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SURVEY ARTICLE

Four-manifolds with positive isotropic curvature

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Abstract We give a survey on 4-dimensional manifolds with positive isotropic curvature. We will introduce the work of B. L. Chen, S. H. Tang and X. P. Zhu on a complete classification theorem on compact four-manifolds with positive isotropic curvature (PIC). Then we review an application of the classification theorem, which is from Chen and Zhu's work. Finally, we discuss our recent result on the path-connectedness of the moduli spaces of Riemannian metrics with positive isotropic curvature.

Keywords Four-manifolds, positive isotropic curvature (PIC), Ricci flow
MSC 53C20, 53C44, 57M50

1 Positive isotropic curvature

Curvature is one of the most fundamental concepts in geometry and it dates back to the work of Gauss and Riemann. Given a manifold M, whether M admits some metrics with prescribed curvature restriction is a fundamental problem in Riemannian geometry. These curvature conditions may include positive scalar curvature, positive Ricci curvature, positive or negative sectional curvature, etc. In this paper, we will concentrate on the positive isotropic curvature (PIC) condition.

The notion of isotropic curvature was introduced by Micallef and Moore [26], it appears naturally in the second variation formula on the areas of surfaces.

Let $g(\cdot,\cdot)$ denote the Riemannian metric and its complex bilinear extension on the complexified tangent bundle $TM\otimes\mathbb{C}$, and we use $\langle\cdot,\cdot\rangle$ to denote the Hermitian extension of $g(\cdot,\cdot)$ on $TM\otimes\mathbb{C}$. Let $\mathscr{R}\colon \Lambda^2TM\to\Lambda^2TM$ be the curvature operator and also its complex linear extension to $\Lambda^2TM\otimes\mathbb{C}$. Suppose that a two-dimensional subspace $W\subset T_pM\otimes\mathbb{C}$ is spanned by a unitary basis

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$v, w \in T_pM \otimes \mathbb{C}$, and we call

$$K_{\mathbb{C}}(W) := \langle \mathscr{R}(v \wedge w), (v \wedge w) \rangle$$

the complex sectional curvature of W. A subspace $W \subset T_pM \otimes \mathbb{C}$ is isotropic if $g(w, w) = 0$ for all $w \in W$. The manifold (M, g) is said to have positive isotropic curvature if $K_{\mathbb{C}}(W) > 0$ whenever W is a two-dimensional isotropic subspace of $T_pM \otimes \mathbb{C}$ for every point $p \in M$.

By linear algebra, it is not hard to prove that, a two-dimensional subspace $W \subset T_pM \otimes \mathbb{C}$ is isotropic if and only if there exist

$$v = e_1 + \sqrt{-1}\, e_2, \quad w = e_3 + \sqrt{-1}\, e_4,$$

such that

$$W = \operatorname{span}\{v, w\},$$

where e_1, e_2, e_3, e_4 are orthonormal vectors. Thus, the isotropic curvature condition is non-vacuous only for $n \geqslant 4$. By expanding the formula for isotropic curvature, we obtain an alternative characterization of PIC: (M^n, g) has PIC if and only if

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$$

for any orthonormal four vectors e_1, e_2, e_3, e_4. Here,

$$R(X, Y, Z, W) = g(\mathscr{R}(X \wedge Y), Z \wedge W)$$

is the curvature tensor.

It is known that some classical curvature conditions such as strictly pointwise $\frac{1}{4}$-pinched sectional curvature and positive curvature operator imply PIC condition (see [26]). On the other hand, PIC implies positive scalar curvature (see [27]). The following diagram shows the relative strength of the positivity for various notions of curvatures:

$$\begin{array}{ccccc} & \text{pointwise } \frac{1}{4} \text{ pinching} & & \\ & \Downarrow & & \\ \mathscr{R} > 0 \Longrightarrow & K_{\mathbb{C}} > 0 & \Longrightarrow sec > 0 \Longrightarrow Ric > 0 \Longrightarrow R > 0 \\ & \Downarrow & & \\ & \text{PIC} & \Longrightarrow R > 0 \end{array}$$

Here, sec denotes the sectional curvature, Ric is the Ricci curvature, R is the scalar curvature on M, and the strictly pointwise $\frac{1}{4}$-pinched sectional curvature means that

$$0 < \max\{sec(\sigma)\} < 4 \min\{sec(\sigma)\}$$

holds for every $p \in M$ and 2-plane $\sigma \subset T_pM$.

Isotropic curvature plays a similar role for surface areas variation to that of sectional curvature for curve lengths. Combining Morse theory and variational theory, Micallef and Moore proved the following elegant theorem.

Theorem 1.1 (Micallef and Moore [26]) *Let M be a compact simply connected Riemannian manifold of dimension $n \geqslant 4$ with PIC. Then M is a homotopy sphere. In particular, M is homeomorphic to S^n.*

At present, we do not know whether a compact simply-connected manifold with PIC is diffeomorphic to S^n.

The basic examples of compact manifolds with PIC are the quotients of the spheres S^n, and the compact quotients of $S^{n-1} \times \mathbb{R}$. Furthermore, we know that the connected sum of these basic pieces also admits metrics with PIC. It is a corollary of the following theorem of Micallef and Wang.

Theorem 1.2 (Micallef and Wang [27]) *Let (M_1, g_1) and (M_2, g_2) be two manifolds of dimension $n \geqslant 4$ with PIC. Then the connected sum $M_1 \# M_2$ also admits a metric with PIC.*

From this result, we know that the fundamental group of a manifold with PIC could be very large. The following conjecture on the fundamental group of a compact Riemannian manifold with PIC was proposed by Gromov [13].

Conjecture 1.3 For $n \geqslant 4$, let M be an n-dimensional compact Riemannian manifold with PIC. Then the fundamental group of M contains a free subgroup of finite index.

The topology of non-simply connected manifolds with PIC is not fully understood. A partial result was obtained by [27] on second Betti numbers.

Theorem 1.4 (Micallef and Wang [27]) *Let M^{2n} be a closed even-dimensional Riemannian manifold with PIC. Then $b_2(M) = 0$.*

As a corollary, a Kähler manifold can never have PIC and also $\Sigma_g \times S^{2k}$, where Σ_g is a Riemann surface of genus $g \geqslant 2$, admits no metric with PIC.

However, a stronger conjecture on the topology of manifolds with PIC can be proposed (see Schoen [33]).

Conjecture 1.5 For $n \geqslant 4$, let M be an n-dimensional compact Riemannian manifold with PIC. Then a finite cover of M is diffeomorphic to a finite connected sum of $S^{n-1} \times S^1$.

For fundamental groups, Fraser has obtained an important result.

Theorem 1.6 (Fraser [12]) *Let M be a compact Riemannian manifold of dimension $n \geqslant 5$ with PIC. Then the fundamental group of M does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

The proof of Theorem 1.6 relies on a delicate study on stable minimal tori. It is difficult to generalize to high genus surfaces. Brendle and Schoen [2] extended Theorem 1.6 to the dimension $n = 4$ case provided that M is orientable.

The PIC condition is also closely related to Ricci flow. One of the most interesting thing is that Ricci flow preserves PIC condition.

Theorem 1.7 (Hamilton [18] (for $n = 4$); Brendle and Schoen [1]; Nguyen [28]) *Let M be a compact manifold of dimension $n \geqslant 4$, and let $g(t)$, $t \in [0, T)$, be a* *solution to the Ricci flow on M. If $(M, g(0))$ has PIC, then $(M, g(t))$ has PIC for all $t \in [0, T)$.*

The PIC condition plays a key role in Brendle-Schoen's proof of the $\frac{1}{4}$-pinched Differentiable Sphere Theorem, see [1]. Here, we will not go into the details of their works. In the rest of the paper, we will focus on dimension $n = 4$ and show that how Ricci flow can be applied to solve Conjectures 1.3 and 1.5 when $n = 4$.

Hamilton [18] initiated the classification for four-manifolds M with PIC without essential incompressible space forms. Here, no essential imcompressible space forms is a condition on the fundamental group, which means that any embedded three-dimensional spherical space form N with injective fundamental group $\pi_1(N)$ into $\pi_1(M)$ satisfies $\pi_1(N) = 0$ or $\pi_1(N) = \mathbb{Z}_2$, and we require that the normal bundle of N is non-orientable in the latter case. Hamilton's classification was completed in Chen-Zhu [8]. The complete classification theorem on compact four-manifolds with PIC was given by [7].

Theorem 1.8 (Chen et al. [7]) *Let M be a compact four-dimensional manifold. Then it admits a metric with PIC if and only if it is diffeomorphic to S^4, $\mathbb{R}\mathrm{P}^4$, $(S^3 \times \mathbb{R})/G$, or a connected sum of them. Here, G is a cocompact fixed-point-free discrete isometric subgroup of the standard $S^3 \times \mathbb{R}$.*

In Section 2, we will give a sketch of the proof of Theorem 1.8.

As an application of Theorem 1.8, Chen and Zhu [9] proved a conformally invariant classification theorem (see Theorem 3.3 below). We will review this result in Section 3. Recently, we ([6]) further investigated the moduli space of metrics with PIC on four-manifolds. We will discuss this recent work in Section 4.

There are other interesting results on manifolds with PIC in general dimensions which we cannot cover here, for example, see the survey articles [2,33].

2 Complete classification theorem on compact four-manifolds with PIC

2.1 PIC condition in four-dimension

Let (M, g) be a four-dimensional Riemannian manifold. The local orientation gives the bundle $\Lambda^2 TM$ a decomposition

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

into its self-dual and anti-self-dual parts. Therefore, the curvature operator has a block decomposition

$$\mathscr{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where

$$A = W_+ + \frac{R}{12} I, \quad C = W_- + \frac{R}{12} I,$$

and B is the traceless part of the Ricci curvature, $W_\pm$ are the self-dual and anti-self-dual Weyl tensors, respectively. Denote the eigenvalues of the matrices A, C, and $\sqrt{BB^t}$ by

$$a_1 \leqslant a_2 \leqslant a_3, \quad c_1 \leqslant c_2 \leqslant c_3, \quad b_1 \leqslant b_2 \leqslant b_3,$$

respectively. It is known that PIC is equivalent to $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$ (see [18]). From this, it is clear that if g is locally conformally flat, then g has positive scalar curvature if and only if g has PIC.

2.2 Notations

We fix some notations which will be used throughout the paper.

2.2.1 Orbifold

We will give some terminologies and notations about orbifolds (see [7]).

