Front. Math. China 2015, 10(4): 777–805
DOI 10.1007/s11464-015-0487-6

RESEARCH ARTICLE

Smoothness of local times and self-intersection local times of Gaussian random fields

Zhenlong CHEN[1], **Dongsheng WU**[2], **Yimin XIAO**[3]

1 School of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China
2 Department of Mathematical Sciences, University of Alabama in Huntsville, Huntsville, AL 35899, USA
3 Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA

Abstract This paper is concerned with the smoothness (in the sense of Meyer-Watanabe) of the local times of Gaussian random fields. Sufficient and necessary conditions for the existence and smoothness of the local times, collision local times, and self-intersection local times are established for a large class of Gaussian random fields, including fractional Brownian motions, fractional Brownian sheets and solutions of stochastic heat equations driven by space-time Gaussian noise.

Keywords Anisotropic Gaussian field, local time, collision local time, intersection local time, self-intersection local time, chaos expansion
MSC 60G15, 60H05, 60H07

1 Introduction

In recent years, Malliavin calculus has been shown to be very useful in stochastic analysis of Gaussian processes (cf. [21]). In particular, many authors have studied the chaos expansion and smoothness in the sense of Meyer-Watanabe of local times and intersection local times of Brownian motion, fractional Brownian motion and related self-similar Gaussian processes. See [4,9,11–14, 16–18,22,24,25,34,35]. However, there have been only a few results on smoothness of local times of Gaussian random fields due to their more complicated dependence structures. We refer to [15,17] for the case of Brownian sheet and to [7,8] for results on fractional Brownian sheets.

Received February 8, 2015; accepted April 13, 2015
Corresponding author: Yimin XIAO, E-mail: xiao@stt.msu.edu

The main purpose of this paper is to study the smoothness in the sense of Meyer-Watanabe of the local times of a large class of Gaussian random fields, including fractional Brownian sheets and solutions of stochastic heat equations driven by space-time Gaussian noise. More specifically, let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ by

$$X(t) = (X_1(t), X_2(t), \ldots, X_d(t)), \quad \forall\, t \in \mathbb{R}^N. \tag{1.1}$$

We will call X an (N, d)-Gaussian random field. We assume that the coordinate fields $X_1, X_2, \ldots, X_d$ are independent copies of a real-valued, centered Gaussian random field $X_0 = \{X_0(t),\ t \in \mathbb{R}^N\}$ with continuous covariance function

$$R(s, t) = \mathbb{E}[X_0(s)X_0(t)].$$

Let $H = (H_1, H_2, \ldots, H_N) \in (0, 1)^N$ be a fixed vector. For $a, b \in \mathbb{R}^N$ with $a_j < b_j$ $(j = 1, 2, \ldots, N)$, let

$$I = [a,\, b] := \prod_{j=1}^{N} [a_j, b_j] \subseteq \mathbb{R}^N$$

be the compact interval (or a rectangle). For simplicity, we will take $I = [0, 1]^N$ throughout this paper. We further assume that $X_0 = \{X_0(t),\ t \in \mathbb{R}^N\}$ satisfies the following conditions:

(C1) there exists a positive and finite constant c_1 such that

$$\mathbb{E}[(X_0(s) - X_0(t))^2] \leqslant c_1 \sum_{j=1}^{N} |s_j - t_j|^{2H_j}, \quad \forall\, s, t \in I; \tag{1.2}$$

(C2) there exists a constant $c_2 > 0$ such that for all $s, t \in I$,

$$\operatorname{Var}(X_0(t) \mid X_0(s)) \geqslant c_2 \sum_{j=1}^{N} \min\{|s_j - t_j|^{2H_j}, |t_j|^{2H_j}\}, \tag{1.3}$$

where $\operatorname{Var}(X_0(t) \mid X_0(s))$ denotes the conditional variance of $X_0(t)$ given $X_0(s)$.

The class of Gaussian random fields that satisfy conditions (C1) and (C2) is large. When $N = 1$, it includes fractional Brownian motion, bi-fractional Brownian motion and related Gaussian processes. For $N \geqslant 2$, this class contains fractional Brownian sheets (cf. [2,29] for verification), solutions to stochastic heat equation driven by space-time Gaussian noises [6,20,26,28] and many more (cf. [32]).

The purpose of this paper is to study the existence and smoothness (in the sense of Meyer-Watanabe) of the local times and the self-intersection local times of Gaussian random fields that satisfy conditions (C1), (C2), and/or (C3) below. Our main results in Sections 2 and 3 unify and extend the previous results in the references mentioned at the beginning of this section. We should also mention that Hölder regularities of local times and their applications to sample path properties of Gaussian random fields have been studied by several authors, including [1,2,3,5,10,23,29–33].

The rest of this paper is organized as follows. In Section 2, we provide a sufficient and necessary condition for the existence, and a sufficient condition for the smoothness (in the sense of Meyer-Watanabe) of the local time at any level $x \in \mathbb{R}^d$ for a large class of Gaussian random fields. We also prove that this condition for the smoothness is also necessary for the local times at $x = 0$. We then apply the conditions to prove the existence and smoothness results for the collision local times and the intersection local times for two independent anisotropic Gaussian random fields.

Section 3 is concerned with self-intersection local times. We establish a sufficient and necessary condition for the existence and smoothness of self-intersection local times on two disjoint intervals. More interestingly, we also consider the analogous problems on two intersecting intervals. We will see that the results in the intersecting cases are different from and more difficult than those in the disjoint case.

Throughout this paper, we will use c to denote unspecified positive finite constants which may be different in each appearance. More specific constants are numbered as $c_1, c_2, \ldots$.

2 Existence and smoothness of local times

This section is concerned with the existence and smoothness of the local times of a Gaussian random field X in the sense of Meyer-Watanabe. We start by recalling the definition of chaos expansion, which is an orthogonal decomposition of $L^2(\Omega, \mathbb{P})$. We refer to [11,19,21,22] and references therein for more information.

Let Ω be the space of continuous $\mathbb{R}^d$-valued functions ω on I. Then Ω is a Banach space with respect to the sup norm. Let $\mathscr{F}$ be the Borel σ-algebra on Ω. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathscr{F})$, and let $\mathbb{E}$ denote the expectation on this probability space. Denote by $L^2(\Omega, \mathbb{P})$ the space of all real (or complex) valued functional on Ω such that

$$\mathbb{E}(F^2) = \int_\Omega |F(\omega)|^2 \mathbb{P}(\mathrm{d}\omega) < +\infty.$$

Let

$$Y = \{(Y_1(t), Y_2(t), \ldots, Y_d(t)),\, t \in I\}$$

be an (N, d)-Gaussian random field, where $Y_1, Y_2, \ldots, Y_d$ are d independent copies of some centered, real-valued Gaussian random field Y_0 on I. Let $p_n(y_1, y_2, \ldots, y_k)$ be a polynomial of degree n of k variables $y_1, y_2, \ldots, y_k$. Then, for any

$t^1, t^2, \ldots, t^k \in I$ and $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, d\}$, $p_n(Y_{i_1}(t^1), Y_{i_2}(t^2), \ldots, Y_{i_k}(t^k))$ is called a *polynomial functional* of Y. Let $\mathscr{P}_n$ be the completion with respect to the $L^2(\Omega, \mathbb{P})$ norm of the set of all polynomials of degree less than or equal to n. Then $\mathscr{P}_n$ is a subspace of $L^2(\Omega, \mathbb{P})$. Let $\mathscr{C}_n$ be the orthogonal complement of $\mathscr{P}_{n-1}$ in $\mathscr{P}_n$. Then $L^2(\Omega, \mathbb{P})$ is the direct sum of $\mathscr{C}_n$, i.e.,

$$L^2(\Omega, \mathbb{P}) = \bigoplus_{n=0}^{+\infty} \mathscr{C}_n.$$

Namely, for any functional $F \in L^2(\Omega, \mathbb{P})$, there exists a sequence $\{F_n\}_{n=0}^{+\infty}$ with $F_n \in \mathscr{C}_n$, such that

$$F = \sum_{n=0}^{+\infty} F_n.$$

This decomposition is called the *chaos expansion* of F, and F_n is called the *n-th chaos* of F. Clearly,

$$F_0 = \mathbb{E}(F), \quad \mathbb{E}(|F|^2) = \sum_{n=0}^{+\infty} \mathbb{E}(|F_n|^2).$$

In Malliavin Calculus, the space of 'smooth' functions in the sense of Meyer-Watanabe (cf. [21,27]) is defined by

$$D_1 := \Big\{ F \in L^2(\Omega, \mathbb{P}),\ F = \sum_{n=0}^{+\infty} F_n,\ \sum_{n=0}^{+\infty} n\mathbb{E}(|F_n|^2) < +\infty \Big\}.$$

For $F \in L^2(\Omega, \mathbb{P})$ with a chaos expansion $F = \sum F_n$, define the operator Γ_u with $u \in [0, 1]$ by

$$\Gamma_u F := \sum_{n=0}^{+\infty} u^n F_n, \tag{2.1}$$

and set

$$\Theta_F(u) := \Gamma_{\sqrt{u}} F.$$

Clearly, $\Theta_F(1) = F$. Define

$$\Phi_{\Theta_F}(u) := \frac{\mathrm{d}}{\mathrm{d}u} \mathbb{E}(|\Theta_F(u)|^2).$$

Then we have

$$\Phi_{\Theta_F}(u) = \sum_{n=1}^{+\infty} n u^{n-1} \mathbb{E}(|F_n|^2).$$

In the following, we provide several technical lemmas which will be useful for proving the existence and smoothness of local times. Lemma 2.1 is similar to [3, Lemma 8.6] whose proof is elementary. Lemmas 2.2 and 2.3 are from Wu and Xiao [30].

Lemma 2.1 *Let α and β be positive constants. Then, for all $A \in (0,1)$,*

$$\int_0^1 \frac{1}{(A+t^\alpha)^\beta}\,\mathrm{d}t \asymp \begin{cases} A^{-(\beta-\frac{1}{\alpha})}, & \alpha\beta > 1, \\ \log(1+A^{-1/\alpha}), & \alpha\beta = 1, \\ 1, & \alpha\beta < 1. \end{cases} \tag{2.2}$$

In the above, $f(A) \asymp g(A)$ means that the ratio $f(A)/g(A)$ is bounded from below and above by positive constants that do not depend on $A \in (0,1)$.

