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RESEARCH ARTICLE

Intrinsic contractivity properties of Feynman-Kac semigroups for symmetric jump processes with infinite range jumps

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Dedicated to Professor Mu-Fa Chen on the occasion of his 70*th birthday*

Abstract Let $(X_t)_{t\geqslant 0}$ be a symmetric strong Markov process generated by non-local regular Dirichlet form $(D, \mathscr{D}(D))$ as follows:

$$D(f,g)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))J(x,y)\mathrm{d}x\mathrm{d}y,\quad f,g\in\mathscr{D}(D),$$

where $J(x,y)$ is a strictly positive and symmetric measurable function on $\mathbb{R}^d\times\mathbb{R}^d$. We study the intrinsic hypercontractivity, intrinsic supercontractivity, and intrinsic ultracontractivity for the Feynman-Kac semigroup

$$T_t^V(f)(x)=\mathbb{E}^x\bigg(\exp\bigg(-\int_0^t V(X_s)\mathrm{d}s\bigg)f(X_t)\bigg),\quad x\in\mathbb{R}^d,\ f\in L^2(\mathbb{R}^d;\mathrm{d}x).$$

In particular, we prove that for $J(x,y)\asymp|x-y|^{-d-\alpha}\mathbb{1}_{\{|x-y|\leqslant 1\}}+\mathrm{e}^{-|x-y|}\mathbb{1}_{\{|x-y|>1\}}$ with $\alpha\in(0,2)$ and $V(x)=|x|^\lambda$ with $\lambda>0$, $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive if and only if $\lambda>1$; and that for symmetric α-stable process $(X_t)_{t\geqslant 0}$ with $\alpha\in(0,2)$ and $V(x)=\log^\lambda(1+|x|)$ with some $\lambda>0$, $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive (or intrinsically supercontractive) if and only if $\lambda>1$, and $(T_t^V)_{t\geqslant 0}$ is intrinsically hypercontractive if and only if $\lambda\geqslant 1$. Besides, we also investigate intrinsic contractivity properties of $(T_t^V)_{t\geqslant 0}$ for the case that $\liminf_{|x|\to+\infty}V(x)<+\infty$.

Keywords Symmetric jump process, Lévy process, Dirichlet form, Feynman-Kac semigroup, intrinsic contractivity
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1 Introduction and main results

1.1 Setting and assumptions

Let $((X_t)_{t\geqslant 0}, \mathbb{P}^x)$ be a symmetric strong Markov process on $\mathbb{R}^d$ generated by the following non-local symmetric regular Dirichlet form:

$$D(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f(x)-f(y))(g(x)-g(y))J(x,y)\mathrm{d}x\mathrm{d}y,$$
$$\mathscr{D}(D) = \overline{C_c^1(\mathbb{R}^d)}^{D_1}.$$

Here, $J(x,y)$ is a strictly positive and symmetric measurable function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying that there exist constants $\alpha_1,\alpha_2\in(0,2)$ with $\alpha_1\leqslant\alpha_2$ and positive constants κ, c_1, c_2 such that

$$c_1|x-y|^{-d-\alpha_1} \leqslant J(x,y) \leqslant c_2|x-y|^{-d-\alpha_2}, \quad 0<|x-y|\leqslant\kappa, \tag{1.1}$$
$$J(x,y)>0, \quad |x-y|>\kappa, \tag{1.2}$$

and

$$\sup_{x\in\mathbb{R}^d}\int_{\{|x-y|>\kappa\}} J(x,y)\mathrm{d}y < +\infty; \tag{1.3}$$

$C_c^1(\mathbb{R}^d)$ denotes the space of C^1 functions on $\mathbb{R}^d$ with compact support, and $\overline{C_c^1(\mathbb{R}^d)}^{D_1}$ denotes the closure of $C_c^1(\mathbb{R}^d)$ under the norm

$$\|f\|_{D_1} := \sqrt{D(f,f)+\int f^2(x)\mathrm{d}x}\,.$$

According to [1, Theorems 1.1, 1.2], $(X_t)_{\geqslant 0}$ has a symmetric, bounded, and positive transition density function $p(t,x,y)$ defined on $[0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, whence the associated strongly continuous Markov semigroup $(T_t)_{t\geqslant 0}$ is given by

$$T_tf(x) = \mathbb{E}^x(f(X_t)) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\mathrm{d}y, \quad x\in\mathbb{R}^d,\ t>0,\ f\in B_b(\mathbb{R}^d),$$

where $\mathbb{E}^x$ denotes the expectation under the probability measure $\mathbb{P}^x$. Throughout this paper, we further assume that for every $t>0$, the function $(x,y)\mapsto p(t,x,y)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$, see [1,5–8] and references therein for sufficient conditions ensuring this property. For symmetric Lévy process $(X_t)_{t\geqslant 0}$, the continuity of density function is equivalent to $\mathrm{e}^{-t\Psi_0(\cdot)}\in L^1(\mathbb{R}^d;\mathrm{d}x)$ for any $t>0$, where Ψ_0 is the characteristic exponent or the symbol of Lévy process $(X_t)_{t\geqslant 0}$,

$$\mathbb{E}^x(\mathrm{e}^{\mathrm{i}\langle\xi,X_t-x\rangle}) = \mathrm{e}^{-t\Psi_0(\xi)}, \quad \xi\in\mathbb{R}^d,\ t>0.$$