For $x \in X$, where X is an n-dimensional orbifold, we use Γ_x to denote the local uniformization group at x, that is, there is an open neighborhood $B_x \ni x$, such that B_x is diffeomorphic to $\mathbb{R}^n/\Gamma_x$, where Γ_x is a finite subgroup of linear transformations of $\mathbb{R}^n$. After conjugating with an element in $\mathrm{GL}(\mathbb{R}^n)$, we can assume $\Gamma_x \subset \mathrm{O}(n)$.

By Lefschetz fixed-point formula, every orientation-reversing diffeomorphism of S^3 has a fixed point (see [34]). Therefore, if X is a four-dimensional orbifold with at most isolated singularities, then, for every point $x \in X$, we have $\Gamma_x \subset \mathrm{SO}(4)$.

We will fix some notations of orbifolds that will appear in this paper (see also [6]).

Suppose that Γ is a fixed-point-free finite subgroup of SO(4) acting on S^3. We write the equation of S^4 as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1. \tag{2.1}$$

Regard S^3 as an equator

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

in S^4, Γ can naturally extend to act isometrically on S^4 by fixing the x_5-axis. We will still use Γ to denote this group action. Γ has exactly two fixed points $(0,0,0,0,1)$ and $(0,0,0,0,-1)$, and thus, the orbifold S^4/Γ has two orbifold singularities with local uniformization group Γ. In this paper, we say that a spherical orbifold is of the form S^4/Γ with $\Gamma \subset \mathrm{SO}(4)$, it means that it is diffeomorphic to an orbifold constructed above.

If S^3/Γ admits a fixed-point-free isometry τ satisfying $\tau^2 = 1$, then we can define an action $\hat{\tau}$ on $S^3/\Gamma \times \mathbb{R}$ by

$$\hat{\tau}(\theta, r) = (\tau(\theta), -r), \quad \theta \in S^3/\Gamma,\ r \in \mathbb{R}.$$

The quotient $(S^3/\Gamma \times \mathbb{R})/\{1, \hat{\tau}\}$ is a smooth four manifold with a neck-like end $S^3/\Gamma \times \mathbb{R}$. We denote this manifold by C_Γ^τ. If we think of S^4 as the compactification of $S^3 \times \mathbb{R}$ by adding two points at infinities of $S^3 \times \mathbb{R}$, then Γ and $\hat{\tau}$ can be naturally regarded as isometries of the standard S^4 . We denote $S^4/\langle\Gamma,\hat{\tau}\rangle$ the resulting orbifold in this paper. Obviously, C^τ_Γ is diffeomorphic to the smooth manifold obtained by removing the orbifold singularity (or a smooth point when Γ is trivial) from $S^4/\langle\Gamma,\hat{\tau}\rangle$. $\mathbb{R}\mathrm{P}^4\setminus\overline{B}^4$ is an example of C^τ_Γ.

In the following, we define topological necks and caps whose meanings will be fixed in our subsequent discussion. A neck is defined to be a manifold diffeomorphic to $S^3/\Gamma\times\mathbb{R}$. For caps, we have smooth caps and orbifold caps. Smooth caps consist of C^τ_Γ and B^4. Our orbifold caps have two types, denoted by C_Γ and C_{II} below.

The orbifold cap of Type I is diffeomorphic to $\mathbb{R}^4/\Gamma$, where $\Gamma\subset\mathrm{SO}(4)$ is a finite subgroup fixing the origin of $\mathbb{R}^4$ and acting freely on the unit three-sphere in $\mathbb{R}^4$. We denote it by C_Γ. C_Γ has a neck-like end $S^3/\Gamma\times\mathbb{R}$ and one orbifold singularity with local uniformization group Γ.

The orbifold cap of type II is constructed as follows. Let the equation of S^3 be $x_1^2+x_2^2+x_3^2+x_4^2=1$. The isometry

$$\gamma\colon (x_1,x_2,x_3,x_4)\to(x_1,-x_2,-x_3,-x_4)$$

has exactly two fixed points $p_1=(1,0,0,0)$ and $p_2=(-1,0,0,0)$, and satisfies $\gamma^2=1$. We define an action $\hat{\gamma}$ on $S^3\times\mathbb{R}$ by $\hat{\gamma}(x,r)=(\gamma(x),-r)$, where $x\in S^3$, $r\in\mathbb{R}$. It is clear $\hat{\gamma}^2=1$, and $(p_1,0),(p_2,0)$ are the only two fixed points of $\hat{\gamma}$. Denote the quotient orbifold $(S^3\times\mathbb{R})/\{1,\hat{\gamma}\}$ by C_{II}, and call it the orbifold cap of Type II. It has a neck-like end $S^3\times\mathbb{R}$ and two orbifold singularities with local uniformization group $\mathbb{Z}_2$. There is another way to understand C_{II}. Let the equation of S^4 be as (2.1). The isometry

$$\zeta\colon (x_1,x_2,x_3,x_4,x_5)\to(-x_1,-x_2,-x_3,-x_4,x_5)$$

has exactly two fixed points $(0,0,0,0,1)$ and $(0,0,0,0,-1)$. Here, we note that $S^4/\{1,\zeta\}$ is a special case of S^4/Γ with $\Gamma\subset\mathrm{SO}(4)$. Removing a smooth point from $S^4/\{1,\zeta\}$, we get an orbifold diffeomorphic to C_{II}.

Finally, we recall a characterization of four-dimensional spherical orbifolds with at most isolated singularities $S^4/\widetilde{\Gamma}$. By studying the corresponding group actions on S^4 (see [7, Lemmas 5.1, 5.2]), one can derive that such an $S^4/\widetilde{\Gamma}$ has no more than two orbifold singularities. So there are three possible diffeomorphism types of $S^4/\widetilde{\Gamma}$ according to the number of singularities: the first type is S^4 or $\mathbb{R}\mathrm{P}^4$, the second type is of the form $S^4/\langle\Gamma,\hat{\tau}\rangle$ with $\Gamma\subset\mathrm{SO}(4)$, and the third case is of the form S^4/Γ with $\Gamma\subset\mathrm{SO}(4)$.

2.2.2 Orbifold connected sum [7]

Suppose that X_1,X_2 are two n-dimensional orbifolds (not necessarily distinct) with at most isolated singularities. Let $x_1\in X_1$, $x_2\in X_2$ be two distinct points (not necessarily singular) such that Γ_{x_1} is conjugate to Γ_{x_2} as subgroups of $\mathrm{GL}(\mathbb{R}^n)$. By choosing new local trivializations, we may assume

$$\Gamma_{x_1}=\Gamma_{x_2}=:\Gamma\subset\mathrm{O}(n).$$

Let $B^n \subset \mathbb{R}^n$ be the unit open ball, $B_{x_1} \approx B^n/\Gamma$ and $B_{x_2} \approx B^n/\Gamma$ are neighborhoods of x_1 and x_2, respectively. Let f be a diffeomorphism from ∂B_{x_1} to ∂B_{x_2}. Remove B_{x_1} and B_{x_2} from X_1 and X_2, and identify the boundary ∂B_{x_1} and ∂B_{x_2} by the diffeomorphism f. The resulting orbifold is denoted by $\#_{f;x_1,x_2}(X_1, X_2)$ or $\#_f(X_1, X_2)$, and is called orbifold connected sum of X_1 and X_2. Note that the diffeomorphism type of the resulting orbifold depends only on the isotopic class of f. When the orientation is taken into account, we adopt the convention that the orientation of ∂B_{x_1} is induced from the orientation of X_1, while the orientation of ∂B_{x_2} is reverse to that induced from X_2.

Suppose that X is diffeomorphic to S^4/Γ, $\Gamma \subset \mathrm{SO}(4)$, with two orbifold singularities p_1 and p_2 (when Γ is trivial, we take p_1 and p_2 to be arbitrary two different smooth points). If we perform an orbifold connected sum on X with itself at p_1 and p_2 by $f \in \mathrm{Diff}(S^3/\Gamma)$, then we obtain the mapping torus of f, and denote it by $S^3/\Gamma \times_f S^1$. $S^3/\Gamma \times_f S^1$ has the structure of a fiber bundle over S^1 with fibers S^3/Γ and the monodromy f. It can be shown that the bundle structure depends only on the isotopic class of f.

2.3 Ricci flow

The Ricci flow equation

$$\frac{\partial g}{\partial t} = -2Ric_g$$

is an evolution equation on Riemannian metrics, where Ric_g is the Ricci curvature of g. This equation was introduced by Hamilton [16].

Let g_0 be a metric with PIC on a compact four-manifold or orbifold M. We evolve g_0 by the Ricci flow. The solution g_t exists for a short time (see [10,16,19]). Hamilton [18] proved that PIC is preserved on 4-d Ricci flow. Moreover, he derived the following improved pinching estimates. Since the maximum principle can still be applied on orbifolds, these estimates are still true for orbifolds.

Theorem 2.1 [18, Theorems B1.1, B2.3] *There exist positive constants ρ, $\Lambda, P < +\infty$ depending only on the initial metric, such that the solution to the Ricci flow satisfies*

$$\begin{gathered} a_1 + \rho > 0, \quad c_1 + \rho > 0, \\ \max\{a_3, b_3, c_3\} \leqslant \Lambda(a_1 + \rho), \quad \max\{a_3, b_3, c_3\} \leqslant \Lambda(c_1 + \rho), \\ \frac{b_3}{\sqrt{(a_1+\rho)(c_1+\rho)}} \leqslant 1 + \frac{\Lambda \mathrm{e}^{Pt}}{\max\{\log\sqrt{(a_1+\rho)(c_1+\rho)}, 2\}}. \end{gathered} \tag{2.2}$$

As a result, any blowing up limit will satisfy the following restricted isotropic curvature pinching condition:

$$a_3 \leqslant \Lambda a_1, \quad c_3 \leqslant \Lambda c_1, \quad b_3^2 \leqslant a_1 c_1. \tag{2.3}$$

We call the solution $g(t)$, $t \in [0, T)$, of Ricci flow is κ non-collapsed at $(x_0, t_0) \in M \times [0, T)$ on the scale r_0 if it satisfies the following: whenever

$|Rm|(x,t) \leqslant r_0^{-2}$ holds for all $t \in [t_0 - r_0^2, t_0]$ and $x \in B_t(x_0, r_0)$, we have

$$\mathrm{Vol}_{t_0}(B_{t_0}(x_0, r_0)) \geqslant \kappa r_0^4.$$

From the evolution equation of the scalar curvature

$$\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}|^2,$$

we have the differential inequality

$$\frac{\mathrm{d}R_{\min}(t)}{\mathrm{d}t} \geqslant \frac{1}{2} R_{\min}^2(t). \tag{2.4}$$

Since $a = R_{\min}(0) > 0$, we derive that $R_{\min}(t) \geqslant 2a/(2 - at)$, and thus, the solution will blow up at a finite time $t_1 \leqslant 2/a$.