Lemma 2.2 *Let α and β be positive constants such that $\alpha\beta \geqslant 1$.*

(i) *If $\alpha\beta > 1$, then there exists a constant $c_3 > 0$ whose value depends on α and β only such that for all $A \in (0,1)$, $r > 0$, $u^* \in \mathbb{R}$, all integers $n \geqslant 1$, and all distinct $u_1, u_2, \dots, u_n \in O(u^*, r)$, we have*

$$\int_{O(u^*,r)} \frac{\mathrm{d}u}{(A+\min_{1\leqslant j\leqslant n}|u-u_j|^\alpha)^\beta} \leqslant c_3 n A^{-(\beta-\frac{1}{\alpha})}. \tag{2.3}$$

where $O(u^*, r)$ denotes a ball centered at u^* with radius r.

(ii) *If $\alpha\beta = 1$, then for any $\kappa \in (0,1)$, there exists a constant $c_4 > 0$ whose value depends on α, β, and κ only such that for all $A \in (0,1)$, $r > 0$, $u^* \in \mathbb{R}$, all integers $n \geqslant 1$, and all distinct $u_1, u_2, \dots, u_n \in O(u^*, r)$, we have*

$$\int_{O(u^*,r)} \frac{\mathrm{d}u}{(A+\min_{1\leqslant j\leqslant n}|u-u_j|^\alpha)^\beta} \leqslant c_4 n \log\Big[\mathrm{e} + \Big(\frac{r}{n} A^{-1/\alpha}\Big)^\kappa\Big]. \tag{2.4}$$

Lemma 2.3 *Let $\beta \in (0,1)$ be a constant. Then there exists a positive constant c_5 such that the following statements hold.*

(i) *For all $r > 0$, $u^* \in \mathbb{R}$, all integers $n \geqslant 1$, and all distinct $u_1, u_2, \dots, u_n \in O(u^*, r)$, we have*

$$\int_{O(u^*,r)} \frac{\mathrm{d}u}{\min_{1\leqslant j\leqslant n}|u-u_j|^\beta} \leqslant c_5 n^\beta r^{-(\beta-1)}. \tag{2.5}$$

(ii) *For all constants $r, M > 0$, all $u^* \in \mathbb{R}$, integers $n \geqslant 1$, and all distinct $u_1, u_2, \dots, u_n \in O(u^*, r)$, we have*

$$\int_{O(u^*,r)} \log\big[\mathrm{e} + M\big(\min_{1\leqslant j\leqslant n}|u-u_j|\big)^{-\beta}\big]\mathrm{d}u \leqslant c_5 r \log\Big[\mathrm{e} + M\Big(\frac{r}{n}\Big)^{-\beta}\Big]. \tag{2.6}$$

2.1 General results

We will apply the following proposition and the method of its proof to study the existence and smoothness of the local times of X.

Proposition 2.4 *Let $X = \{X(t),\, t \in I\}$ be an (N,d)-Gaussian field defined by* (1.1), *and assume that X_0 satisfies conditions* (C1) *and* (C2) *with index $H \in (0,1)^N$. Then, for any $\gamma > 0$, $\lambda \geqslant 0$,*

$$\int_{I^2} \frac{|[\mathbb{E}(X_0(s)X_0(t))]|^{\lambda}}{[\det \mathrm{Cov}(X_0(s), X_0(t))]^{\gamma/2}}\, \mathrm{d}s\mathrm{d}t < +\infty \tag{2.7}$$

if and only if

$$\sum_{\ell=1}^{N} \frac{1}{H_\ell} > \gamma. \tag{2.8}$$

Proof First, we prove the sufficiency. By (C2), we have

$$\mathrm{Var}(X_0(s)) \geqslant \mathrm{Var}\Big(X_0(s) \,\Big|\, X_0\Big(\frac{s}{2}\Big)\Big) \geqslant c_2 2^{-2} \sum_{j=1}^{N} s_j^{2H_j}, \quad \forall\, s \in I. \tag{2.9}$$

This and the fact that

$$\det \mathrm{Cov}(X_0(s), X_0(t)) = \mathrm{Var}(X_0(s))\mathrm{Var}(X_0(t) \mid X_0(s)) \tag{2.10}$$

imply

$$\det \mathrm{Cov}(X_0(s), X_0(t)) \geqslant c\Big(\sum_{j=1}^{N} s_j^{2H_j}\Big)\Big(\sum_{j=1}^{N} \min\{|s_j - t_j|^{2H_j}, t_j^{2H_j}\}\Big). \tag{2.11}$$

On the other hand, it follows from the Cauchy-Schwarz inequality and the continuity of the covariance function $R(s,t)$ that

$$|[\mathbb{E}(X_0(s)X_0(t))]|^{\lambda} \leqslant c, \quad \forall\, s, t \in I. \tag{2.12}$$

Hence, for proving the sufficiency, it suffices to verify that if (2.8) is satisfied, then

$$\int_{I^2} \frac{\mathrm{d}s\mathrm{d}t}{(\sum_{j=1}^{N} s_j^{2H_j})^{\gamma/2}(\sum_{j=1}^{N} \min\{|s_j - t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} < +\infty. \tag{2.13}$$

To estimate the integral in (2.13), we will assume that

$$0 < H_1 \leqslant H_2 \leqslant \cdots \leqslant H_N < 1 \tag{2.14}$$

and integrate in the order of $\mathrm{d}t_1, \mathrm{d}t_2, \ldots, \mathrm{d}t_N$, $\mathrm{d}s_1, \mathrm{d}s_2, \ldots, \mathrm{d}s_N$. When (2.8) is satisfied, there exists $k \in \{1, 2, \ldots, N\}$ such that

$$\sum_{j=1}^{k-1} \frac{1}{H_j} \leqslant \gamma < \sum_{j=1}^{k} \frac{1}{H_j}. \tag{2.15}$$

Note that

$$\int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\sum_{j=1}^N \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} \leqslant \int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\sum_{j=1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}}. \tag{2.16}$$

We distinguish two cases:

(i) $\sum_{j=1}^{k-1}\frac{1}{H_j} < \gamma < \sum_{j=1}^{k}\frac{1}{H_j}$,

(ii) $\sum_{j=1}^{k-1}\frac{1}{H_j} = \gamma < \sum_{j=1}^{k}\frac{1}{H_j}$,

and show that the last integral in (2.16) is bounded by a constant that is independent of $s \in I$.

Case (i). If $k = 1$, then $H_1\gamma < 1$. We apply Lemma 2.3 (i) to derive

$$\int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\sum_{j=1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} \leqslant \int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\min\{|s_1-t_1|^{2H_1}, t_1^{2H_1}\})^{\gamma/2}} \leqslant c_6,$$

as desired.

If $k > 1$, then $H_1\gamma > 1$. We first apply Lemma 2.2 (i) with

$$\alpha = 2H_1, \quad \beta = \frac{\gamma}{2}, \quad A = \sum_{j=2}^N \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\}$$

to deduce that

$$\int_0^1 \frac{\mathrm{d}t_1}{(\min\{|s_1-t_1|^{2H_1}, t_1^{2H_1}\} + \sum_{j=2}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} \leqslant \frac{c_7}{(\sum_{j=2}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\frac{1}{2}(\gamma-\frac{1}{H_1})}}, \tag{2.17}$$

where c_7 is a constant which only depends on H_1 and γ. By repeatedly using Lemma 2.2 (i) as in (2.17), after $k-1$ steps, we obtain that

$$\int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\sum_{j=1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} \leqslant c\int_0^1 \frac{\mathrm{d}t_k}{(\min\{|s_k-t_k|^{2H_k}, t_k^{2H_k}\})^{\frac{1}{2}(\gamma-\sum_{j=1}^{k-1}\frac{1}{H_j})}}. \tag{2.18}$$

Noting that

$$H_k\biggl(\gamma - \sum_{j=1}^{k-1}\frac{1}{H_j}\biggr) < 1,$$

by applying Lemma 2.3 (i) to the last integral in (2.18), we see from (2.16) that in Case (i),

$$\int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{\big(\sum_{j=1}^N \min\big\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\big\}\big)^{\gamma/2}} \leqslant c_8. \tag{2.19}$$

Case (ii). Notice that $k>1$ in (2.16). We integrate in order of $\mathrm{d}t_1, \mathrm{d}t_2, \ldots, \mathrm{d}t_N$ and repeatedly apply Lemma 2.2 (i) for $k-2$ steps to get

$$\begin{aligned}&\int_I \frac{\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_N}{(\sum_{j=1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}}\\ &\leqslant c\int_0^1\int_0^1 \frac{\mathrm{d}t_{k-1}\,\mathrm{d}t_k}{(\sum_{j=k-1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\frac{1}{2}(\gamma-\sum_{j=1}^{k-2}\frac{1}{H_j})}}.\end{aligned} \tag{2.20}$$

Note that

$$H_{k-1}\Big(\gamma-\sum_{j=1}^{k-2}\frac{1}{H_j}\Big)=1.$$

By applying Lemma 2.2 (ii) with $A=\min\{|s_k-t_k|^{2H_k}, t_k^{2H_k}\}$ and Lemma 2.3 (ii), we derive

$$\begin{aligned}&\int_0^1\int_0^1 \frac{\mathrm{d}t_{k-1}\mathrm{d}t_k}{(\sum_{j=k-1}^k \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\frac{1}{2}(\gamma-\sum_{j=1}^{k-2}\frac{1}{H_j})}}\\ &\leqslant c\int_0^1 \log\Big[\mathrm{e}+\Big(\frac{1}{2}(\min\{|s_k-t_k|^{2H_k}, t_k^{2H_k}\})^{-\frac{1}{2H_{k-1}}}\Big)^{\kappa}\Big]\mathrm{d}t_k\\ &\leqslant c\log(\mathrm{e}+2^{H_k-H_{k-1}}),\end{aligned} \tag{2.21}$$

where $\kappa\in(0,1)$ is a constant and we have used the fact that $H_k\geqslant H_{k-1}$. It follows from (2.20) and (2.21) that (2.19) also holds in Case (ii).

Hence, by (2.13) and (2.19), we have

$$\int_{I^2}\frac{\mathrm{d}s\mathrm{d}t}{(\sum_{j=1}^N s_j^{2H_j})^{\gamma/2}(\sum_{j=1}^N \min\{|s_j-t_j|^{2H_j}, t_j^{2H_j}\})^{\gamma/2}} \leqslant c\int_I \frac{\mathrm{d}s}{(\sum_{j=1}^N s_j^{H_j})^{\gamma}}.$$

It is elementary to verify, by using Lemma 2.1, that the last integral is finite provided $\sum_{j=1}^N \frac{1}{H_j}>\gamma$. This proves (2.13), and thus the sufficiency.

