac{1}{6}+\varepsilon})^6 \ll T^\varepsilon. \tag{24}$$

Now, we can deduce that

$$I_1 \ll x^{\frac{7}{12}+\varepsilon} + x^{\frac{7}{12}+\varepsilon}T^\varepsilon. \tag{25}$$

For I_2 and I_3, we have

$$I_2 + I_3 \ll \sup_{\frac{7}{12}+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^\sigma T^{-1}|M_{K,f_\chi}(\sigma + \mathrm{i}T)|$$
$$\ll \sup_{\frac{7}{12}+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^\sigma T^{-1} T^{(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3})(1-\sigma)+\varepsilon}$$
$$\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{7}{12}+\varepsilon}T^{-\frac{1}{6}+\varepsilon}. \tag{26}$$

From (22), (25), and (26), we have

$$\sum_{n\leqslant x} a_K(n) f_\chi(n) = xP_4(\log x) + O(x^{\frac{7}{12}+\varepsilon}T^\varepsilon) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big). \tag{27}$$

Taking $T = x^{\frac{5}{12}+\varepsilon}$ in (27), we have

$$\sum_{n\leqslant x} a_K(n) f_\chi(n) = xP_4(\log x) + O(x^{\frac{7}{12}+\varepsilon}).$$

We complete the proof of Theorem 1. □

Proof of Theorem 2 Now, assume that K is a cubic non-normal extension over $\mathbb{Q}$. According to Lemma 3 and (18), we have

$$a_K(p)f_\chi(p) = 1 + \chi(p) + M(p) + \chi(p)M(p), \tag{28}$$

where p is a prime.

By virtue of (28), we have the relation

$$L_{K,f_\chi}(s) = \zeta(s)L(s,\chi)L(s,f)L(s,f\times\chi)\cdot U_2(s),$$

where $L(s,f\times\chi)$ is the Rankin-Selberg convolution L-function of $L(s,f)$ and $L(s,\chi)$, and $U_2(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > 1/2$. Therefore, the function $L_{K,f_\chi}(s)$ admits an analytic continuation into the half-plane $\sigma > 1/2$, having as its only singularity a pole of order 3 at $s = 1$, because $L(s,f\times\chi)$ has no poles at $s = 1$.

Similarly, as the proof of Theorem 1, by using Perron's formula, we have

$$\sum_{n\leqslant x} a_K(n)f_\chi(n) = \frac{1}{2\pi \mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} L_{K,f_\chi}(s)\frac{x^s}{s}\,\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where $b = 1+\varepsilon$ and $1 \leqslant T \leqslant x$ is a parameter to be chosen later.

Then we move the integration to the segment parallel with $\mathrm{Re}\, s = \frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\leqslant x} a_K(n)f_\chi(n) &= \frac{1}{2\pi \mathrm{i}}\Big\{\int_{\frac12+\varepsilon-\mathrm{i}T}^{\frac12+\varepsilon+\mathrm{i}T} + \int_{\frac12+\varepsilon+\mathrm{i}T}^{b+iT} + \int_{b-\mathrm{i}T}^{\frac12+\varepsilon-\mathrm{i}T}\Big\} L_{K,f_\chi}(s)\frac{x^s}{s}\,\mathrm{d}s\\ &\quad + \operatorname*{Res}_{s=1} L_{K,f_\chi}(s)\frac{x^s}{s} + O\Big(\frac{x^{1+\varepsilon}}{T}\Big)\\ &=: J_1 + J_2 + J_3 + xP_3(\log x) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),\end{aligned} \tag{29}$$

where $P_3(t)$ is a polynomial in t with degree 2.

Let

$$s_{1/2} = \frac{1}{2} + \varepsilon + \mathrm{i}t.$$

Then we have

$$\begin{aligned} J_1 &\ll x^{\frac12+\varepsilon} + x^{\frac12+\varepsilon}\int_1^T |\zeta(s_{1/2})L(s_{1/2},\chi)L(s_{1/2},f)L(s_{1/2},f\times\chi)|t^{-1}\mathrm{d}t\\ &\ll x^{\frac12+\varepsilon} + x^{\frac12+\varepsilon}\log T\cdot T_1^{-1}H_1(T_1)^{1/4}H_2(T_1)^{1/4}H_3(T_1)^{1/4}H_4(T_1)^{1/4}, \end{aligned} \tag{30}$$

where

$$H_1(T_1) = \int_{T_1}^{2T_1} \Big|\zeta\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t\Big)\Big|^4 \mathrm{d}t,$$

$$H_2(T_1) = \int_{T_1}^{2T_1} \Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi\Big)\Big|^4 \mathrm{d}t,$$

$$H_3(T_1) = \int_{T_1}^{2T_1} \Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,f\Big)\Big|^4 \mathrm{d}t,$$

$$H_4(T_1) = \int_{T_1}^{2T_1} \Big|L\Big(\frac{1}{2}+\varepsilon+\mathrm{i}t,f\times\chi\Big)\Big|^4 \mathrm{d}t.$$