2.4 Ancient solution

In order to understand the singularities of the Ricci flow, it is important to investigate the structures of ancient κ-orbifold solutions.

Definition 2.2 We say that a solution to the Ricci flow is an ancient κ-orbifold solution if it is a smooth complete nonflat solution to the Ricci flow on a four-orbifold X with at most isolated singularities satisfying the following three conditions:

(1) the solution exists on time interval $t \in (-\infty, 0]$,

(2) it has positive isotropic curvature and bounded curvature, and satisfies the restricted isotropic curvature pinching condition (2.3),

(3) it is κ-noncollapsed on all scales for some $\kappa > 0$.

The structure of ancient κ-orbifold solutions was thoroughly studied in [7, Section 3], we summarize the results in the following theorem.

Theorem 2.3 [7, Theorems 3.4–3.10] *For a four-dimensional ancient κ-orbifold solution (X, g_t), $t \in (-\infty, 0]$,*

(1) *if the curvature operator has nontrivial null eigenvectors somewhere, then X is isometric to $S^3/\Gamma \times \mathbb{R}$, C_Γ^τ, or C_{II}, with the induced metric from the product metric on $S^3 \times \mathbb{R}$;*

(2) *if the curvature operator is strictly positive everywhere, then either X is compact and diffeomorphic to a spherical space form S^4/Γ with at most isolated singularities, or X is noncompact and diffeomorphic to $\mathbb{R}^4$ or C_Γ.*

Furthermore, for $\varepsilon > 0$ small enough, one can find positive constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$, such that for every (x,t), there is a radius r,

$$\frac{1}{C_1}(R(x,t))^{-1/2} < r < C_1(R(x,t))^{-1/2},$$

so that some open neighborhood $B_t(x, r) \subset B \subset B_t(x, 2r)$ falls into one of the following categories:

(i) *B is an evolving ε-neck around (x,t) (in the sense that it is the time slice at time t of the parabolic region $\{(x',t') \mid x' \in B,\ t' \in [t-R(x,t)^{-1}, t]\}$, where the solution is well defined on the whole parabolic neighborhood and is, after scaling with factor $R(x,t)$ and shifting the time t to zero, ε-close to the corresponding subset of the evolving round cylinder $S^3/\Gamma \times \mathbb{R}$ with scalar curvature 1 at the time zero);*

(ii) *B is an evolving ε-cap (in the sense that it is the time slice at the time t of an evolving metric on open caps $\mathbb{R}^4$, C_Γ^τ, C_Γ, or C_{II} such that the region outside some suitable compact subset is an evolving ε-neck);*

(iii) *at time t, x is contained in a connected compact component with positive curvature operator.*

Moreover, the scalar curvature of B in cases (i) *and* (ii) *at time t is between $C_2^{-1}R(x,t)$ and $C_2R(x,t)$.*

We remark that conclusions in Theorem 2.3 in smooth manifold case were proved by Chen and Zhu [8]. In the orbifold case, due to the possible collapsing of the solution in the presence of orbifold singularities with big local uniformization groups, some analysis in the proof of the smooth case cannot go through directly. For example, if we follow the argument in [8] directly, the constants C_1, C_2 may depend on the noncollapsing constant κ. The idea to solve this problem in [7] is to lift the ancient κ-orbifold solution to its universal cover so that we can make use of the results in the manifold case. See [7, Section 3] for more details.

2.5 Hamilton's surgery process

Hamilton [18] initiated the surgery process to handle the higher curvature part of Ricci flow.

Suppose that h is a metric on $N = S^3 \times (-4,4)$ such that h is ε-close to h_{std} in $C^{[1/\varepsilon]}$-topology, where h_{std} is the standard round cylinder metric on N of scalar curvature 1. Denote the coordinate of the second factor by s. Let f be a smooth function defined by

$$f(s) = \begin{cases} 0, & s \leqslant 0, \\ c\mathrm{e}^{-q/s}, & s > 0, \end{cases} \tag{2.5}$$

where $c, q > 0$. Hamilton [18, Section D 3.1] showed that if c is small enough and q large enough (independent of ε), then the metric $\hat{h} = \mathrm{e}^{-2f}h$ satisfies Hamilton's improved pinching estimates (see Theorem 2.1) on $s \in [0,4]$, and has positive curvature operator for $s \in (1,4]$, if h is ε-close to h_{std} with ε sufficiently small. And we will fix such a small c and a large q. Now, let $\alpha\colon \mathbb{R} \to [0,1]$ be a fixed smooth cutoff function such that $\alpha(s) \equiv 1$ if $s \leqslant 2$ and $\alpha(s) \equiv 0$ if $s \geqslant 3$. Then there exists a universal constant ε_1 such that, if h is ε-close to h_{std} with $\varepsilon < \varepsilon_1$, then the metric

$$\check{h} = \mathrm{e}^{-2f}[\alpha(s)h + (1-\alpha(s))h_{\mathrm{std}}]$$

satisfies Hamilton's improved pinching estimates on $s \in [0,4]$ and has positive curvature operator on $s \in (1,4]$. From the above construction, $\check{h}$ on the part $s \in [3,4]$ is independent of h, and then we extend $\check{h}$ by gluing a fixed suitably chosen rotationally symmetric cap with positive curvature operator and satisfying Hamilton's improved pinching estimates. We denote the resulting Riemaninan manifold by $(\mathscr{S}, h_{\rm surg})$. Note that in this case, $\mathscr{S}$ is diffeomorphic to $\mathbb{R}^4$.

When N is $S^3/\Gamma \times (-4,4)$ with nontrivial Γ, and h is a metric close to $h_{\rm std}$ on N, the above construction can also be applied. But in this case, we will obtain a Riemannian orbifold $(\mathscr{S}, h_{\rm surg})$ with $\mathscr{S}$ diffeomorphic to C_Γ.

We remark that no essential imcompressible space forms condition in [18] and [8] prevents the appearance of orbifold singularities. In general, if there is no any topological assumption on the initial manifold M, isolated orbifold singularities may appear after performing surgeries.

2.6 Ricci flow with surgery and proof of Theorem 1.8

The existence of Ricci flow with surgery on four-dimensional orbifold with isolated singularities and with PIC metric was established in [7, Section 4]. We summarize the results in the following theorem.

Theorem 2.4 *Let g_0 be a PIC metric on M (M may be a four-dimensional manifold or orbifold with isolated singularities). There exist two sequences of non-increasing small positive numbers $\{r_i\}$, $\{\delta_i\}$, and a Ricci flow with surgery on orbifolds with at most isolated singularities $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$, $0 \leqslant i \leqslant p$, such that*

1) *$X_0 = M$ and $g_0(0) = g_0$;*

2) *the flow becomes extinct at a finite time $T = t_{p+1}$;*

3) *for every $0 \leqslant i \leqslant p$, the flow $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$ satisfies the ε-canonical neighborhood assumption with parameter r_i and the pinching assumption*;

4) *for every $0 \leqslant i \leqslant p-1$, $(X_{i+1}, g_{i+1}(t_{i+1}))$ is obtained from $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$ by doing surgery at singular time t_{i+1} with parameters r_i and δ_i.*

In [8, Section 5], Theorem 2.4 was established for smooth manifolds under suitable topological assumptions, while in [7, Section 4], it was established in a general form. We remark that in the general case, due to the possible existence of orbifold singularities with big uniformizaton group, the canonical neighborhoods may be sufficiently collapsed, which bring difficulties to establish the long time existence of Ricci flow with surgery. Thus, there need some work to overcome this problem, see [7] for details. We will not include the very detailed and lengthy argument here, except explain the terminologies appear in Theorem 2.4.

Definition 2.5 Let $\varepsilon > 0$ be a small constant, and let $r\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function. We say that a solution to the Ricci flow with surgery $g(t)$, $t \in [0,b]$, satisfies ε-canonical neighborhood assumption with parameter

r, if there exist two constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$ depending only on ε such that every point $(x,t) \in M \times [0,b]$ with $R(x,t) \geqslant r(t)^{-2}$ has an open neighborhood B, called an ε-canonical neighborhood, satisfying the properties that $B_t(x,r) \subset B \subset B_t(x,2r)$ with $0 < r < C_1 R(x,t)^{-1/2}$, and one of the following conditions:

(a) B is an ε-neck around (x,t);

(b) B is an ε-cap $(\mathscr{C}, g)$, and thus, B is diffeomorphic to B^4, C_Γ^τ, C_Γ, or C_{II} such that the region outside some suitable compact subset is an ε-neck;

(c) at time t, x lies in a compact connected component with positive curvature operator.

Moreover, for cases (a) and (b), the scalar curvature in B at time t is between $C_2^{-1}R(x,t)$ and $C_2 R(x,t)$, and satisfies the estimate

$$|\nabla R| < \eta R^{3/2}, \quad \left|\frac{\partial R}{\partial t}\right| < \eta R^2,$$

where η is a universal constant.

At each singular time t_{i+1}, for those components of X_i with positive curvature operator at time t_{i+1}, we know that these components are spherical by Hamilton's result (see [18]). Denote the union of the remaining components of X_i by Ω. Let

$$\rho = \delta_i r_i, \quad \Omega_\rho = \left\{x \in \Omega \mid \lim_{t \to t_{i+1}} R(x,t) \leqslant \rho^{-2}\right\}.$$

Then every point of Ω outside Ω_ρ has an ε-neck or ε-cap neighborhood by the canonical neighborhood assumption with parameter r_i. When $\Omega_\rho \neq \emptyset$, there are finitely many connected components of $\Omega \setminus \Omega_\rho$ whose one end is in Ω_ρ, another end has unbounded curvature. These components are called ε-horns and we denote them by $H_j, 1 \leqslant j \leqslant k$. Each of these components is diffeomorphic to $S^3/\Gamma \times (0,1)$ for some $\Gamma \subset \mathrm{SO}(4)$. By [7, Proposition 4.4], there exists $h \in (0, \delta_i \rho_i)$ such that every point on an ε-horn with curvature $\geqslant h^{-2}$ is a center of a δ_i-neck. For every j, we select such an x_j on H_j with curvature $\geqslant h^{-2}$, and x_j is a center of a δ_i-neck N_j, and denote the center slice of N_j by S_j. Let $\widetilde{\Omega}$ be the union of the connected components of $\Omega \setminus \cup S_j$ with finite curvature at t_{i+1}, and denote $\hat{\Omega} = \Omega \backslash \overline{\widetilde{\Omega}}$. Now, we cut off the δ_i-neck N_j along S_j, and glue back surgery caps to the boundaries of $\widetilde{\Omega}$. The resulting Riemannian orbifold is just $(X_{i+1}, g_{i+1}(t_{i+1}))$, which have bounded curvature and the Ricci flow can be resumed until it hits another singular time. This is what we mean by doing surgery at singular time t_{i+1} with parameters r_i and δ_i.