To prove the necessity, we prove that if $\sum_{j=1}^N \frac{1}{H_j}\leqslant\gamma$, then

$$\int_{I^2}\frac{|[\mathbb{E}(X_0(s)X_0(t))]|^{\lambda}}{[\det\mathrm{Cov}(X_0(s),X_0(t))]^{\gamma/2}}\,\mathrm{d}s\mathrm{d}t=+\infty. \tag{2.22}$$

For $\varepsilon_0\in(0,1/2)$, let $I_{\varepsilon_0}:=[\varepsilon_0,1]^N$. (2.9) and the uniform continuity of $R(s,t)$ on $I_{\varepsilon_0}^2$ imply that there exists a constant $\delta_0>0$ such that for all $s,t\in[\varepsilon_0,\varepsilon_0+\delta_0]^N$,

$$\mathbb{E}(X_0(s)X_0(t))\geqslant\frac{1}{2}\,\mathbb{E}(X_0^2(t))\geqslant c_9>0. \tag{2.23}$$

On the other hand, it follows from (2.10) and condition (C1) that for all $s, t \in I$,

$$\det \mathrm{Cov}(X_0(s), X_0(t)) \leqslant c \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \tag{2.24}$$

By (2.23) and (2.24), we derive

$$\int_{I^2} \frac{|\mathbb{E}(X_0(s)X_0(t))|^{\lambda}}{[\det \mathrm{Cov}(X_0(s), X_0(t))]^{\gamma/2}} \mathrm{d}s\mathrm{d}t \geqslant c \int_{[\varepsilon_0, \varepsilon_0+\delta_0]^{2N}} \frac{\mathrm{d}s\mathrm{d}t}{(\sum_{j=1}^{N} |s_j - t_j|^{H_j})^{\gamma}}.$$

By using Lemma 2.1 again, it is elementary to verify that the last integral is infinite when $\sum_{j=1}^{N} \frac{1}{H_j} \leqslant \gamma$. This proves the necessity of the proposition. □

In the following, we consider the existence of the local time of a Gaussian random field satisfying (C1) and (C2). Instead of using a Fourier analytic argument as in [32] (see [10] for a systematic account), we approximate the Dirac delta function by the heat kernel

$$p_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{d/2}} \exp\Big(- \frac{\|x\|^2}{2\varepsilon}\Big), \quad x \in \mathbb{R}^d, \tag{2.25}$$

and let

$$\begin{aligned} L_\varepsilon(x, I, X) &= \int_I p_\varepsilon(X(s) - x)\mathrm{d}s \\ &= \frac{1}{(2\pi)^d} \int_I \int_{\mathbb{R}^d} \exp\Big(\mathrm{i}\langle \xi,\ X(s) - x\rangle - \frac{\varepsilon\|\xi\|^2}{2}\Big)\mathrm{d}\xi\mathrm{d}s. \end{aligned} \tag{2.26}$$

The following is a general result on existence of local times.

Lemma 2.5 *Let $Y = \{(Y_1(t), Y_2(t), \ldots, Y_d(t)),\ t \in I\}$ be an (N, d)-Gaussian random field, where $Y_1, Y_2, \ldots, Y_d$ are d independent copies of a centered, real-valued Gaussian random field Y_0 on I. Then, for any $y \in \mathbb{R}^d$, as $\varepsilon \to 0$, $L_\varepsilon(y, I, Y)$ converges to a limit $L(y, I, Y)$ in $L^2(\Omega, \mathbb{P})$ if and only if*

$$\int_{I^2} \exp\Big(- \frac{\|y\|^2 \mathbb{E}[(Y_0(s) - Y_0(t))^2]}{\det \mathrm{Cov}(Y_0(t), Y_0(s))}\Big) \frac{\mathrm{d}s\mathrm{d}t}{[\det \mathrm{Cov}(Y_0(t), Y_0(s))]^{d/2}} < +\infty. \tag{2.27}$$

Proof Let I_{2d} be the identity matrix of order $2d$, and let

$$\Gamma_{\varepsilon,d}(s, t) = \varepsilon I_{2d} + \mathrm{Cov}(Y(s), Y(t)).$$

For any $y \in \mathbb{R}^d$ and $\varepsilon > 0$, Fubini's theorem and (2.26) imply

$$\begin{aligned} &\mathbb{E}(|L_\varepsilon(y, I, Y)|^2) \\ &= \frac{1}{(2\pi)^{2d}} \int_{I^2} \mathrm{d}s\mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\varepsilon(\|\xi\|^2+\|\eta\|^2)/2} \\ &\quad \times \mathbb{E} \exp(\mathrm{i}\langle \xi, Y(s) - y\rangle - \mathrm{i}\langle \eta, Y(t) - y\rangle)\mathrm{d}\xi\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^{2d}} \int_{I^2} \mathrm{d}s\mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}\langle \xi-\eta, y\rangle} \exp\Big(- \frac{1}{2} (\xi, \eta)\Gamma_{\varepsilon,d}(s, t)(\xi, \eta)^{\mathrm{T}}\Big)\mathrm{d}\xi\mathrm{d}\eta \\ &= \frac{1}{(2\pi)^{2d}} \int_{I^2} \exp\Big(- \frac{1}{2} (y, y)\Gamma_{\varepsilon,d}^{-1}(s, t)(y, y)^{\mathrm{T}}\Big) \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\det \Gamma_{\varepsilon,d}(s, t)}}. \end{aligned} \tag{2.28}$$

Since the coordinate processes of Y are independent copies of Y_0, we have

$$\det\Gamma_{\varepsilon,d}(s,t)=[\det\Gamma_{\varepsilon,1}(s,t)]^d \tag{2.29}$$

and

$$\frac{1}{2}\,(y,y)\Gamma^{-1}_{\varepsilon,d}(s,t)(y,y)^{\mathrm{T}}=\frac{\|y\|^2(2\varepsilon+\mathbb{E}[(Y_0(s)-Y_0(t))^2])}{\det\Gamma_{\varepsilon,1}(s,t)}, \tag{2.30}$$

where

$$\Gamma_{\varepsilon,1}(s,t)=\varepsilon I_2+\mathrm{Cov}(Y_0(s),Y_0(t)).$$

It follows from (2.29), (2.30), and the dominated convergence theorem that

$$\begin{aligned}&\lim_{\varepsilon\to0}\mathbb{E}(|L_\varepsilon(y,I,Y)|^2)\\&\quad=\frac{1}{(2\pi)^{2d}}\int_{I^2}\exp\Big(-\frac{\|y\|^2\mathbb{E}[(Y_0(s)-Y_0(t))^2]}{\det\Gamma_{0,1}(s,t)}\Big)\frac{\mathrm{d}s\mathrm{d}t}{[\det\Gamma_{0,1}(s,t)]^{d/2}}.\end{aligned} \tag{2.31}$$

Next, we show that $\{L_\varepsilon(y,I,Y),\,\varepsilon>0\}$ is a Cauchy sequence in $L^2(\Omega,\mathbb{P})$ if and only if (2.27) holds. For all integers $m,n\geqslant1$, we assume, without loss of generality, that $m=n+p$ for some integer p. Let

$$\Gamma_{n+p}(s,t)=(n+p)^{-1}I_{2d}+\mathrm{Cov}(Y(s),Y(t)),$$
$$\Gamma_n(s,t)=n^{-1}I_{2d}+\mathrm{Cov}(Y(s),Y(t)),$$

and

$$\Gamma_{n+p,n}(s,t)=\begin{pmatrix}(n+p)^{-1}I_d & 0\\ 0 & n^{-1}I_d\end{pmatrix}+\mathrm{Cov}(Y(s),Y(t)).$$

Then, it follows from Fubini's theorem and (2.26) that

$$\begin{aligned}&\mathbb{E}[(L_{1/(n+p)}(y,I,Y)-L_{1/n}(y,I,Y))^2]\\&\quad=\frac{1}{(2\pi)^{2d}}\int_{I^2}\mathrm{d}s\mathrm{d}t\int_{\mathbb{R}^{2d}}\mathrm{e}^{-\mathrm{i}\langle\xi-\eta,y\rangle}\Big\{\exp\Big(-\frac{1}{2}\,(\xi,\eta)\Gamma_{n+p}(s,t)(\xi,\eta)^{\mathrm{T}}\Big)\\&\qquad+\exp\Big(-\frac{1}{2}\,(\xi,\eta)\Gamma_n(s,t)(\xi,\eta)^{\mathrm{T}}\Big)\\&\qquad-2\exp\Big(-\frac{1}{2}\,(\xi,\eta)\Gamma_{n+p,n}(s,t)(\xi,\eta)^{\mathrm{T}}\Big)\Big\}\mathrm{d}\xi\mathrm{d}\eta\\&\quad=\frac{1}{(2\pi)^{2d}}\int_{I^2}\Big\{\frac{1}{\sqrt{\det\Gamma_{n+p}(s,t)}}\exp\Big(-\frac{1}{2}\,(y,y)\Gamma^{-1}_{n+p}(s,t)(y,y)^{\mathrm{T}}\Big)\\&\qquad+\frac{1}{\sqrt{\det\Gamma_n(s,t)}}\exp\Big(-\frac{1}{2}\,(y,y)\Gamma^{-1}_n(s,t)(y,y)^{\mathrm{T}}\Big)\\&\qquad-\frac{2}{\sqrt{\det\Gamma_{n+p,,n}(s,t)}}\exp\Big(-\frac{1}{2}\,(y,y)\Gamma^{-1}_{n+p,n}(s,t)(y,y)^{\mathrm{T}}\Big)\Big\}\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Similar to (2.31), we can verify that

$$\lim_{n\to+\infty} \mathbb{E}[(L_{1/(n+p)}(y,I,Y) - L_{1/n}(y,I,Y))^2] = 0$$

if and only if (2.27) holds. Then $\{L_\varepsilon(y,I,Y),\ \varepsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega,\mathbb{P})$ if and only if that (2.27) holds. This finishes the proof. $\square$

Now, we provide a sufficient and necessary condition for the existence of the local time of X, which complements [32, Theorem 8.1] and [18, Theorem 3.1].