By using (14), it is easily to get

$$H_3(T_1) \ll T_1^{\frac{3}{2}+\varepsilon}, \quad H_4(T_1) \ll T_1^{\frac{3}{2}+\varepsilon}.$$

So that we have

$$J_1 \ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon}T^{\frac{1}{4}+\varepsilon}. \tag{31}$$

For J_2 and J_3, let $s_\sigma = \sigma + \mathrm{i}T$. Then we have

$$\begin{aligned} J_2 + J_3 &\ll \sup_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^\sigma T^{-1}|\zeta(s_\sigma)L(s_\sigma,\chi)L(s_\sigma,f)L(s_\sigma,f\times\chi)| \\ &\ll \sup_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} x^\sigma T^{-1}T^{(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3})(1-\sigma)+\varepsilon} \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^\varepsilon. \end{aligned} \tag{32}$$

From (29), (31), and (32), we have

$$\sum_{n\leqslant x} a_K(n)f_\chi(n) = xP_3(\log x) + O(x^{\frac{1}{2}+\varepsilon}T^{\frac{1}{4}+\varepsilon}) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big). \tag{33}$$

Taking $T = x^{\frac{2}{5}+\varepsilon}$ in (33), we have

$$\sum_{n\leqslant x} a_K(n)f_\chi(n) = xP_3(\log x) + O(x^{\frac{3}{5}+\varepsilon}). \qquad \square$$

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References

1. Chandrasekharan K, Good A. On the number of integral ideals in Galois extensions. Monatsh Math, 1983, 95(2): 99–109

2. Deligne P, Serre J P. Formes modulaires de poids 1. Ann Sci Éc Norm Supér (4), 1975, 7: 507–530
3. Fomenko O M. Distribution of lattice points on surfaces of second order. J Math Sci, 1997, 83: 795–815
4. Fomenko O M. Mean values connected with the Dedekind zeta function. J Math Sci, 2008, 150(3): 2115–2122
5. Fröhlich A, Taylor M J. Algebraic Number Theory. Cambridge Stud Adv Math, Vol 27. Cambridge: Cambridge Univ Press, 1993
6. Good A. The square mean of Dirichlet series associated with cusp forms. Mathematika, 1982, 29(2): 278–295
7. Heath-Brown D R. The growth rate of the Dedekind zeta function on the critical line. Acta Arith, 1988, 49(4): 323–339
8. Ivić A. The Riemann Zeta-function. The Theory of the Riemann Zeta-function with Applications. New York: John Wiley and Sons, Inc, 1985
9. Iwaniec H, Kowalski E. Analytic Number Theory. Amer Math Soc Colloq Publ, Vol 53. Providence: Amer Math Soc, 1997
10. Jutila M. Lectures on a Method in the Theory of Exponential Sums. Tata Inst Fund Res Lectures on Math and Phys, Vol 80. Berlin; Springer, 1987
11. Kim H H. Functoriality and number of solutions of congruences. Acta Arith, 2007, 128(3): 235–243
12. Landau E. Einführung in die elementare and analytische Theorie der algebraischen Zahlen und der Ideale. Teubner, 1927
13. Lü G. Mean values connected with the Dedekind zeta-function of a non-normal cubic field. Cent Eur J Math, 2013, 11(2): 274–282
14. Lü G, Wang Y. Note on the number of integral ideals in Galois extension. Sci China Math, 2010, 53(9): 2417–2424
15. Lü G, Yang Z. The average behavior of the coefficients of Dedekind zeta function over square numbers. J Number Theory, 2011, 131: 1924–1938
16. Nowak W G. On the distribution of integral ideals in algebraic number theory fields. Math Nachr, 1993, 161: 59–74
17. Müller W. On the distribution of ideals in cubic number fields. Monatsh Math, 1988, 106(3): 211–219
18. Tenenbaum G. Introduction to Analytic and Probabilistic Number Theory. Cambridge Stud Adv Math, Vol 46. Cambridge: Cambridge Univ Press, 1995