As the surgeries are done at the points lying deeply in the ε-horns, the minimum of the scalar curvature $R_{\min}(t)$ of the solution to the Ricci flow with surgery at each time-slice is achieved in the region unaffected by the surgeries. Then, from (2.4), we know that there will be a $t_{p+1} = T < +\infty$ such that every point is covered by a canonical neighborhood. In this case, every connected component of X_p can be well characterized, thus we stop the Ricci flow and say that it is extinct at time T.

With Theorem 2.4 in hand, we can prove Theorem 1.8. The proof is by backward induction.

Because at the extinct time t_{p+1}, every point is covered by a canonical neighborhood, we know that every connected component of X_p is diffeomorphic to a spherical orbifold, or an $(S^3 \times \mathbb{R})/G$, or $\#(S^4/\{1,\zeta\}, S^4/\{1,\zeta\})$, or $\#(S^4/\{1,\zeta\}, \mathbb{R}\mathrm{P}^4)$, where the connected sum is performed at smooth points. Topologically, these connected components can be obtained by performing suitable orbifold connected sum on a finite number four-dimensional spherical orbifolds $S^4/\widetilde{\Gamma}$.

At every surgery time t_i, $i < p+1$, we can recover the topology of X_{i-1} from that of X_i, because X_i is obtained from X_{i-1} by first throwing away those component with positive curvature operator and then performing surgery, and the canonical neighborhood assumptions enable us to know the topological types of the higher curvature part. By backward inductions, we can argue that each connected component of X_{i-1} is a suitable orbifold connected sum of a finite number four-dimensional spherical orbifolds.

Since there are no orbifold singularities on the initial manifold M, by the characterization of four-dimensional spherical orbifolds with isolated singularities (see [7, Lemmas 5.1, 5.2]), one can show that the conclusion of Theorem 1.8 hold. See [7] for details.

3 A conformally invariant classification theorem

Suppose that M^n is a closed manifold of dimension n with $n \geqslant 3$. Given a metric g on M^n, let $\mathscr{C}_g = \{\rho g \mid \rho > 0\}$ be the class of metrics conformal to g.

Define

$$\mathscr{Y}(M^n, \mathscr{C}_g) = \inf_{g' \in \mathscr{C}_g} \frac{\int_{M^n} R_{g'} \mathrm{d}v_{g'}}{\left(\int_{M^n} \mathrm{d}v_{g'}\right)^{(n-2)/n}},$$

and the Yamabe invariant $\mathscr{Y}(M^n)$ of the manifold is defined to be

$$\mathscr{Y}(M^n) = \sup_{\mathscr{C}} \mathscr{Y}(M^n, \mathscr{C}),$$

where the superum is taken over all conformal classes of Riemannian metrics on M^n.

An interesting question is to classify manifolds with positive Yamabe invariant. But it turns out that when dimension $n \geqslant 4$, the Yamabe invariant alone is too weak to control the whole topology of manifolds. One needs additional conditions to investigate the topology of the manifolds with positive Yamabe invariant.

In dimension 4, recall that the Gauss-Bonnet-Chern Theorem says

$$\frac{1}{8\pi^2}\int_{M^4}\Big(|W_+|^2+|W_-|^2+\frac{R^2}{24}-\frac{|\overset{\circ}{Ric}|^2}{2}\Big)\mathrm{d}v_g=\chi(M^4), \tag{3.1}$$

where $W_\pm$ are the self-dual and anti-self-dual Weyl tensors, respectively, see Section 2.1.

Chang et al. [5] proved a conformally invariant sphere theorem in dimension 4, where besides the positivity of the Yamabe invariant, they assumed that the Weyl curvature is suitably controlled in L^2 sense by the Euler characteristic $\chi(M^4)$ of the manifold.

Theorem 3.1 (Chang, Gursky, and Yang [5]) *Let (M^4,g) be a compact four-dimensional Riemannian manifold. Suppose that we have*

(1) $\mathscr{Y}(M^4,\mathscr{C}_g)>0$;

(2) $\int_{M^4}\big(|W_+|^2+|W_-|^2\big)\mathrm{d}v_g<4\pi^2\chi(M^4)$.

Then M^4 is diffeomorphic to S^4 or $\mathbb{R}\mathrm{P}^4$.

Note that conditions (1) and (2) are invariant under conformal change of the metric.

In [5], the authors also obtained the following rigidity theorem that shows the pinching condition (2) is sharp.

Theorem 3.2 (Chang, Gursky, and Yang [5]) *Let (M^4,g) be a compact four-dimensional Riemannian manifold which is not diffeomorphic to S^4 or $\mathbb{R}\mathrm{P}^4$. Suppose that we have*

(1) $\mathscr{Y}(M^4,\mathscr{C}_g)>0$;

(2) $\int_{M^4}(|W_+|^2+|W_-|^2)\mathrm{d}v_g=4\pi^2\chi(M^4)$.

Then

(a) *(M^4,g) is conformal to $\mathbb{C}\mathrm{P}^2$ with the Fubini-Study metric; or*

(b) *(M^4,g) is conformal to a manifold which is isometrically covered by $S^3\times S^1$ endowed with the standard product metric.*

It is obvious that $\chi(M^4)$ is positive in Theorem 3.1 and is non-negative in Theorem 3.2.

3.1 Conformally invariant classification theorems

Chen and Zhu [9] generalized the above sharp conformally invariant sphere theorems to manifolds with possibly non-positive Euler characteristic as follows.

Theorem 3.3 (Chen and Zhu [9]) *Let (M^4,g) be a compact four-dimensional Riemannian manifold satisfying*

(1) $\mathscr{Y}(M^4,\mathscr{C}_g)>0$;

(2)

$$\int_{M^4}[\max\{\lambda_{\max}(W_+),\lambda_{\max}(W_-)\}]^2\mathrm{d}v_g<\frac{1}{36}\mathscr{Y}(M^4,\mathscr{C}_g)^2,$$

where $\lambda_{\max}(W_\pm)$ *is the largest eigenvalue of* $W_\pm$, *respectively. Then* M^4 *is diffeomorphic to a connected sum*

$$S^4\#m\mathbb{R}\mathrm{P}^4\#(S^3\times\mathbb{R})/\Gamma_1\#\cdots\#(S^3\times\mathbb{R})/\Gamma_k,$$

where k *is a non-negative integer,* $m=0$ *or* 1, *and each* Γ_i *is a cocompact discrete subgroup of the isometric group of* $S^3\times\mathbb{R}$.

Theorem 3.4 (Chen and Zhu [9]) *Let* (M^4,g) *be a compact four-dimensional Riemannian manifold satisfying*

(1) $\mathscr{Y}(M^4,\mathscr{C}_g)>0$;

(2) $\int_{M^4}[\max\{\lambda_{\max}(W_+),\lambda_{\max}(W_-)\}]^2\mathrm{d}v_g=\frac{1}{36}\mathscr{Y}(M^4,\mathscr{C}_g)^2$.

If M^4 *is not diffeomorphic to*

$$S^4\#m\mathbb{R}\mathrm{P}^4\#(S^3\times\mathbb{R})/\Gamma_1\#\cdots\#(S^3\times\mathbb{R})/\Gamma_k$$

for all $m=0,1$ *and non-negative integer* k, *then*

(a) (M^4,g) *is conformal to* $\mathbb{C}\mathrm{P}^2$ *with the Fubini-Study metric*; *or*

(b) *the universal cover of* (M^4,g) *is conformal to* $(\Sigma_1,g_1)\times(\Sigma_2,g_2)$, *where the surface* (Σ_i,g_i) *has constant Gaussian curvature* k_i, *and* $k_1+k_2>0$.

Clearly, conditions (1) and (2) in Theorem 3.3 are conformally invariant. We now show that Theorem 3.1 can be deduced from Theorem 3.3.

By Schoen's solution of Yamabe problem [32], there is a metric $\widetilde{g}\in\mathscr{C}_g$ such that $\widetilde{g}$ has constant scalar curvature and

$$\mathscr{Y}(M^4,\mathscr{C}_g)=\frac{\int_{M^4}R_{\widetilde{g}}\mathrm{d}v_{\widetilde{g}}}{\big(\int_{M^4}\mathrm{d}v_{\widetilde{g}}\big)^{1/2}}.$$

Note that, by (3.1), condition (2) in Theorem 3.1 is equivalent to

$$\int_{M^4}(|W_+|^2+|W_-|^2)\mathrm{d}v_g<\int_{M^4}\Big(\frac{R^2}{24}-\frac{|\overset{\circ}{Ric}|^2}{2}\Big)\mathrm{d}v_g. \tag{3.2}$$

Since the LHS of (3.2) is conformally invariant, we know the RHS of (3.2) is also conformally invariant. By (3.2), we have

$$\begin{aligned}\int_{M^4}(|W_+|^2+|W_-|^2)\mathrm{d}v_g&<\int_{M^4}\Big(\frac{R_{\widetilde{g}}^2}{24}-\frac{|\overset{\circ}{Ric}_{\widetilde{g}}|_{\widetilde{g}}^2}{2}\Big)\mathrm{d}v_{\widetilde{g}}\\&\leqslant\int_{M^4}\frac{R_{\widetilde{g}}^2}{24}\,\mathrm{d}v_{\widetilde{g}}\\&=\frac{\mathscr{Y}(M^4,\mathscr{C}_g)^2}{24}.\end{aligned} \tag{3.3}$$

On the other hand, let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ be the eigenvalues of W_+. Since W_+ is trace free, we have $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and thus,

$$|W_+|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geqslant \lambda_1^2 + \frac{1}{2}(\lambda_2 + \lambda_3)^2 = \frac{3}{2}|\lambda_{\max}(W_+)|^2. \tag{3.4}$$

Similarly, we have

$$|W_-|^2 \geqslant \frac{3}{2}|\lambda_{\max}(W_-)|^2. \tag{3.5}$$

Combining (3.4), (3.5) with (3.3), we have

$$\int_{M^4}(|\lambda_{\max}(W_+)|^2 + |\lambda_{\max}(W_-)|^2)\mathrm{d}v_g < \frac{\mathscr{Y}(M^4, \mathscr{C}_g)^2}{36}, \tag{3.6}$$

which clearly implies condition (2) in Theorem 3.3. Thus, it follows from the conclusion of Theorem 3.3 that M^4 has a finite cover $\widetilde{M}^4$ diffeomorphic to $S^4\#k(S^3 \times S^1)$, where

$$k = b_1(\widetilde{M}^4), \quad b_2(\widetilde{M}^4) = 0.$$

Obviously, condition (2) in Theorem 3.1 implies that $\chi(\widetilde{M}^4) > 0$. Noting that $\chi(\widetilde{M}^4) = 2 - 2k$, we have $k = 0$. Thus, $\widetilde{M}^4$ is diffeomorphic to S^4 and M^4 is diffeomorphic to S^4 or $\mathbb{R}\mathrm{P}^4$. Hence, we have proved that Theorem 3.1 can be deduced from Theorem 3.3.