Theorem 2.6 *Let $X = \{X(t),\ t \in I\}$ be an (N,d)-Gaussian random field defined by* (1.1) *and assume that X_0 has mean zero, continuous covariance function, and satisfies conditions* (C1) *and* (C2) *with index $H \in (0,1)^N$. Then, for every $x \in \mathbb{R}^d$, $L_\varepsilon(x,I,X)$ converges in $L^2(\Omega,\mathbb{P})$ sense, to a limit $L(x,I,X)$ as $\varepsilon \to 0$ if and only if $\sum_{j=1}^N 1/H_j > d$.*

Proof By Lemma 2.5, we only need to verify that for any $x \in \mathbb{R}^d$,

$$\mathscr{M} := \int_{I^2} \exp\Big(-\frac{\|x\|^2 \mathbb{E}[(X_0(s) - X_0(t))^2]}{\det \mathrm{Cov}(X_0(t), X_0(s))}\Big) \frac{\mathrm{d}s\mathrm{d}t}{[\det \mathrm{Cov}(X_0(t), X_0(s))]^{d/2}}$$

is finite if and only if $\sum_{j=1}^N \frac{1}{H_j} > d$.

The sufficiency follows immediately from Proposition 2.4. To prove the necessity, we derive from (2.9), (C1), and (C2) that, for any $\varepsilon_0 \in (0,1)$, there exist constants $c_{11} \geqslant 1$ and $c_{12} > 0$ such that

$$c_{11}^{-1} \leqslant \mathrm{Var}(X_0(s)) \leqslant c_{11}$$

and

$$\mathrm{Var}(X_0(t) \mid X_0(s)) \geqslant c_{12} \sum_{j=1}^N |s_j - t_j|^{2H_j}$$

for all $s,t \in [\varepsilon_0,1]^N$. These inequalities and (2.10) imply

$$\frac{\mathbb{E}[(X_0(s) - X_0(t))^2]}{\det \Gamma_{0,1}(s,t)} \asymp 1, \quad \forall\, s,t \in [\varepsilon_0,1]^N. \tag{2.32}$$

It follows from (2.32) that

$$\mathscr{M} \geqslant c \int_{[\varepsilon_0,1]^{2N}} \frac{\mathrm{d}s\mathrm{d}t}{[\det \mathrm{Cov}(X_0(s), X_0(t))]^{d/2}}.$$

From the proof of Proposition 2.4 with $\gamma = d$ and $\lambda = 0$, we see that the last integral is infinite if $\sum_{j=1}^N \frac{1}{H_j} \leqslant d$. This proves the necessity and hence the theorem. $\square$

In order to study the smoothness of the local times, we will make use of the following lemmas. Lemma 2.7 is from Hu [11], and Lemma 2.8 is from Chen and Yan [4].

Lemma 2.7 *Let $F \in L^2(\Omega, \mathbb{P})$. Then $F \in D_1$ if and only if $\Phi_\Theta(1) < +\infty$.*

Lemma 2.8 *For any $d \in \mathbb{N}$, we have, for $x \in [-1, 1)$,*

$$\sum_{n=1}^{+\infty} \sum_{\substack{0 \leqslant k_1, k_2, \ldots, k_d \leqslant n \\ k_1 + k_2 + \cdots + k_d = n}} \prod_{j=1}^{d} \frac{(2k_j - 1)!!}{(2k_j)!!} 2nx^n \asymp x(1-x)^{-(\frac{d}{2}+1)}.$$

Recall that the Hermite polynomial of degree n is defined by

$$H_n(x) = \frac{(-1)^n}{n!} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2/2}, \quad n \in \mathbb{Z}_+.$$

It is known that (cf. [21]) for any centered Gaussian random vector (ξ, η) with $\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = 1$, we have

$$\mathbb{E}[H_n(\xi) H_m(\eta)] = \begin{cases} 0, & m \neq n, \\ \dfrac{1}{n!} [\mathbb{E}(\xi\eta)]^n, & m = n, \end{cases} \tag{2.33}$$

and for all $z \in \mathbb{C}$ and $x \in \mathbb{R}$,

$$\mathrm{e}^{zx - \frac{z^2}{2}} = \sum_{n=0}^{+\infty} z^n H_n(x). \tag{2.34}$$

We will make use of the following lemma.

Lemma 2.9 *Let $Y = \{(Y_1(t), Y_2(t), \ldots, Y_d(t)),\ t \in I\}$ be an (N, d)-Gaussian random field, where $Y_1, Y_2, \ldots, Y_d$ are d independent copies of a centered, real-valued Gaussian random field Y_0 on I. Suppose that its local time $L(y, I, Y) \in L^2(\Omega, \mathbb{P})$. Then,*

(i) *$L(0, I, Y) \in D_1$ if and only if*

$$\int_{I^2} \frac{[\mathbb{E}(Y_0(s) Y_0(t))]^2}{[\det \mathrm{Cov}(Y_0(t), Y_0(s))]^{\frac{d}{2}+1}} \mathrm{d}s\mathrm{d}t < +\infty; \tag{2.35}$$

(ii) *if (2.35) holds, then $L(y, I, Y) \in D_1$ for every $y \in \mathbb{R}^d \backslash \{0\}$.*

Proof The proof is similar to that of [4, Lemma 3.2], see also [11]. Let $L_\varepsilon(y, I, Y)$ be as in (2.26) (by replacing X by Y). Thanks to (2.26) and (2.34), we can write

$$\begin{aligned} L_\varepsilon(y, I, Y) &= \frac{1}{(2\pi)^d} \int_I \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \xi, y \rangle} \exp\Big(\mathrm{i}\langle \xi, Y(s) \rangle - \varepsilon \frac{\|\xi\|^2}{2}\Big) \mathrm{d}\xi \mathrm{d}s \\ &= \frac{1}{(2\pi)^d} \int_I \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \xi, y \rangle} \exp\Big(-\frac{1}{2}(\mathbb{E}(Y_0^2(s)) + \varepsilon)\|\xi\|^2\Big) \\ &\quad \times \sum_{n=0}^{+\infty} \mathrm{i}^n (\mathbb{E}(Y_0^2(s))\|\xi\|^2)^{n/2} H_n\Big(\frac{\langle \xi, Y(s) \rangle}{\sqrt{\mathbb{E}(Y_0^2(s))\|\xi\|^2}}\Big) \mathrm{d}\xi \mathrm{d}s \\ &=: \sum_{n=0}^{+\infty} F_n^{y,\varepsilon}. \end{aligned} \tag{2.36}$$

Denote

$$\Phi_{\Theta_{y,\varepsilon}}(u) = \mathbb{E}(|\Gamma_{\sqrt{u}} L_\varepsilon(y, I, Y)|^2), \quad \Phi_{\Theta_y}(u) = \mathbb{E}(|\Gamma_{\sqrt{u}} L(y, I, Y)|^2).$$

Also, for simplicity of notation, let

$$a^2 = \mathbb{E}(Y_0^2(s)) + \varepsilon, \quad b^2 = \mathbb{E}(Y_0^2(t)) + \varepsilon.$$

It follows from (2.36) and (2.33) that

$$\begin{aligned}
\Phi_{\Theta_{y,\varepsilon}}(1) &= \sum_{n=0}^{+\infty} n\mathbb{E}(|F_n^{y,\varepsilon}|^2) \\
&= \sum_{n=0}^{+\infty} \frac{n}{(2\pi)^{2d}} \mathbb{E}\Big[\int_{I^2}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}\langle \xi-\eta, y\rangle} [\mathbb{E}(Y_0^2(s))\mathbb{E}(Y_0^2(t))\|\xi\|^2\|\eta\|^2]^{n/2} \\
&\quad \times \exp\Big(-\frac{1}{2}[a^2\|\xi\|^2 + b^2\|\eta\|^2]\Big) \\
&\quad \times H_n\Big(\frac{\langle \xi, Y(s)\rangle}{\sqrt{\mathbb{E}(Y_0^2(s))\|\xi\|^2}}\Big) H_n\Big(\frac{\langle \eta, Y(t)\rangle}{\sqrt{\mathbb{E}(Y_0^2(t))\|\eta\|^2}}\Big) \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}s\mathrm{d}t\Big] \\
&= \sum_{n=1}^{+\infty} \frac{1}{(2\pi)^{2d}(n-1)!} \int_{I^2} [\mathbb{E}(Y_0(s)Y_0(t))]^n \mathrm{d}s\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}\langle \xi-\eta, y\rangle} \langle \xi, \eta\rangle^n \exp\Big(-\frac{1}{2}[a^2\|\xi\|^2 + b^2|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta. \qquad (2.37)
\end{aligned}$$

If $y = 0$, then the integrals in (2.37) become 0 for all odd numbers n. Hence,

$$\begin{aligned}
\Phi_{\Theta_{0,\varepsilon}}(1) &= \sum_{n=1}^{+\infty} \frac{1}{(2\pi)^{2d}(2n-1)!} \int_{I^2} [\mathbb{E}(Y_0(s)Y_0(t))]^{2n} \mathrm{d}s\mathrm{d}t \\
&\quad \times \int_{\mathbb{R}^{2d}} \langle \xi, \eta\rangle^{2n} \exp\Big(-\frac{1}{2}[a^2\|\xi\|^2 + b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta. \qquad (2.38)
\end{aligned}$$

By using the fact that for $k \in \mathbb{Z}_+$, $\gamma > 0$,

$$\int_{\mathbb{R}} v^{2k} \exp\Big(-\frac{\gamma v^2}{2}\Big)\mathrm{d}v = \sqrt{2\pi}\,(2k-1)!!\,\gamma^{-(k+\frac{1}{2})}$$

and the same argument as [4, p. 1010], we obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{\langle \xi, \eta\rangle^{2n}}{(2n-1)!} \exp\Big(-\frac{1}{2}[a^2\|\xi\|^2 + b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta \\
&= \sum_{\substack{0\leqslant k_1,k_2,\dots,k_d\leqslant n \\ k_1+k_2+\cdots+k_d=n}} \prod_{j=1}^{d} \frac{(2k_j-1)!!}{(2k_j)!!} \frac{2n}{[(\mathbb{E}(Y_0^2(s))+\varepsilon)(\mathbb{E}(Y_0^2(t))+\varepsilon)]^{n+\frac{d}{2}}}. \qquad (2.39)
\end{aligned}$$

(This can be verified by using induction.) Combining (2.38) and (2.39), and applying Lemma 2.8 and the monotone convergence theorem, we obtain

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\frac{1}{(2n-1)!}\int_{I^2}[\mathbb{E}(Y_0(s)Y_0(t))]^{2n}\,\mathrm{d}s\mathrm{d}t\int_{\mathbb{R}^{2d}}\langle\xi,\eta\rangle^{2n}\\
&\qquad\times\exp\Big(-\frac{1}{2}\,[a^2\|\xi\|^2+b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta\\
&\asymp\int_{I^2}\frac{[\mathbb{E}(Y_0(s)Y_0(t))]^2}{\{a^2b^2-[\mathbb{E}(Y_0(s)Y_0(t))]^2\}^{\frac{d}{2}+1}}\,\mathrm{d}s\mathrm{d}t\\
&\overset{\text{as }\varepsilon\to0}{\longrightarrow}\int_{I^2}\frac{[\mathbb{E}(Y_0(s)Y_0(t))]^2}{\{\mathbb{E}(Y_0^2(s))\mathbb{E}(Y_0^2(t))-[\mathbb{E}(Y_0(s)Y_0(t))]^2\}^{\frac{d}{2}+1}}\,\mathrm{d}s\mathrm{d}t\\
&=\int_{I^2}\frac{[\mathbb{E}(Y_0(t)Y_0(s))]^2}{[\det\mathrm{Cov}(Y_0(s),Y_0(t))]^{\frac{d}{2}+1}}\,\mathrm{d}s\mathrm{d}t,
\end{aligned}\tag{2.40}$$

which proves part (i) of the lemma, thanks to Lemma 2.7.