Furthermore, it follows from Theorem 3.3 that if $\chi(M^4) = 0$, then M^4 has a finite cover diffeomorphic to $S^3 \times S^1$, and if $k \geqslant 2$, then we have $\chi(M^4) < 0$. Thus, Theorem 3.3 generalizes Theorem 3.1 to manifolds with negative Euler characteristic numbers.

3.2 Proofs of Theorems 3.3 and 3.4

We give a sketch on the proofs of Theorems 3.3 and 3.4 in this subsection.

Analogous to the Yamabe invariant, Chen and Zhu [9] defined a conformal invariant $\mathscr{GY}(M^4, \mathscr{C}_g)$ and a differentiable invariant $\mathscr{GY}(M^4)$ on the four-manifold M^4 as follows:

$$\mathscr{GY}(M^4, \mathscr{C}_g) := \inf_{g' \in \mathscr{C}_g} \frac{\int_{M^4}(R_{g'} - 6\max\{\lambda_{\max}(W_+^{g'}), \lambda_{\max}(W_-^{g'})\})\mathrm{d}v_{g'}}{(\int_{M^4}\mathrm{d}v_{g'})^{1/2}},$$

$$\mathscr{GY}(M^4) := \sup_{\mathscr{C}} \mathscr{GY}(M^4, \mathscr{C}).$$

Then the following theorem was established.

Theorem 3.5 [9, Theorem 1.7] *Let M^4 be a compact four-dimensional manifold with $\mathscr{GY}(M^4) > 0$. Then M^4 is diffeomorphic to a connected sum*

$$S^4\#m\mathbb{R}\mathrm{P}^4\#(S^3 \times \mathbb{R})/\Gamma_1\#\cdots\#(S^3 \times \mathbb{R})/\Gamma_k,$$

where k is a non-negative integer, $m = 0$ or 1, and each Γ_i is a cocompact discrete subgroup of the isometric group of $S^3 \times \mathbb{R}$.

Now, we give a sketch of the proof of Theorem 3.5. A key step is to transform the integral condition $\mathscr{GY}(M^4) > 0$ to a pointwise curvature condition by solving a semi-linear elliptic equation.

If we define

$$\sigma_g := R_g - 6\max\{\lambda_{\max}(W_+), \lambda_{\max}(W_-)\}, \tag{3.7}$$

where $\lambda_{\max}(W_\pm)$ are the largest eigenvalue of $W_\pm$, respectively. Since both W_+ and W_- are trace free, it is easy to see that the condition $\sigma_g > 0$ is equivalent to $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$, that is to say, (M^4, g) has PIC (see Section 2.1).

Let $\widetilde{g} = u^2 g$, $u \in C^\infty(M^4)$, $u > 0$. By direct computations, we obtain the following relations:

$$R_{\widetilde{g}} = u^{-3}(-6\Delta u + R_g u), \tag{3.8}$$

$$\sigma_{\widetilde{g}} = u^{-3}(-6\Delta u + \sigma_g u). \tag{3.9}$$

Proposition 3.6 *For any given conformal class $\mathscr{C}_g$ of Riemannian metrics on M^4, $\mathscr{GY}(M^4, \mathscr{C}_g)$ can be achieved by some $\widetilde{g} \in \mathscr{C}_g$ such that $\sigma_{\widetilde{g}}$ is a constant.*

We have the following corollary.

Corollary 3.7 *If $\mathscr{GY}(M^4, \mathscr{C}_g) > 0$, then there exists $\widetilde{g} \in \mathscr{C}_g$ such that $\sigma_{\widetilde{g}} > 0$.*

Indeed, Proposition 3.6 and Corollary 3.7 are some kind of generalized Yamabe problem. They can be stated in a more general form, see [9, Lemma 2.1, Corollary 2.2] and [15, Proposition 3]. The proof of Proposition 3.6 is similar to that of the Yamabe problem.

Since the metric $\widetilde{g}$ in Corollary 3.7 has PIC, we can apply the classification theorem (Theorem 1.8) and finish the proof of Theorem 3.5.

With Theorem 3.5 in hand, Theorem 3.3 can actually be proved by verifying $\mathscr{GY}(M^4) > 0$.

Proof of Theorem 3.3 By Proposition 3.6, there is a metric $\widetilde{g}$ of unit volume in $\mathscr{C}_g$ achieving $\mathscr{GY}(M^4, \mathscr{C}_g)$ and $\sigma_{\widetilde{g}} \equiv \mathscr{GY}(M^4, \mathscr{C}_g)$. Hence, we have

$$\begin{aligned}
\mathscr{GY}(M^4, \mathscr{C}_g) &= \int_{M^4} (R_{\widetilde{g}} - 6\max\{\lambda_{\max}(W_+^{\widetilde{g}}), \lambda_{\max}(W_-^{\widetilde{g}})\}) \mathrm{d}v_{\widetilde{g}} \\
&\overset{\text{(i)}}{\geqslant} \mathscr{Y}(M^4, \mathscr{C}_g) - 6\Big(\int_{M^4} [\max\{\lambda_{\max}(W_+^{\widetilde{g}}), \lambda_{\max}(W_-^{\widetilde{g}})\}]^2 \mathrm{d}v_{\widetilde{g}}\Big)^{1/2} \\
&\overset{\text{(ii)}}{=} \mathscr{Y}(M^4, \mathscr{C}_g) - 6\Big(\int_{M^4} [\max\{\lambda_{\max}(W_+), \lambda_{\max}(W_-)\}]^2 \mathrm{d}v_g\Big)^{1/2} \\
&\overset{\text{(iii)}}{>} 0,
\end{aligned} \tag{3.10}$$

where we have used the Schwarz inequality in (i), conformal invariance of $W_\pm$ in (ii), and assumptions (1) and (2) of Theorem 3.3 in (iii). The conclusion of Theorem 3.3 follows from Theorem 3.5. □

For the proof of Theorem 3.4, we first mimic the above argument, but we have '=' in (iii) of (3.10) instead of '>'. Also, by the assumptions of Theorem 3.4, we know that (i) of (3.10) must be an equality, which implies that $\widetilde{g}$ satisfies

$$R_{\widetilde{g}} = 6\max\{\lambda_{\max}(W_+^{\widetilde{g}}), \lambda_{\max}(W_-^{\widetilde{g}})\} \equiv \text{const.} > 0.$$

Note that $(M^4, \widetilde{g})$ has non-negative isotropic curvature. Then we can apply the results of Micallef-Wang [27]. See [9] for details.

4 Moduli spaces of metrics with PIC

As is known, the existence of certain Riemannian metrics with various curvature conditions is an important problem in geometry. On the other hand, when the manifold M admits such a metric, it will be a very interesting problem to investigate the topology of the moduli space of such metrics.

It is easy to see that on a given manifold M, the space of all Riemennian metrics, equipped with the C^∞-topology, is star-shaped, hence contractible. To the contrary, when we are restricted to the subspace of metrics with certain curvature conditions, the topology of this space or its corresponding moduli space may become difficult to study.

There are many works studying the moduli space of metrics with certain curvature restrictions. Let us denote the set of Riemannian metrics g with positive scalar curvature by $\mathscr{R}_+(M)$. The group of diffeomorphisms on M, denoted by $\mathrm{Diff}(M)$, acts on $\mathscr{R}_+(M)$ naturally. In 1916, Weyl [36] proved that $\mathscr{R}_+(S^2)$ is path-connected. Rosenberg and Stolz [30] further showed that $\mathscr{R}_+(S^2)$ is contractible. When dimension $n \geqslant 7$, there are many examples with disconnected $\mathscr{R}_+(M^n)$ or even the moduli spaces $\mathscr{R}_+(M^n)/\mathrm{Diff}(M^n)$, see [3,14,20,22,29], etc. When dimension $n = 3$, Marques [24] proved recently that the moduli space $\mathscr{R}_+(M)/\mathrm{Diff}(M)$ is path-connected if M is compact orientable with $\mathscr{R}_+(M) \neq \emptyset$. Combining the result of Cerf [4] on $\mathrm{Diff}(S^3)$, Marques [24] further argued that $\mathscr{R}_+(S^3)$ is path-connected. In dimension 4, not much is known about the moduli space $\mathscr{R}_+(M)/\mathrm{Diff}(M)$. There is an example of a 4-manifold, due to Ruberman [31], for which the moduli space $\mathscr{R}_+(M)/\mathrm{Diff}(M)$ is disconnected. For the four-sphere S^4, we remark that whether the moduli space $\mathscr{R}_+(S^4)/\mathrm{Diff}(S^4)$ is connected is still an open problem. There are many other interesting results about the connectedness or disconnectedness of moduli spaces of metrics satisfying certain geometric conditions, see [11,21,23], etc.

The topological classification problem for compact four-manifolds with PIC is completed by Theorem 1.8. Thus, it is a natural question to study the space of metrics g with PIC, denoted by $\mathrm{PIC}(M)$, and its corresponding moduli space $\mathrm{PIC}(M)/\mathrm{Diff}(M)$. In this section, we introduce our recent work [6] on the path-connectedness of $\mathrm{PIC}(M)/\mathrm{Diff}(M)$. The main theorem in [6] is as follows.

Theorem 4.1 (Chen and Huang [6]) *The moduli space* $\mathrm{PIC}(M)/\mathrm{Diff}(M)$ *is path-connected if M is orientable and diffeomorphic to one of the following* *manifolds*:

(1) S^4;

(2) $(S^3 \times \mathbb{R})/G$, *where* G *is a cocompact fixed-point-free discrete isometric subgroup of* $S^3 \times \mathbb{R}$;

(3) *a finite connected sum* $(S^3/\Gamma_1 \times S^1)\# \cdots \#(S^3/\Gamma_k \times S^1)$, *where* Γ_i $(1 \leqslant i \leqslant k)$ *is either the trivial group or a non-cyclic isometric group of* S^3.

Let g and g' be metrics with PIC. We say that g is isotopic to g' if there exists a continuous path g_μ, $\mu \in [0,1]$, such that $g_0 = g$, $g_1 = g'$, and $g_\mu \in \mathrm{PIC}(M)$ for every $\mu \in [0,1]$. The path-connectedness of $\mathrm{PIC}(M)/\mathrm{Diff}(M)$ just means that for any two $g_1, g_2 \in \mathrm{PIC}(M)$, there is a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ such that g_1 is isotopic to $\varphi^* g_2$.

The proof of Theorem 4.1 is mainly using Ricci flow. The idea is to use Ricci flow to deform the initial metric $g_0 = g$. By Theorem 1.7, once the solution g_t is well defined, it gives a curve in $\mathrm{PIC}(M)$. By Theorem 2.4, Ricci flow with surgery is well-defined: at every surgery time t_{i+1}, we obtain $(X_{i+1}, g_{i+1}(t_{i+1}))$ from $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$ by doing surgery at singular time t_{i+1} with parameters r_i, δ_i, and the flow becomes extinct at a finite time $T = t_{p+1}$. Here, we remark that, after doing surgery, the underling orbifold will change, and the behavior of metrics on the parts cut or glued in the surgery process will also bring difficulties in our analysis. This requires more careful investigations on the surgery procedure.

Following the strategy as in Marques's paper [24] for dimension 3, our arguments consist of two steps. In the first step, we obtain the following result.

Theorem 4.2 [6, Theorem 1.2] *Let* M *be a compact four-dimensional manifold with* $\mathrm{PIC}(M) \neq \emptyset$. *If* $g \in \mathrm{PIC}(M)$, *then there is a path of metrics* g_μ, $\mu \in [0,1]$, *such that* $g_0 = g$, g_1 *is a canonical metric, and* $g_\mu \in \mathrm{PIC}(M)$ *for all* $\mu \in [0,1]$.

The precise definition of a canonical metric will be given in Section 4.1.3.

The second step is to prove two different canonical metrics $\widetilde{g}$ and $\widetilde{g}'$ are isotopic to each other modulo diffeomorphisms. This turns out to be a topological problem.

We will provide some details for these two steps in the remaining subsections.

4.1 M-W connected sum, standard metrics, and canonical metrics

4.1.1 M-W connected sum

We first need a connected sum construction which will be served as the inverse process of the surgery in Ricci flow. Recall that Micallef and Wang [27] proved that the connected sum of two manifolds admitting metrics with PIC also admits metrics with PIC (see Theorem 1.2). We give a short description of the proof of Theorem 1.2 here to indicate how the connected sum construction is performed. The construction depends smoothly on many parameters, the key point is that the resulting metrics are isotopic to each other modulo diffeomorphisms.

Sketch of proof of Theorem 1.2 Let (M_1, g_1) be an n-dimensional Riemannian manifold with PIC. Given $p_1 \in M_1$, denote the geodesic ball of radius r around p_1 by $B_r(p_1)$, and let $r(x) = d(x, p_1)$ be the distance from x to p_1. Given an orthonormal frame $\{e_i^{(1)}\}_{1 \leqslant i \leqslant n} \subset T_{p_1}M_1$, we have a natural isometry from $\mathbb{R}^n$ to $T_{p_1}M_1$. Denote the ball of radius r in $\mathbb{R}^n$ by D_r. Denote by $\mathrm{d}\theta^2$ the metric on S^{n-1} induced from the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n$. The exponential map

$$\exp_{p_1}^{g_1} : \mathbb{R}^n \cong T_{p_1}M_1 \to M_1$$

gives a diffeomorphism from $D_r \subset \mathbb{R}^n$ to $B_r(p_1)$ for small $r > 0$, and gives a local coordinate system (r, θ) around p_1.

By careful computations (see [27] for details), Micallef and Wang showed that, there exist small positive numbers r_0 and ρ_0 depending only on g_1, such that for every $\rho < \rho_0$, we can always find a function $u \colon [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

1) the metric $g_1'(x) = u^2(r(x))g_1(x)$ is a complete metric with PIC on $M_1 \setminus \{p_1\}$;

2) g_1' coincides with g_1 outside $B_{r_0}(p_1)$;

3) there is a small neighborhood $U_1 \subset M_1$ of p_1 such that g_1' on $U_1 \setminus \{p_1\}$ is C^2-close to the product metric $\mathrm{d}s^2 + \rho^2 \mathrm{d}\theta^2$ on $S^{n-1} \times (0, +\infty)$.

Since g_1' on $U_1 \setminus \{p_1\}$ is C^2-close to the product metric, we can slightly modify g_1' on $U_1 \setminus \{p_1\}$ to obtain a metric $\widetilde{g}_1$ with PIC such that $\widetilde{g}_1$ is just g_1 outside $B_{r_0}(p_1)$ and the product metric $\mathrm{d}s^2 + \rho^2 \mathrm{d}\theta^2$ near p_1, where $s = -\rho \log r$.

Suppose that we have another n-dimensional manifold (M_2, g_2) with PIC. Fix a point $p_2 \in M_2$ and an orthonormal frame $\{e_i^{(2)}\}_{1 \leqslant i \leqslant n} \subset T_{p_2}M_2$ at p_2. This gives a geodesic polar coordinate system (r, θ) around p_2 induced by the exponential map and the frame $\{e_i^{(2)}\}_{1 \leqslant i \leqslant n}$. We can do the same construction as above to obtain a new metric $\widetilde{g}_2$ on $M_2 \setminus \{p_2\}$ such that $\widetilde{g}_2$ has PIC, which coincides with g_2 outside a small ball of p_2 and is a product metric $\mathrm{d}s^2 + \rho^2 \mathrm{d}\theta^2$ on a punctured small neighborhood U_2 of p_2, where $s = -\rho \log r$.

Note that we can choose the same small number ρ in the both constructions on M_1 and M_2.

We cut the half-cylinder ends from both $(M_1 \setminus \{p_1\}, \widetilde{g}_1)$ and $(M_2 \setminus \{p_2\}, \widetilde{g}_2)$ for fixed large $s = s_0$. Note that the boundaries of the remaining parts are isometric. Fix an isometry f of the boundaries, and glue the two truncated manifolds together by f along the boundaries. It results in a manifold $\#_{f;p_1,p_2}(M_1, M_2)$ with a metric $\#_{f;p_1,p_2}(g_1, g_2)$ of PIC. $\square$

The gluing map f in the above proof actually is a diffeomorphism between two geodesic spheres (of original metrics) around p_1 and p_2. In the polar coordinates, f can be viewed as an element in $\mathrm{Isom}(S^3)$. If there is no danger of confusion, we use the notations $\#_{f;p_1,p_2}(M_1, M_2)$, $\#_{f;p_1,p_2}(g_1, g_2)$, where $f \in \mathrm{Isom}(S^3)$, to denote above constructed manifolds and metrics, respectively, without mentioning the choices of polar coordinates.

The resulting metric $\#_{f;p_1,p_2}(g_1,g_2)$ in the above construction is not unique. From the argument in [27], we know that it depends on our choice of the small parameters r_0,ρ, the function u, the bases $\{e_i^{(1)}\}$, $\{e_i^{(2)}\}$, and the isometry f, but the resulting metrics with different but continuous choices of these parameters are isotopic to each other modulo diffeomorphisms. Furthermore, it is not hard to see that, the connected sum construction can be applied to a continuous family of metrics with PIC. Moreover, it can be generalized to orbifold connected sums. The readers can refer to [6, Remark 2.2] for a precise description. In conclusion, we have the following result.

Proposition 4.3 *The* M-W *connected sum can be performed continuously on orbifolds such that the resulting metrics have PIC and vary continuously with the parameters.*

We call such a procedure an M-W connected sum for short.

4.1.2 Standard metrics

Let $h_{\rm std}=\mathrm{d}s^2+\mathrm{d}\theta^2$ be the standard cylindrical metric on $S^3\times\mathbb{R}$, where $\mathrm{d}\theta^2$ is the metric induced from the inclusion $S^3\subset\mathbb{R}^4$. Let G be a cocompact fixed-point-free discrete subgroup of the isometric group of $(S^3\times\mathbb{R},h_{\rm std})$. We call the quotient metric on $(S^3\times\mathbb{R})/G$, denoted also by $h_{\rm std}$, a standard metric. For a manifold M and a diffeomorphism $\Psi\colon M\to(S^3\times\mathbb{R})/G$, we also call $\Psi^*(h_{\rm std})$ a standard metric on M.

In [6, Section 3.1], we have recalled some basic facts on the geometries on $(S^3\times\mathbb{R})/G$: they are diffeomorphic either to $S^3/\Gamma\times_f S^1$ or $\#_f(S^4/\langle\Gamma,\hat\tau_1\rangle, S^4/\langle\Gamma,\hat\tau_2\rangle)$. Note that manifolds diffeomorphic to $\#_f(S^4/\langle\Gamma,\hat\tau_1\rangle,S^4/\langle\Gamma,\hat\tau_2\rangle)$ are not orientable.

Let $h_{\rm round}$ be the standard metric of S^4 induced by the inclusion $S^4\hookrightarrow\mathbb{R}^5$. If $\widetilde\Gamma$ is a discrete subgroup of isometries of $(S^4,h_{\rm round})$, we will also call the induced metric on $S^4/\widetilde\Gamma$ a standard metric, and still denote it by $h_{\rm round}$. For an orbifold M and a diffeomorphism $\Psi\colon M\to S^4/\widetilde\Gamma$, we also call $\Psi^*(h_{\rm round})$ a standard metric on M.

In conclusion, we have defined standard metrics on manifolds or orbifolds diffeomorphic to one of the following types:

(1) $S^3/\Gamma\times_f S^1$, $\#_f(S^4/\langle\Gamma,\hat\tau_1\rangle,S^4/\langle\Gamma,\hat\tau_2\rangle)$;

(2) $S^4/\widetilde\Gamma$.

4.1.3 Canonical metrics

Let $p_1,p_2,\dots,p_k,q_1,q_2,\dots,q_l$ be $(k+l)$ distinct points on $(S^4,h_{\rm round})$. Let (M_i,h_i), $i=1,2,\dots,k$, be compact Riemannian manifolds isometric to $(S^3\times\mathbb{R})/G_i$ with standard metrics, with $p_i'\in M_i$. Let $(X_j,\widetilde h_j)$, $j=1,2,\dots,l$, be Riemannian orbifolds isometric to spherical orbifolds S^4/Γ_j with isolated singularities and with standard metrics, where $\Gamma_j\subset\mathrm{O}(5)$, $\Gamma_j\neq\{1\}$. Let $q_j'\in X_j$ be smooth points of X_j for $j=1,2,\dots,l$. If we perform the M-W connected sum operation on $(S^4,h_{\rm round},p_1,p_2,\dots,p_k,q_1,q_2,\dots,q_l)$ with those (M_i,h_i,p_i') and $(X_j,\widetilde h_j,q_j')$, we obtain a Riemannian orbifold $(\hat X,\hat g)$. Clearly, the resulting metric $\hat{g}$ has PIC. The sphere (S^4, h_{round}) in the above construction will be called a principal sphere. (M_i, h_i) and $(X_j, \widetilde{h}_j)$ in the above construction will be called subcomponents of $\hat{X}$. The decomposition of $\hat{X}$ as the connected sum of subcomponents of above types is called a canonical decomposition of $\hat{X}$.