Now, we prove part (ii) of the lemma. Notice that for $y\in\mathbb{R}^d\backslash\{0\}$, it does not seem easy to compute the integrals in the last equality of (2.37) explicitly. So we turn to the following upper bound:

$$\begin{aligned}
\Phi_{\Theta_{y,\varepsilon}}(1)\leqslant&\sum_{n=1}^{+\infty}\frac{1}{(2\pi)^{2d}(n-1)!}\int_{I^2}|\mathbb{E}(Y_0(s)Y_0(t))|^n\mathrm{d}s\mathrm{d}t\\
&\times\int_{\mathbb{R}^{2d}}|\langle\xi,\eta\rangle|^n\exp\Big(-\frac{1}{2}\,[a^2\|\xi\|^2+b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}\tag{2.41}$$

The sum over even integers in (2.41) is the same as in (2.40). So we only need to consider the terms over odd integers. For this purpose, let

$$\begin{aligned}
J_{2n+1}=&\frac{1}{(2n)!}\int_{I^2}|\mathbb{E}(Y_0(s)Y_0(t))|^{2n+1}\mathrm{d}s\mathrm{d}t\\
&\times\int_{\mathbb{R}^{2d}}|\langle\xi,\eta\rangle|^{2n+1}\exp\Big(-\frac{1}{2}\,[a^2\|\xi\|^2+b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}\tag{2.42}$$

By using the Cauchy-Schwarz inequality and the elementary inequality

$$x\mathrm{e}^{-\frac{\beta}{2n}x^2}\leqslant\sqrt{\frac{n}{\mathrm{e}\beta}},\quad\forall\,\beta,x>0,$$

we have

$$|\langle\xi,\eta\rangle|\mathrm{e}^{-\frac{1}{2n}[a^2\|\xi\|^2+b^2\|\eta\|^2]}\leqslant\frac{n}{\mathrm{e}ab}.$$

Plugging this into (2.42) yields

$$\begin{aligned}
J_{2n+1}\leqslant&\frac{1}{2\mathrm{e}(2n-1)!}\int_{I^2}|\mathbb{E}(Y_0(s)Y_0(t))|^{2n}\mathrm{d}s\mathrm{d}t\\
&\times\int_{\mathbb{R}^{2d}}|\langle\xi,\eta\rangle|^{2n}\exp\Big(-\frac{n-1}{2n}\,[a^2\|\xi\|^2+b^2\|\eta\|^2]\Big)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}\tag{2.43}$$

The same argument for (2.39) gives

$$\int_{\mathbb{R}^{2d}} \frac{|\langle \xi, \eta\rangle|^{2n}}{(2n-1)!} \exp\Big(-\frac{n-1}{2n}[a^2\|\xi\|^2 + b^2\|\eta\|^2]\Big) \mathrm{d}\xi \mathrm{d}\eta$$
$$= \frac{(2\pi)^d}{(1-n^{-1})^{n+\frac{d}{2}}} \sum_{\substack{0\leqslant k_1,k_2,\dots,k_d\leqslant n\\ k_1+k_2+\cdots+k_d=n}} \prod_{j=1}^{d} \frac{(2k_j-1)!!}{(2k_j)!!} \frac{2n}{(a^2b^2)^{n+\frac{d}{2}}}. \tag{2.44}$$

Combining (2.42)–(2.44) with (2.41), and using the same argument as in (2.40), we derive $\Phi_{\Theta_y}(1) < +\infty$ under (2.35). This finishes the proof of part (ii). □

The following is the main theorem of this section.

Theorem 2.10 *Let $X = \{X(t),\, t \in I\}$ be an (N, d)-Gaussian field defined by* (1.1), *and assume that X_0 satisfies* (C1) *and* (C2) *with index $H \in (0,1)^N$. Then the following statements hold*:

(i) $L(0, I, X) \in D_1$ *if and only if* $\sum_{j=1}^{N} \frac{1}{H_j} > d + 2$;

(ii) *if* $\sum_{j=1}^{N} \frac{1}{H_j} > d + 2$, *then* $L(x, I, X) \in D_1$ *for every* $x \in \mathbb{R}^d \backslash \{0\}$.

Proof By Theorem 2.6 and Lemma 2.9, it is sufficient for us to verify that

$$\int_{I^2} \frac{[\mathbb{E}(X_0(s)X_0(t))]^2}{[\det \mathrm{Cov}(X_0(t), X_0(s))]^{\frac{d}{2}+1}} \mathrm{d}s\mathrm{d}t < +\infty \tag{2.45}$$

if and only if $\sum_{j=1}^{N} \frac{1}{H_j} > d+2$. This follows from Proposition 2.4 with $\gamma = d+2$ and $\lambda = 2$ immediately. □

Remark 2.11 As we mentioned in Section 1, the class of Gaussian random fields that satisfy (C1) and (C2) is large, including fractional Brownian sheets and the solutions to stochastic heat equation driven by various space-time Gaussian noises. In particular, Theorem 2.10 recovers [7, Theorem 11] and [9, Theorem 2.1] with $\alpha = 1$.

In the following, we apply Theorems 2.6 and 2.10 to study the collision and intersection local times of independent Gaussian fields. Theorems 2.12 and 2.13 below generalize the results of [4,34,35] for fractional Brownian motion and related Gaussian processes.

2.2 Smoothness of collision local time

Given

$$H = (H_1, H_2, \dots, H_N) \in (0,1)^N, \quad K = (K_1, K_2, \dots, K_N) \in (0,1)^N,$$

let

$$X^H = \{X^H(s),\, s \in \mathbb{R}^N\}, \quad X^K = \{X^K(t),\, t \in \mathbb{R}^N\}$$

be two independent Gaussian random fields with values in $\mathbb{R}^d$ as defined in (1.1). We assume that the associate real-valued random fields X_0^H and X_0^K

satisfy conditions (C1) and (C2) on interval $I \subseteq \mathbb{R}^N$, respectively, with indices H and with indices K.

The collision local time of X^H and X^K on I is formally defined by

$$L_C(X^H, X^K) := \int_I \delta(X^H(s) - X^K(s))\mathrm{d}s. \tag{2.46}$$

Theorem 2.12 *Let $L_C(X^H, X^K)$ be the collision local time of X^H and X^K as above. Then*

(i) *$L_C(X^H, X^K) \in L^2(\Omega, \mathbb{P})$ if and only if $\sum_{j=1}^N \frac{1}{H_j \wedge K_j} > d$;*

(ii) *$L_C(X^H, X^K) \in D_1$ if and only if $\sum_{j=1}^N \frac{1}{H_j \wedge K_j} > d + 2$.*

Proof Consider the (N, d) Gaussian field $Z = \{Z(t),\ t \in I\}$ defined by

$$Z(t) := X^H(t) - X^K(t), \quad \forall\, t \in I.$$

Then the collision local time of X^H and X^K on I is nothing but $L(0, I, Z)$, the local time of Z on I at $x = 0$. Hence, the theorem follows from Theorems 2.6 and 2.10 once we verify that the real-valued Gaussian field $Z_0(t) = X_0(t) - Y_0(t)$ satisfies (C1) and (C2) in the interval I with indices

$$(H_1 \wedge K_1, H_2 \wedge K_2, \ldots, H_N \wedge K_N) \in (0, 1)^N.$$

Since it is easy to show that Z_0 satisfies (C1), we verify (C2) only. By the definition of conditional variance and independence of X^H and X^K, we have

$$\begin{aligned}
&\mathrm{Var}(Z_0(t) \mid Z_0(s)) \\
&\quad = \inf_{a\in\mathbb{R}} \mathbb{E}[(X_0^H(t) - aX_0^H(s))^2 + (X_0^K(t) - aX_0^K(s))^2] \\
&\quad \geqslant \inf_{a\in\mathbb{R}} \mathbb{E}[(X_0^H(t) - aX_0^H(s))^2] + \inf_{b\in\mathbb{R}} \mathbb{E}[(X_0^K(t) - bX_0^K(s))^2] \\
&\quad = \mathrm{Var}(X_0^H(t) \mid X_0^H(s)) + \mathrm{Var}(X_0^K(t) \mid X_0^K(s)) \\
&\quad \geqslant c\Big(\sum_{j=1}^N \min\{|s_j - t_j|^{2H_j}, t_j^{2H_j}\} + \sum_{j=1}^N \min\{|s_j - t_j|^{2K_j}, t_j^{2K_j}\}\Big) \\
&\quad \geqslant c\sum_{j=1}^N \min\{|s_j - t_j|^{2(H_j\wedge K_j)}, t_j^{2(H_j\wedge K_j)}\}, \quad \forall\, s, t \in I,
\end{aligned}$$

for some constant $c > 0$. This verifies that Z_0 satisfies condition (C2). □

2.3 Smoothness of intersection local time

Let

$$H = (H_1, H_2, \ldots, H_{N_1}) \in (0, 1)^{N_1}, \quad K = (K_1, K_2, \ldots, K_{N_2}) \in (0, 1)^{N_2}$$

be two constant vectors. Let

$$X^H = \{X^H(s),\ s \in \mathbb{R}^{N_1}\}, \quad X^K = \{X^K(t),\ t \in \mathbb{R}^{N_2}\}$$

be two independent Gaussian random fields with values in $\mathbb{R}^d$ as defined in (1.1). We assume that the associate real-valued random fields X_0^H and X_0^K satisfy conditions (C1) and (C2), respectively, on interval $I_1 \subseteq \mathbb{R}^{N_1}$ with indices $H = (H_1, H_2, \ldots, H_{N_1})$ and on $I_2 \subseteq \mathbb{R}^{N_2}$ with indices $K = (K_1, K_2, \ldots, K_{N_2})$. Then the intersection local time of X^H and X^K is formally defined by

$$L_I(X^H, X^K) := \int_{I_{N_1} \times I_{N_2}} \delta(X^H(s) - X^K(t)) \mathrm{d}s\mathrm{d}t. \tag{2.47}$$