Suppose that (M, g) is a Riemannian orbifold. We call g a canonical metric on M if there is a diffeomorphism $\Psi\colon M \to \hat{X}$ such that $\Psi^*(\hat{g}) = g$, where $(\hat{X}, \hat{g})$ is a Riemannian orbifold constructed as above.

Because of the presence of the principal sphere, a canonical metric on $(S^3 \times \mathbb{R})/G$ (or S^4/Γ with $\Gamma \neq \{1\}$) is not isometric to the standard metric induced from the $(S^3 \times \mathbb{R}, h_{\text{std}})$ (or (S^4, h_{round})).

The orbifolds admitting a canonical decomposition will naturally appear in the process of Ricci flow with surgery.

We remark that given a manifold M, a priori, there might exist different canonical metrics on M. One of the reasons is that if we have two different diffeomorphisms $\Psi_1\colon M \to \hat{X}_1$, $\Psi_2\colon M \to \hat{X}_2$, such that $(\hat{X}_1, \hat{g}_1)$, $(\hat{X}_2, \hat{g}_2)$ are Riemannian orbifolds with canonical metrics constructed as above, then the pull-backs $\Psi_1^*(\hat{g}_1)$ and $\Psi_2^*(\hat{g}_2)$ are both canonical metrics on M, but we even do not know whether topologically $\hat{X}_1$ and $\hat{X}_2$ have the same canonical decompositions, i.e., the subcomponents of the decompositions of $\hat{X}_1$ and $\hat{X}_2$ are diffeomorphic to each other up to a reordering.

4.2 Proof of Theorem 4.2

The proof of Theorem 4.2 is lengthy and detailed. Except for Ricci flow, we need some other techniques to deform the metrics to standard ones or canonical ones. For example, we use the conformal method. The conformal method also plays an important role in the work of Weyl [36] and Marques [24]. The following proposition and its proof give a model for applying the conformal method.

Proposition 4.4 [6, Proposition 4.1] *Let (M, g) be a compact four-orbifold with PIC. Then the space* $\mathrm{PIC}(M) \cap \{\widetilde{g} \mid \widetilde{g} = u^2 g,\ u \in C^\infty(M),\ u > 0\}$ *of metrics is star-shaped, hence contractible.*

Proof Recall that we have defined σ_g in (3.7), and we know that $\sigma_g > 0$ is equivalent to PIC.

Let $\widetilde{g} = u^2 g$, $u \in C^\infty(M)$, $u > 0$. By (3.9), we have

$$\sigma_{\widetilde{g}} = u^{-3}(-6\Delta u + \sigma_g u).$$

Denote

$$u_\mu = 1 - \mu + \mu u, \quad g_\mu = u_\mu^2 g, \quad \sigma_\mu = \sigma_{g_\mu}, \quad \mu \in [0, 1].$$

Then $g_0 = g$, $g_1 = \widetilde{g}$. Suppose that both σ_g and $\sigma_{\widetilde{g}}$ are positive. Then we have

$$u_\mu^3 \sigma_\mu = \mu(-6\Delta u + \sigma_g u) + (1 - \mu)\sigma_g > 0.$$

Hence, every g_μ has PIC and we finish the proof. □

Making use of the properties of M-W connected sum (Proposition 4.3) and conformal method as above, we have the following results.

Proposition 4.5 [6, Proposition 4.7] *Let (M,g) be a Riemannian orbifold isometric to one of S^4/G, $S^3/\Gamma\times_f S^1$, or $S^4/\langle\Gamma,\hat{\tau}_1\rangle\#_f S^4/\langle\Gamma,\hat{\tau}_2\rangle$, equipped with a standard metric. Suppose that $(\widetilde{M},\widetilde{g})$ is the canonical metric obtained from an* M-W *connected sum of (M,g) with a sphere $(S^4,h_{\rm round})$. Then $\widetilde{g}$ is isotopic to g modulo a diffeomorphism.*

Proposition 4.6 [6, Proposition 4.12] *Suppose that (X_1,g_1), (X_2,g_2), $\ldots$, (X_l,g_l) are orbifolds endowed with canonical metrics. If (X,g) is the Riemannian orbifold obtained by performing* M-W *connected sums between them (we allow performing connected sums on some X_i with itself), then g is isotopic to a canonical metric on X.*

The readers can refer to [6, Section 4] for proofs of the above propositions.

In [6], M-W connected sum is served as the inverse process of the surgery in Ricci flow. The influence of compositions of M-W connected sums and Hamilton's surgeries on the metrics needs a careful study. We summarize it in the following proposition.

Proposition 4.7 (see [6, Lemma 5.5, Remark 5.6]) *Suppose that (M,g) is a Riemannian orbifold containing a region N which is ε-close to a standard neck $(S^3/\Gamma\times(-4,4),h_{\rm std})$ with $\varepsilon<\varepsilon_2$, where ε_2 is some universal positive constant. Let $\psi_N\colon S^3/\Gamma\times(-4,4)\to N$ be the corresponding parametrization, and let $s\colon N\to\mathbb{R}$ be the function such that $s(\psi_N(\theta,t))=t$. Suppose that we perform Hamilton's surgery along the central slice of N and then perform* M-W *connected sum along the tips of surgery caps, where we use a suitable choice of the isotopic class of the gluing map $f\in{\rm Isom}(S^3/\Gamma)$ in the* M-W *connected sum such that the resulting orbifold is diffeomorphic to M. Denote the resulting orbifold by (M',g') and the diffeomorphism by $\Psi\colon M\to M'$. Then $\Psi^*(g')$ is isotopic to g, and the isotopy preserves g outside $s^{-1}([-3,3])$.*

The idea of the proof of Proposition 4.7 is as follows. We assume that Γ is trivial for simplicity. Since the metric $h=\psi_N^*g$ is sufficiently close to $h_{\rm std}$ on $S^3\times(-4,4)$, the linear homotopy

$$h_\mu=\mu h_{\rm std}+(1-\mu)h,\quad \mu\in[0,1],$$

provides an isotopy between h and $h_{\rm std}$. We perform Hamilton's surgery on $(S^3\times(-4,4),h_\mu)$ uniformly at the central slice, and then perform M-W connected sum at the tips of the added caps uniformly, these procedures provide an isotopy. Note that performing Hamilton's surgery and then performing M-W connected sum at the tips of the added caps on $(S^3\times(-4,4),h_{\rm std})$ gives a rotationally symmetric metric on $S^3\times(-4,4)$, which is isotopic to $h_{\rm std}$ by the conformal method in Proposition 4.4, and then isotopic to h via $h_{1-\mu}$. This is sufficient to prove Proposition 4.7.

In the surgery process, we cut or glue something to change the topology and metrics of the original orbifolds. Since we are concerned not only with the topology but also the geometry of the manifold M, we have to choose the parameters in the surgery process more carefully, and investigate closely the geometric structure of the pieces left by surgeries. See [6, Section 6] for some refined properties of Ricci flow with surgery.

Combining Propositions 4.5 to 4.7 with the properties of Ricci flow with surgery obtained in [6], we derive the following proposition.

Proposition 4.8 [6, Proposition 7.1] *Suppose that (X, g) is a compact connected four-orbifold with isolated singularities and with PIC such that every point $x \in X$ has an ε-canonical neighborhood. Then g is isotopic to a canonical metric, and X is diffeomorphic to one of the following orbifolds*:

1) S^4/Γ or $S^4/\langle\Gamma, \hat{\tau}\rangle$, where $\Gamma \subset \mathrm{SO}(4)$ (Γ may be trivial);

2) $(S^3 \times \mathbb{R})/G$, where G is a cocompact fixed point free discrete isometric subgroup of $(S^3 \times \mathbb{R}, h_{\mathrm{std}})$;

3) $\#(S^4/\{1, \zeta\}, S^4/\{1, \zeta\})$ or $\#(S^4/\{1, \zeta\}, \mathbb{R}\mathrm{P}^4)$, where the connected sum is performed at smooth points.

The idea in the proof of Proposition 4.8 is as follows. Since every point $x \in X$ has an ε-canonical neighborhood (see [6, Section 6.3] for description on ε-canonical neighborhoods), we can suitably perform Hamilton's surgeries on (X, g) to decompose it into finitely many pieces which is isotopic to some standard ones. Perform suitable M-W connected sums to glue these pieces together, and then by Propositions 4.6 and 4.7, we are not hard to prove that g itself is isotopic to a canonical metric. See [6, Section 7] for details.

Now, we begin to prove Theorem 4.2.

Proof of Theorem 4.2 Let g_0 be a PIC metric on M. By Theorem 2.4, there exist two sequences of non-increasing small positive numbers $\{r_i\}, \{\delta_i\}$, and a Ricci flow with surgery on orbifolds with at most isolated singularities $(X_i,\, g_i(t))_{t\in[t_i,t_{i+1})}$, $0 \leqslant i \leqslant p$, such that

1) $X_0 = M$ and $g_0(0) = g_0$;

2) the flow becomes extinct at a finite time $T = t_{p+1}$;

3) for every $0 \leqslant i \leqslant p$, the flow $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$ satisfies the ε-canonical neighborhood assumption with parameter r_i and the pinching assumption;

4) for every $0 \leqslant i \leqslant p-1$, $(X_{i+1}, g_{i+1}(t_{i+1}))$ is obtained from $(X_i, g_i(t))_{t\in[t_i,t_{i+1})}$ by doing surgery at singular time t_{i+1} with parameters r_i and δ_i.

Let A_i be the assertion that the restriction of $g_i(t_i)$ to each component of X_i is isotopic to a canonical metric. We will prove the theorem by backward induction on i.

First, since the flow becomes extinct at T, at a time $t' \in [t_p, T)$ sufficiently close to T, every point is covered by a canonical neighborhood. By Proposition 4.8, each connected component in $(X_p, g_p(t'))$ is isotopic to a canonical metric. By the Ricci flow equation, $g_p(t_p)$ is isotopic to $g_p(t')$. Hence, on each connected component of X_p, $g_p(t_p)$ is isotopic to a canonical metric and A_p is proven.