Theorem 2.13 *Let $L_I(X^H, X^K)$ be the intersection local time of X^H and X^K as above. Then*

(i) $L_I(X^H, X^K) \in L^2(\Omega, \mathbb{P})$ *if and only if*

$$\sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j} > d;$$

(ii) $L_I(X^H, X^K) \in D_1$ *if and only if*

$$\sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j} > d + 2.$$

Proof Let

$$U = \{U(s,t), \, (s,t) \in I_{N_1} \times I_{N_2}\}$$

be the $(N_1 + N_2, d)$-Gaussian random field with mean 0 defined by

$$U(s,t) = X^H(s) - X^K(t), \quad \forall\, s \in I_{N_1}, \forall\, t \in I_{N_2}.$$

Clearly, the intersection local time of X^H and X^K is nothing but $L(0, I_{N_1} \times I_{N_2}, U)$, the local time of U on $I_{N_1} \times I_{N_2}$ at $x = 0$. One can verify that the Gaussian random field

$$U_0(s,t) = X_0^H(s) - X_0^K(t)$$

satisfies conditions (C1) and (C2) on the interval $I_{N_1} \times I_{N_2}$ with indices

$$(H_1, H_2, \ldots, H_{N_1}, K_1, K_2, \ldots, K_{N_2}) \in (0,1)^{N_1 + N_2}.$$

Therefore, the conclusions follow from Theorems 2.6 and 2.10. □

3 Self-intersection local times

In this section, we study the existence and smoothness of self-intersection local times of an (N, d)-Gaussian random field $X = \{X(t), \, t \in \mathbb{R}^N\}$ as in (1.1). These problems are more involved than the collision or intersection local times of independent Gaussian random fields, due to complexity of dependence structures of X. For earlier results for the Brownian sheet, fractional Brownian motion and related self-similar Gaussian processes, we refer to [11,16–18]. Their methods rely on special properties of the Brownian sheet or fractional Brownian motion. Our approach below is based on a weak form of local nondeterminism and is more general.

For any two compact intervals $I, J \subseteq \mathbb{R}^N$, the self-intersection local times of $X = \{X(t),\, t \in \mathbb{R}^N\}$ on I and J is formally defined by

$$L_S(X, I \times J) = \int_{I\times J} \delta(X(s) - X(t))\mathrm{d}s\mathrm{d}t. \tag{3.1}$$

Define a $(2N, d)$-Gaussian random field

$$V = \{V(s,t),\, (s,t) \in \mathbb{R}^{2N}\}$$

by

$$V(s,t) := X(s) - X(t), \quad s, t \in \mathbb{R}^N. \tag{3.2}$$

Then the self-intersection local time of X is $L(0, I \times J, V)$, the local time of V on $I \times J$ at $x = 0$.

Under the condition that X_0 satisfies conditions (C1) and (C2) on both intervals I and J, the Gaussian field

$$V_0(s,t) = X_0(s) - X_0(t)$$

may not satisfy the corresponding (C2) on $I \times J$. Therefore, we cannot apply Theorems 2.6 and 2.10 directly. To overcome this difficulty, we will make use of the following condition:

(C3) there exists a positive constant c_{12} such that for all $u, t^1, t^2, t^3 \in [0,1]^N$,

$$\mathrm{Var}(X_0(u) \mid X_0(t^1), X_0(t^2), X_0(t^3)) \geqslant c_{12} \sum_{j=1}^N \min_{0\leqslant k\leqslant 3} |u_j - t_j^k|^{2H_j}, \tag{3.3}$$

where $t_j^0 = 0$, $j = 1, 2, \ldots, N$.

Clearly, condition (C2) is a special case of condition (C3). It is known that multiparameter fractional Brownian motion and fractional Brownian sheets satisfy conditions (C1) and (C3); see [23,29]. More examples can be found in [32].

For two compact intervals $I, J \subseteq [0,1]^N$, we call them *separated* if

$$\inf_{s\in I,\, t\in J} |s_j - t_j| > 0 \quad \text{for some } j = 1, 2, \ldots, N. \tag{3.4}$$

Let $S \subseteq \{1, 2, \ldots, N\}$ be the collection of all j's that satisfy (3.4) and let $S^c = \{1, 2, \ldots, N\} \setminus S$. Because I and J are compact, there exists $\varepsilon_0 > 0$ such that

$$\inf_{s\in I,\, t\in J} |s_j - t_j| \geqslant \varepsilon_0, \quad j \in S. \tag{3.5}$$

We further call I and J *partially separated* if both S and S^c are nonempty, *well separated* if $S^c = \emptyset$, and *not separated* if $S = \emptyset$. Clearly, I and J are not separated if and only if $I \cap J \neq \emptyset$.

Similar to Imkeller and Weisz [17] for the Brownian sheet, we consider the self-intersection local times of X on I and J by distinguishing three cases:

Case 1 $I, J \subseteq [0,1]^N$ are well separated;

Case 2 $I, J \subseteq [0,1]^N$ are partially separated;

Case 3 $I, J \subseteq [0,1]^N$ are not separated.

In Case 1, we have the following theorem.

Theorem 3.1 *Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be an (N,d)-Gaussian random field defined by* (1.1) *with X_0 satisfying conditions* (C1) *and* (C3), *and let $L_S(X, I \times J)$ be the self-intersection local time of X on I and J. If I and J are well separated, then the following statements hold*:

(i) $L_S(X, I \times J) \in L^2(\Omega, \mathbb{P})$ *if and only if* $2\sum_{j=1}^N \frac{1}{H_j} > d$;

(ii) $L_S(X, I \times J) \in D_1$ *if and only if* $2\sum_{j=1}^N \frac{1}{H_j} > d + 2$.

Proof Since the Gaussian field X_0 satisfies (C1) on I and J, we see that for any $(s,t), (s',t') \in I \times J$,

$$\mathbb{E}[(V_0(s,t) - V_0(s',t'))^2] \leqslant c\biggl(\sum_{j=1}^N |s_j - s_j'|^{2H_j} + \sum_{j=1}^N |t_j - t_j'|^{2H_j}\biggr). \tag{3.6}$$

Thus, the Gaussian field

$$V_0(s,t) = X_0^H(s) - X_0^H(t)$$

satisfies (C1) on $I \times J$ with indices $(H_1, H_2, \ldots, H_N, H_1, H_2, \ldots, H_N) \in (0,1)^{2N}$. To verify that V_0 also satisfies (C2), we see that (C3) implies

$$\begin{aligned}\mathrm{Var}(V_0(s,t) \mid V_0(s',t')) &\geqslant \mathrm{Var}(X_0(t) \mid X_0(s), X_0(s'), X_0(t')) \\ &\geqslant c_{12} \sum_{j=1}^N \min\{|t_j - s_j|^{2H_j}, |t_j - s_j'|^{2H_j}, |t_j - t_j'|^{2H_j}, t_j^{2H_j}\} \\ &\geqslant c_{13} \sum_{j=1}^N \min\{|t_j - t_j'|^{2H_j}, t_j^{2H_j}\}\end{aligned}$$

thanks to the facts that $|t_j - s_j| \geqslant \varepsilon_0$ and $|t_j - s_j'| \geqslant \varepsilon_0$. Here, the constant c_{13} depends on ε_0. By the same token, we have

$$\mathrm{Var}(V_0(s,t) \mid V_0(s',t')) \geqslant c_{13} \sum_{j=1}^N \min\{|s_j - s_j'|^{2H_j}, s_j^{2H_j}\}.$$

Adding up these two inequalities shows

$$\begin{aligned}&\mathrm{Var}(V_0(s,t)\mid V_0(s',t'))\\&\geqslant \frac{c_{13}}{2}\bigg(\sum_{j=1}^{N}\min\{|s_j-s_j'|^{2H_j},s_j^{2H_j}\}+\sum_{j=1}^{N}\min\{|t_j-t_j'|^{2H_j},t_j^{2H_j}\}\bigg).\end{aligned}$$

This proves that V_0 satisfies (C2) on $I\times J$ with

$$(H_1,H_2,\ldots,H_N,H_1,H_2,\ldots,H_N)\in(0,1)^{2N}.$$

Therefore, the conclusions follow from Theorems 2.6 and 2.10. □

Now, we consider Case 2, e.g., the two compact intervals I and J are partially separated. In this case, both S and S^c are nonempty sets. For concreteness, we may assume that

$$I=[a,a+\langle h\rangle],\quad J=[b,b+\langle h\rangle],$$

where $b_j>a_j+h$ for $j\in S$ and $a_j=b_j$ for $j\in S^c$. Then (3.5) holds with

$$\varepsilon_0=\min\{b_j-a_j-h,\ j\in S\}.$$

Note that, when X is the (N,d) Brownian sheet, the existence condition in (i) in the following theorem coincides with that in Theorem 3 of [16,17].

Theorem 3.2 *Let $X=\{X(t),\ t\in\mathbb{R}^N\}$ be an (N,d)-Gaussian random field as in Theorem* 3.1. *Let I and J be partially separated as described above. Then the following statements hold*:

(i) $L_S(X,I\times J)\in L^2(\Omega,\mathbb{P})$ *if*

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d;$$

(ii) $L_S(X,I\times J)\notin L^2(\Omega,\mathbb{P})$ *if*

$$2\sum_{j=1}^{N}\frac{1}{H_j}\leqslant d;$$

(iii) $L_S(X,I\times J)\in D_1$ *if*

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d+2;$$

(iv) $L_S(X,I\times J)\notin D_1$ *if*

$$2\sum_{j=1}^{N}\frac{1}{H_j}\leqslant d+2.$$

Proof We prove part (i) at first. By Lemma 2.5, we only need to prove that if

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d,$$

then

$$\mathscr{J}:=\int_{(I\times J)^2}\frac{\mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'}{[\det\mathrm{Cov}(V_0(s,t),V_0(s',t'))]^{d/2}}<+\infty. \tag{3.7}$$

By the definition of conditional variance and (C3), we see that for any (s,t), $(s',t')\in I\times J$,

$$\begin{aligned}&\mathrm{Var}(V_0(s,t)\mid V_0(s',t'))\\&\quad\geqslant\mathrm{Var}(X_0(t)\mid X_0(s),X_0(s'),X_0(t'))\\&\quad\geqslant c_{12}\sum_{j=1}^{N}\min\{|t_j-s_j|^{2H_j},|t_j-s_j'|^{2H_j},|t_j-t_j'|^{2H_j},t_j^{2H_j}\}\\&\quad\geqslant c_{14}\bigg(\sum_{j\in S}\min\{|t_j-t_j'|^{2H_j},t_j^{2H_j}\}\\&\qquad+\sum_{j\in S^c}\min\{|t_j-s_j|^{2H_j},|t_j-s_j'|^{2H_j},|t_j-t_j'|^{2H_j},t_j^{2H_j}\}\bigg),\end{aligned} \tag{3.8}$$

thanks to the fact that if $j\in S$, then $|t_j-s_j|\geqslant\varepsilon_0$ and $|t_j-s_j'|\geqslant\varepsilon_0$. By the same token, we have

$$\begin{aligned}&\mathrm{Var}(V_0(s,t)\mid V_0(s',t'))\\&\quad\geqslant c_{14}\bigg(\sum_{j\in S}\min\{|s_j-s_j'|^{2H_j},s_j^{2H_j}\}\\&\qquad+\sum_{j\in S^c}\min\{|s_j-t_j|^{2H_j},|s_j-s_j'|^{2H_j},|s_j-t_j'|^{2H_j},s_j^{2H_j}\}\bigg).\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\begin{aligned}&\mathrm{Var}(V_0(s,t)\mid V_0(s',t'))\\&\quad\geqslant c_{14}\bigg[\sum_{j\in S}(\min\{|t_j-t_j'|^{2H_j},t_j^{2H_j}\}+\min\{|s_j-s_j'|^{2H_j},s_j^{2H_j}\})\\&\qquad+\sum_{j\in S^c}\min\{|s_j-t_j|^{2H_j},|s_j-s_j'|^{2H_j},|s_j-t_j'|^{2H_j},s_j^{2H_j}\}\bigg].\end{aligned} \tag{3.10}$$

Note that, in (3.10), only one sum over S^c in (3.8) and (3.9) is kept. This is due to the fact that, when integrating $\mathrm{d}s_j$ for $j\in S^c$, all the other variables, s_j', t_j, and t_j', will disappear (see Lemma 2.2). This situation is different from the case when we integrate $\mathrm{d}s_j$ for $j\in S$.

Since I and J are partially separated (i.e., $S \neq \emptyset$), we have

$$\mathrm{Var}(V_0(s',t')) = \mathbb{E}[(X_0(s') - X_0(t'))^2] \asymp 1, \quad \forall\, s' \in I, \forall\, t' \in J. \tag{3.11}$$

It follows from (3.10) and (3.11) that the integral $\mathscr{J}$ in (3.7) is at most

$$\int_{(I\times J)^2} \bigg[\sum_{j\in S}(\min\{|t_j - t'_j|^{H_j}, t_j^{H_j}\} + \min\{|s_j - s'_j|^{H_j}, s_j^{H_j}\}) \\ + \sum_{j\in S^c} \min\{|s_j - t_j|^{H_j}, |s_j - s'_j|^{H_j}, |s_j - t'_j|^{H_j}, s_j^{H_j}\}\bigg]^{-d} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'. \tag{3.12}$$

Similar to the argument in the proofs of (2.13) and (2.22), we integrate iteratively and apply Lemmas 2.1–2.3 to show that the integral in (3.12) is finite if

$$2\sum_{j\in S}\frac{1}{H_j} + \sum_{j\in S^c}\frac{1}{H_j} > d.$$

This proves the sufficiency in part (i).

Next, we prove part (ii). For any $(s,t), (s',t') \in I \times J$, condition (C1) implies that

$$\begin{aligned}\mathrm{Var}(V_0(s,t)|V_0(s',t')) &\leqslant \mathbb{E}[(X_0(s) - X_0(t) - X_0(s') + X_0(t'))^2] \\ &\leqslant c\sum_{j=1}^{N}(|s_j - s'_j|^{2H_j} + |t_j - t'_j|^{2H_j}).\end{aligned} \tag{3.13}$$

It follows from (3.11) and (3.13) that

$$\det \mathrm{Cov}(V_0(s,t), V_0(s',t')) \leqslant c\sum_{j=1}^{N}(|s_j - s'_j|^{2H_j} + |t_j - t'_j|^{2H_j}). \tag{3.14}$$

This implies, by using Lemma 2.1 repeatedly, that the integral $\mathscr{J}$ in (3.7) is infinite if $2\sum_{j=1}^{N}\frac{1}{H_j} \leqslant d$.

In order to prove part (iii), by Lemma 2.9, it suffices to show that, if

$$2\sum_{j\in S}\frac{1}{H_j} + \sum_{j\in S^c}\frac{1}{H_j} > d + 2,$$

then

$$\mathscr{K} = \int_{(I\times J)^2} \frac{[\mathbb{E}(V_0(s,t)V_0(s',t'))]^2}{[\det \mathrm{Cov}(V_0(s,t), V_0(s',t'))]^{(d+2)/2}}\, \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' < +\infty. \tag{3.15}$$

For any $(s,t), (s',t') \in I \times J$, we use the Cauchy-Schwarz inequality and (C1) again to show that

$$[\mathbb{E}(V_0(s,t)V_0(s',t'))]^2 \leqslant \mathbb{E}[V_0^2(s,t)]\mathbb{E}[V_0^2(s',t')] \leqslant c. \tag{3.16}$$

Similar to the argument in (3.11) and (3.12), we derive from (3.16) that the integral $\mathscr{K}$ in (3.15) is, up to a constant, bounded from above by

$$\int_{(I\times J)^2}\Bigg[\sum_{j\in S}(\min\{|t_j-t_j'|^{H_j},t_j^{H_j}\}+\min\{|s_j-s_j'|^{H_j},s_j^{H_j}\})$$
$$+\sum_{j\in S^c}\min\{|s_j-s_j|^{H_j},|s_j-s_j'|^{H_j},|s_j-t_j'|^{H_j},s_j^{H_j}\}\Bigg]^{-(d+2)}\mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'. \quad (3.17)$$

Again, exactly like what we did in the proof of (2.13), we can show that the integral in (3.17) is finite provided

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d+2.$$

This proves part (iii).

Since the function $\mathbb{E}(V_0(s,t)V_0(s',t'))$ is uniformly continuous for $(s,t,s',t')\in(I\times J)^2$ and (3.11) holds, there exist positive constants δ and c such that

$$\mathbb{E}(V_0(s,t)V_0(s',t'))\geqslant c$$

for all $(s,t,s',t')\in(I\times J)^2$ stisfying $|(s,t)-(s',t')|\leqslant\delta$. Hence, the proof of part (iv) is quite similar to the proof of part (ii). We leave the details to the interested reader. □

Parts (ii) and (iv) in Theorem 3.2 can be improved if we have more information on the dependence structure of $V_0(s,t)=X_0(s)-X_0(t)$, as shown by the following theorem.

Theorem 3.3 *If, in addition to the assumptions of Theorem* 3.2, X_0 *satisfies the following condition*:

(C4) *there exists a positive constant* c_{15} *such that for all* $(s,t),(s',t')\in I\times J$,

$$\begin{aligned}&\mathrm{Var}(X_0(s)-X_0(t)\mid X_0(s')-X_0(t'))\\&\quad\leqslant c_{15}\bigg(\sum_{j\in S}(|t_j-t_j'|^{2H_j}+|s_j-s_j'|^{2H_j})+\sum_{j\in S^c}|s_j-t_j|^{2H_j}\bigg),\end{aligned} \quad (3.18)$$

then the following statements hold:

(i) $L_S(X^H,I\times J)\in L^2(\Omega,\mathbb{P})$ *if and only if*

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d;$$

(ii) $L_S(X^H,I\times J)\in D_1$ *if and only if*

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}>d+2.$$

Remark 3.4 Observe that condition (C4) is automatically satisfied if $S=\emptyset$. If $X_0=\{X_0(t),\, t\in\mathbb{R}^N\}$ is an 'additive fractional Brownian motion' defined by

$$X_0(t)=B^{H_1}(t_1)+B^{H_2}(t_2)+\cdots+B^{H_N}(t_N),\quad \forall\, t\in\mathbb{R}^N,$$

where $B^{H_1},B^{H_2},\dots,B^{H_N}$ are independent fractional Brownian motions with indices $H_1,H_2,\dots,H_N$, respectively. Then it is easy to see that condition (C4) is satisfied. If X_0 is the Brownian sheet, then by using the independence of increments over intervals, one can check that (C4) also holds.

Proof of Theorem 3.3 Sufficiencies of the conditions in (i) and (ii) have been proved in Theorem 3.2. Note that (C4) and (3.11) imply

$$\mathscr{J}\geqslant\int_{(I\times J)^2}\frac{\mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'}{[\sum_{j\in S}(|s_j-s_j'|^{H_j}+|t_j-t_j'|^{H_j})+\sum_{j\in S^c}|s_j-t_j|^{H_j}]^d}. \tag{3.19}$$

By applying Lemma 2.1, it can be verified that the last integral diverges when

$$2\sum_{j\in S}\frac{1}{H_j}+\sum_{j\in S^c}\frac{1}{H_j}\leqslant d.$$

This proves the necessity in (i). The proof of necessity in (ii) is similar and is omitted. □

Finally, we consider Case 3, e.g., the two compact intervals I and J are not separated in any direction. So $S=\emptyset$. Compared with Case 2, we note that, on one hand, (3.11) fails and, on the other hand, condition (C4) holds automatically. For concreteness, we assume that $I=J=[0,1]^N$.

Theorem 3.5 *Let $X=\{X(t),\, t\in\mathbb{R}^N\}$ be an (N,d)-Gaussian random field as in Theorem* 3.1. *Then the following statements hold*:

(i) $L_S(X,I\times I)\in L^2(\Omega,\mathbb{P})$ *if and only if* $\sum_{j=1}^N\frac{1}{H_j}>d$;

(ii) $L_S(X,I\times I)\in D_1$ *if* $\sum_{j=1}^N\frac{1}{H_j}>d+2$;

(iii) $L_S(X^H,I\times I)\notin D_1$ *if*

$$\sum_{j=1}^N\frac{1}{H_j}\leqslant\max\Big\{\frac{d+2}{2},\frac{2d}{3}\Big\}. \tag{3.20}$$

Before proving this theorem, we compare its conditions with the results in [16,17,11].

Remark 3.6 (a) If X is the (N,d) Brownian sheet, then our existence condition in (i) coincides with that in Theorem 1 of [16,17]. When X is a fractional Brownian motion $B^H=\{B^H(t),\, t\in\mathbb{R}\}$, our condition in (ii) becomes $H(d+2)<1$, which is stronger than that in [11, Theorem 3.2].