In the following, providing A_{i+1} is true for some $0 \leqslant i \leqslant p-1$, we will prove that A_i is true. Let us recall how the Ricci flow can be extended across the time t_{i+1}. Denote

$$g_i(t_{i+1}^-) = \lim_{t \nearrow t_{i+1}} g_i(t).$$

Then $g_i(t_{i+1}^-)$ is a metric with unbounded curvature on X_i.

Note that X_i may contain several compact connected components. For those components of X_i with positive curvature operator at time t_{i+1}, we know that the metrics $g_i(t_{i+1}^-)$ on these components are isotopic to spherical metrics (see [17]). Denote the union of the remaining components of X_i by Ω. Let $\rho = \delta_i r_i$, and let

$$\Omega_\rho = \big\{x \in \Omega \mid \lim_{t \to t_{i+1}} R(x,t) \leqslant \rho^{-2}\big\}.$$

Then every point of Ω outside Ω_ρ has an ε-neck or ε-cap neighborhood. There are finitely many ε-horns, denoted by H_j, $1 \leqslant j \leqslant k$, which are connected components of $\Omega \setminus \Omega_\rho$ with one end in Ω_ρ and another end has unbounded curvature. Each of these ε-horns is diffeomorphic to $S^3/\Gamma \times (0,1)$ for some $\Gamma \subset \mathrm{SO}(4)$. On each ε-horn H_j, find a δ_i-neck N_j with center x_j where the curvature $\geqslant {h_i}^{-2}$, where $h \in (0, \delta_i \rho_i)$ is the constant as in [7, Proposition 4.4]. Denote the center slice of N_j by S_j. Let $\widetilde{\Omega}$ be the union of the connected components of $\Omega \setminus (\cup S_j)$ with finite curvature at t_{i+1} for the metric $g_i(t_{i+1}^-)$, and denote $\hat{\Omega} = \Omega \setminus \overline{\widetilde{\Omega}}$. Now, we cut off the δ_i-neck N_j along S_j, and glue back surgery caps to the boundaries of $\widetilde{\Omega}$. Then we obtain $(X_{i+1}, g_{i+1}(t_{i+1}))$.

On the other hand, for $t' \in (t_i, t_{i+1})$ sufficiently close to t_{i+1}, the family of metrics $(1-\mu)g_i(t') + \mu g_i(t_{i+1}^-)$ ($\mu \in [0,1]$) on $\widetilde{\Omega} \cup (\cup_j N_j)$ have positive isotropic curvature and has δ_i-neck structures on each N_j. Gluing surgery caps at the slices S_j on this family of metrics, we know that $(X_{i+1}, g_i(t')_{\mathrm{surg}})$ is isotopic to $(X_{i+1}, g_{i+1}(t_{i+1}))$. By induction assumption A_{i+1}, we conclude that on each connected component of X_{i+1}, $g_i(t')_{\mathrm{surg}}$ is isotopic to a canonical metric.

Moreover, at time t', if we glue back surgery caps to the boundary necks of $\hat{\Omega}$, then we get a (possibly disconnected) closed orbifold $(Y_{i+1}, g_i(t')_{\mathrm{surg}})$. Since every point of Y_{i+1} has a canonical neighborhood, by Proposition 4.8, on each connected component of Y_{i+1}, $g_i(t')_{\mathrm{surg}}$ is isotopic to a canonical metric.

Finally, by Proposition 4.7, if we perform suitable M-W connected sums at the tips of the surgery caps of $(X_{i+1}, g_i(t')_{\mathrm{surg}})$ and $(Y_{i+1}, g_i(t')_{\mathrm{surg}})$, then the resulting metric will be isotopic to $g_i(t')$ modulo a diffeomorphism. Hence, the metric $g_i(t')$ on Ω is isotopic to a canonical metric by Proposition 4.6. By the Ricci flow equation, $g_i(t_i)$ is isotopic to $g_i(t')$. This proves A_i.

Repeating the above procedure, we know that g_0 is isotopic to a canonical metric on M. Since M is itself a manifold, we know that there is no subcomponent in the canonical decomposition of M containing orbifold singularities. Therefore, every subcomponent is either diffeomorphic to $\mathbb{RP}^4$ or $(S^3 \times \mathbb{R})/G$. The proof is completed. □

4.3 Proof of Theorem 4.1

Let M be a compact orientable four-manifold, and let g_1, g_2 be two different metrics on M with PIC. By Theorem 4.2, there are two canonical metrics

$\widetilde{g}_1, \widetilde{g}_2 \in \mathrm{PIC}(M)$ such that g_i is isotopic to $\widetilde{g}_i$ for $i = 1, 2$. Let

$$S^4 \# M_1 \# M_2 \cdots \# M_k, \quad S^4 \# N_1 \# N_2 \cdots \# N_l$$

be the canonical decompositions associated to $\widetilde{g}_1$ and $\widetilde{g}_2$, respectively, where M_i and N_i are diffeomorphic to manifolds of the form $S^3/\Gamma \times_f S^1$. We first show that this decomposition is unique in the fundamental group level.

Proposition 4.9 [6, Theorem 8.1] *Suppose that (M, g) admits two canonical decompositions $S^4 \# M_1 \# M_2 \cdots \# M_k$ and $S^4 \# N_1 \# N_2 \cdots \# N_l$. Then $k = l$, and there is a permutation $\sigma \in S_k$ such that $\pi_1(M_i) \cong \pi_1(N_{\sigma(i)})$ for all $1 \leqslant i \leqslant k$.*

The observation in the proof of Proposition 4.9 is that an orientable manifold diffeomorphic to $S^3/\Gamma \times_f S^1$ has a freely indecomposable fundamental group. Thus, we can apply Kurosh's theorem on the uniqueness of free product decomposition of groups. See [6, Section 8] for details.

When $k = 0$, M is diffeomorphic to S^4. By Proposition 4.5, the canonical metric on S^4 is isotopic to the round metric, and thus, the conclusion of Theorem 4.1 is clearly true in this case. The $k = 1$ case was solved in [6], whose proof is omitted here.

Proposition 4.10 [6, Proposition 8.4] *Let M be an orientable four-manifold equipped with two canonical metrics $\widetilde{g}_1$, $\widetilde{g}_2$ and the associated canonical decompositions $S^4 \# M_1$ and $S^4 \# N_1$ have only one nontrivial piece. Then there is a diffeomorphism $\Psi \in \mathrm{Diff}(M)$ such that $\widetilde{g}_2$ is isotopic to $\Psi^*(\widetilde{g}_1)$. In particular, $\mathrm{PIC}(M)/\mathrm{Diff}(M)$ is path-connected.*

When $k \geqslant 2$, we have the following proposition.

Proposition 4.11 *Let M be an orientable manifold diffeomorphic to a finite connected sum of $S^3/\Gamma_i \times S^1$, $1 \leqslant i \leqslant k$, where Γ_i is either the trivial group or a non-cyclic discrete isometric group of S^3. Then $\mathrm{PIC}(M)/\mathrm{Diff}(M)$ is path-connected.*

Proof As before, let $g_1, g_2 \in \mathrm{PIC}(M)$. By Theorem 4.2, there are two canonical metrics $\widetilde{g}_1, \widetilde{g}_2 \in \mathrm{PIC}(M)$ such that g_i is isotopic to $\widetilde{g}_i$ for $i = 1, 2$. Suppose that $\Psi_i \colon M \to \hat{X}_i$ is a diffeomorphism such that $\widetilde{g}_i = \Psi_i^*(\hat{g}_i)$ $(i = 1, 2)$, where $(\hat{X}_1, \hat{g}_1)$ is obtained from M-W connected sum between M_j $(1 \leqslant j \leqslant k)$ equipped with standard metric h_j and S^4 with round metric, while $(\hat{X}_2, \hat{g}_2)$ is obtained from M-W connected sum between N_j $(1 \leqslant j \leqslant k)$ equipped with standard metric h'_j and S^4 with round metric. We require that when doing M-W connected sum, the orientation on M_j are consistent with the orientation of $\hat{X}_1$, while the orientation on N_j are consistent with the orientation of $\hat{X}_2$.

Since M is diffeomorphic to a finite connected sum of $S^3/\Gamma_i \times S^1$, we know

$$\pi_1(M) \cong (\Gamma_1 \times \mathbb{Z}) * (\Gamma_2 \times \mathbb{Z}) * \cdots * (\Gamma_k \times \mathbb{Z}).$$

From Theorem 4.9, we may assume

$$\pi_1(M_j) \cong \pi_1(N_j) \cong \Gamma_j \times \mathbb{Z}, \quad 1 \leqslant j \leqslant k.$$

Note that if two spherical 3-manifolds F_1, F_2 have isomorphic fundamental groups, and $\pi_1(F_1) \cong \pi_1(F_2)$ is either trivial or non-cyclic, then F_1, F_2 are diffeomorphic to each other (see [25]). Combining these facts with [35, Proposition 8], we know that both M_j and N_j are diffeomorphic to $S^3/\Gamma_j \times S^1$.

From Proposition 4.10, we know that both h_j and h'_j are isotopic to some pull-back metrics from $(S^3/\Gamma_j \times S^1, \overline{g}_j)$, where $\overline{g}_j$ is the standard product metric on $S^3/\Gamma_j \times S^1$. Without loss of generality, we may assume that both h_j and h'_j are pull-back metrics from $(S^3/\Gamma_j \times S^1, \overline{g}_j)$. Thus, there is an isometry between (M_j, h_j) and (N_j, h'_j). Note that there is an orientation reversing isometry on $(S^3/\Gamma_j \times S^1, \overline{g}_j)$ defined by

$$\chi\colon S^3/\Gamma_j \times S^1 \to S^3/\Gamma_j \times S^1,$$
$$(\theta, s) \mapsto (\theta, -s).$$

Hence, there is always an orientation preserving diffeomorphism $\varphi_j\colon M_j \to N_j$ such that $\varphi_j^*(h'_j) = h_j$ for $j = 1, 2, \dots, k$.

From Proposition 4.3, during making M-W connected sums, these φ_j's may be glued together to give a global diffeomorphism $\Phi\colon \hat{X}_1 \to \hat{X}_2$ such that $\Phi^*(\hat{g}_2)$ is isotopic to $\hat{g}_1$. From this, we know that $(\Psi_2^{-1}\Phi\Psi_1)^*(\widetilde{g}_2)$ is isotopic to $\widetilde{g}_1$. Hence, $(\Psi_2^{-1}\Phi\Psi_1)^*(g_2)$ is isotopic to g_1. The proof is completed. □

Theorem 4.1 follows from Propositions 4.10 and 4.11.

Finally, we remark that the result of Theorem 4.1 does not include all orientable four-manifolds with $\mathrm{PIC}(M) \neq \emptyset$. We conjecture that the conclusion of Theorem 4.1 should hold for all orientable four-manifolds admitting metrics with PIC.

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