(b) Little has been known about optimal necessary condition for $L_S(X, I\times I)\in D_1$ for a Gaussian random field X. Our condition (3.20) is the first general result of this kind. When X is a fractional Brownian motion B^H, (3.20) becomes

$$H \geqslant \min\Big\{\frac{3}{2d}, \frac{2}{d+2}\Big\},$$

which is the complement of the sufficient condition of [11, Theorem 3.2]. Hence, we have proven that, for fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}\}$ in $\mathbb{R}^d$ and $I = [0,1]$, $L_S(B^H, I \times I) \in D_1$ if and only if

$$H < \min\Big\{\frac{3}{2d}, \frac{2}{d+2}\Big\}.$$

Proof of Theorem 3.5 We prove part (i) at first. Notice that by Lemma 2.5, we only need to prove that

$$\mathscr{J} = \int_{I^4} \frac{\mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'}{[\det \mathrm{Cov}(V_0(s,t), V_0(s',t'))]^{d/2}} < +\infty \tag{3.21}$$

if and only if $\sum_{j=1}^N \frac{1}{H_j} > d$. For any $(s,t), (s',t') \in I \times I$, we use condition (C1) to obtain that

$$\begin{aligned}\det \mathrm{Cov}(V_0(s,t), V_0(s',t')) &\leqslant \mathbb{E}(V_0^2(s,t))\mathbb{E}(V_0^2(s',t')) \\ &\leqslant c\Big(\sum_{j=1}^N |s_j - t_j|^{2H_j}\Big)\Big(\sum_{j=1}^N |s_j' - t_j'|^{2H_j}\Big). \end{aligned}\tag{3.22}$$

This, together with (3.21), implies

$$\begin{aligned}\mathscr{J} &\geqslant c\int_{I^2} \frac{\mathrm{d}s\mathrm{d}t}{(\sum_{j=1}^N |s_j - t_j|^{2H_j})^{d/2}} \int_{I^2} \frac{\mathrm{d}s'\mathrm{d}t'}{(\sum_{j=1}^N |s_j' - t_j'|^{2H_j})^{d/2}} \\ &\geqslant c\Big(\int_I \frac{\mathrm{d}t}{(\sum_{\ell=1}^N t_\ell^{H_\ell})^d}\Big)^2. \end{aligned}\tag{3.23}$$

By using Lemma 2.1, it is elementary to verify that the last integral in (3.23) is infinite provided $\sum_{j=1}^N 1/H_j \leqslant d$. Hence, we prove the necessity of part (i).

To prove the sufficiency in parts (i), we apply condition (C3) to see that for any $(s,t), (s',t') \in I \times I$,

$$\begin{aligned}\mathrm{Var}(V_0(s,t) \mid V_0(s',t')) &\geqslant \mathrm{Var}(X_0(t)|X_0(s), X_0(s'), X_0(t')) \\ &\geqslant c\sum_{j=1}^N \min\{|t_j - s_j|^{2H_j}, |t_j - s_j'|^{2H_j}, |t_j - t_j'|^{2H_j}, t_j^{2H_j}\}.\end{aligned}$$

Moreover, we also have

$$\mathrm{Var}(V_0(s',t')) \geqslant \mathrm{Var}(X_0(t') \mid X_0(s')) \geqslant c\sum_{j=1}^N \min\{|t_j' - s_j'|^{2H_j}, {t'}_j^{2H_j}\}.$$

Combining the above two inequalities with (3.21) yields

$$\mathscr{J} \leqslant c \int_{I^4} \Big(\sum_{j=1}^{N} \min\{|t_j - s_j|^{H_j}, |t_j - s'_j|^{H_j}, |t_j - t'_j|^{H_j}, t_j^{H_j}\} \Big)^{-d} \times \Big(\sum_{j=1}^{N} \min\{|t'_j - s'_j|^{H_j}, {t'}_j^{H_j}\} \Big)^{-d} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'. \tag{3.24}$$

Similar to the proof of (2.13), we integrate $\mathrm{d}t_1, \mathrm{d}t_2, \ldots, \mathrm{d}t_N, \mathrm{d}t'_1, \mathrm{d}t'_2, \ldots, \mathrm{d}t'_N$ to show that the integral in (3.24) is finite provided $\sum_{\ell=1}^{N} 1/H_\ell > d$. This proves the sufficiency of part (i).

In order to prove part (ii), by Lemma 2.9, it suffices to verify that if

$$\sum_{j=1}^{N} \frac{1}{H_j} > d + 2,$$

then

$$\mathscr{K} = \int_{I^4} \frac{[\mathbb{E}(V_0(s,t)V_0(s',t'))]^2}{[\det \mathrm{Cov}(V_0(s,t), V_0(s',t'))]^{(d+2)/2}} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t' < +\infty. \tag{3.25}$$

For any $(s,t), (s',t') \in I \times I$, the Cauchy-Schwarz inequality and (C1) imply

$$[\mathbb{E}(V_0(s,t)V_0(s',t'))]^2 \leqslant \mathbb{E}[V_0^2(s,t)]\mathbb{E}[V_0^2(s',t')] \leqslant c. \tag{3.26}$$

Similar to the argument in part (i), we derive from (3.26) that

$$\mathscr{J} \leqslant c \int_{I^4} \Big(\sum_{j=1}^{N} \min\{|t_j - s_j|^{H_j}, |t_j - s'_j|^{H_j}, |t_j - t'_j|^{H_j}, t_j^{H_j}\} \Big)^{-(d+2)} \times \Big(\sum_{j=1}^{N} \min\{|t'_j - s'_j|^{H_j}, {t'}_j^{H_j}\} \Big)^{-(d+2)} \mathrm{d}s\mathrm{d}t\mathrm{d}s'\mathrm{d}t'. \tag{3.27}$$

Again, we integrate in the order of $\mathrm{d}t_1, \mathrm{d}t_2, \ldots, \mathrm{d}t_N, \mathrm{d}t'_1, \mathrm{d}t'_2, \ldots, \mathrm{d}t'_N$ to show that the integral in (3.27) is finite provided $\sum_{\ell=1}^{N} 1/H_\ell > d + 2$. This proves (3.25) and hence part (ii).

Finally, we prove part (iii). By taking two disjoint sub-intervals and argue as in the proof of Theorem 3.1, one can see easily that if $2\sum_{j=1}^{N} \frac{1}{H_j} \leqslant d + 2$, then the integral $\mathscr{K}$ in (3.25) diverges and, consequently, $L_S(X^H, I \times I) \notin D_1$.

It remains to show that the integral $\mathscr{K}$ also diverges if $3\sum_{j=1}^{N} \frac{1}{H_j} \leqslant 2d$. To this end, we write

$$\rho(s,t) = \sqrt{\mathbb{E}(V_0(s,t)^2)}\,.$$

It will be useful to note that $\rho(s,t)$ is a pseudo-metric on $\mathbb{R}^{2N}$. Since

$$\mathbb{E}((V_0(s,t) - V_0(s',t'))^2) \leqslant 2(\rho(s,t)^2 + \rho(s',t')^2),$$

we see that, if

$$\rho(s,s') \leqslant \frac{1}{2}\,\rho(s,t), \quad \rho(t,t') \leqslant \frac{1}{2}\,\rho(s,t),$$

then

$$\begin{aligned}\mathbb{E}(V_0(s,t)V_0(s',t')) &= \frac{1}{2}\,[\mathbb{E}(V_0(s,t)^2) + \mathbb{E}(V_0(s',t')^2) - \mathbb{E}((V_0(s,t) - V_0(s',t'))^2)]\\ &\geqslant \frac{1}{2}\,\mathbb{E}(V_0(s',t')^2). \end{aligned} \tag{3.28}$$

Let

$$B_\rho(s,t) = \Big\{(s',t') \in I^2\colon \rho(s,s') \leqslant \frac{1}{2}\,\rho(s,t),\, \rho(t,t') \leqslant \frac{1}{2}\,\rho(s,t)\Big\}.$$

It follows from (3.25), (3.22), and (3.28) that

$$\mathscr{K} \geqslant c\int_{I^2} \frac{\mathrm{d}s\mathrm{d}t}{\rho(s,t)^{d+2}} \int_{B_\rho(s,t)} \frac{\mathrm{d}s'\mathrm{d}t'}{\rho(s',t')^{d-2}} \geqslant \int_{I^2} \frac{\mathrm{d}s\mathrm{d}t}{\rho(s,t)^{2(d-Q)}}, \tag{3.29}$$

where

$$Q = \sum_{j=1}^{N} \frac{1}{H_j}.$$

In obtaining the last inequality, we have used the fact that

$$\rho(s',t') \leqslant 2\rho(s,t), \quad \forall\,(s',t') \in B_\rho(s,t),$$

and the Lebesgue measure of $B_\rho(s,t)$ is $c\,\rho(s,t)^{2Q}$. Under conditions (C1),

$$\rho(s,t) \leqslant c_1 \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \quad \forall\, s,t \in I^N.$$

We can apply Lemma 2.1 to show that the last integral in (3.29) diverges if and only if $Q \leqslant 2(d-Q)$. This proves $\mathscr{K} = +\infty$ when $3Q \leqslant 2d$. The proof of Theorem 3.5 is finished. □

The following are concluding remarks.

Remark 3.7 (a) It is known that conditions (C1) and (C3) are satisfied by a large class of Gaussian random fields including N-parameter fractional Brownian motion [23], fractional Brownian sheets [29,32], and stochastic heat equation driver by space-time Gaussian noises [6,26]. Hence, Theorems 3.1 and 3.2 can be applied directly to these Gaussian random fields. However, despite the conditions given by Theorems 3.3 and 3.5, the problem for finding necessary and sufficient conditions for $L_S(X, I \times I) \in D_1$ is still open for a general Gaussian random field. It would be interesting to solve this problem.

(b) Another interesting question is to remove the i.i.d. assumption on the coordinate random fields $X_1, X_2, \ldots, X_d$ in (1.1). While the results of this paper can be extended to Gaussian random fields with independent, but non-identically distributed components, it seems more difficult to remove the independence assumption. Some preliminary results have been proved by Eddahbi et al. [7,8] for vector-valued fractional Brownian sheets, but their conditions may not be optimal.

Acknowledgements Research of Z. Chen and D. Wu was partially supported by the National Natural Science Foundation of China (Grant No. 11371321). Research of Y. Xiao was partially supported by the NSF Grants DMS-1307470 and DMS-1309856.

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