Let V be a non-negative and locally bounded measurable (potential) function on $\mathbb{R}^d$. Define the Feynman-Kac semigroup $(T_t^V)_{t\geqslant 0}$:

$$T_t^V(f)(x) = \mathbb{E}^x\bigg(\exp\bigg(-\int_0^t V(X_s)\mathrm{d}s\bigg)f(X_t)\bigg), \quad x\in\mathbb{R}^d,\ f\in L^2(\mathbb{R}^d;\mathrm{d}x).$$

It is easy to check that $(T_t^V)_{t\geqslant 0}$ is a bounded symmetric semigroup on $L^2(\mathbb{R}^d;\mathrm{d}x)$. By assumptions of $(X_t)_{t\geqslant 0}$, for each $t>0$, T_t^V is also bounded from $L^1(\mathbb{R}^d;\mathrm{d}x)$ to $L^\infty(\mathbb{R}^d;\mathrm{d}x)$, and there exists a symmetric, bounded, and positive transition density function $p^V(t,x,y)$ such that for every $t>0$, the function $(x,y)\mapsto p^V(t,x,y)$ is continuous, and for every $1\leqslant p\leqslant +\infty$,

$$T_t^V f(x) = \int_{\mathbb{R}^d} p^V(t,x,y)f(y)\mathrm{d}y, \quad x\in\mathbb{R}^d,\ f\in L^p(\mathbb{R}^d;\mathrm{d}x),$$

see, e.g., [10, Section 3.2]. Suppose that for every $r>0$,

$$|\{x\in\mathbb{R}^d\colon V(x)\leqslant r\}| < +\infty, \tag{1.4}$$

where $|A|$ denotes the Lebesgue measure of Borel set A. According to [4, Proposition 1.1] (which is essentially based on [21, Corollary 1.3]), the semigroup $(T_t^V)_{t\geqslant 0}$ is compact. By general theory of semigroups for compact operators, there exists an orthonormal basis in $L^2(\mathbb{R}^d;\mathrm{d}x)$ of eigenfunctions $\{\phi_n\}_{n=1}^{+\infty}$ associated with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{+\infty}$ satisfying

$$0<\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots, \quad \lim_{n\to+\infty}\lambda_n=+\infty.$$

That is,

$$L_V\phi_n = -\lambda_n\phi_n, \quad T_t^V\phi_n = \mathrm{e}^{-\lambda_n t}\phi_n,$$

where $(L_V,\mathscr{D}(L_V))$ is the infinitesimal generator of the semigroup $(T_t^V)_{t\geqslant 0}$. The first eigenfunction ϕ_1 is called ground state in the literature. Indeed, in our setting, there exists a version of ϕ_1 which is bounded, continuous, and strictly positive, see, e.g., [4, Proposition 1.2].

In the following, we always assume that (1.1)–(1.4) hold, and that the ground state ϕ_1 is bounded, continuous, and strictly positive.

1.2 Previous work and motivation

We are concerned with intrinsic contractivity properties for the semigroup $(T_t^V)_{t\geqslant 0}$. We first recall the definitions of these properties introduced in [11]. The semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive if and only if for any $t>0$, there exists a constant $C_t>0$ such that for all $x,y\in\mathbb{R}^d$,

$$p^V(t,x,y)\leqslant C_t\phi_1(x)\phi_1(y).$$

Define

$$\widetilde{T}_t^V f(x) = \frac{\mathrm{e}^{\lambda_1 t}}{\phi_1(x)}T_t^V((\phi_1 f))(x), \quad t>0, \tag{1.5}$$

which is a Markov semigroup on $L^2(\mathbb{R}^d;\phi_1^2(x)\mathrm{d}x)$. Then, $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive if and only if $(\widetilde{T}_t^V)_{t\geqslant 0}$ ultracontractive, i.e., for every $t>0$, $\widetilde{T}_t^V$ is a bounded operator from $L^2(\mathbb{R}^d;\phi_1^2(x)\mathrm{d}x)$ to $L^\infty(\mathbb{R}^d;\phi_1^2(x)\mathrm{d}x)$. If for every $p\in(2,+\infty)$, there exists a constant $t_0(p)\geqslant 0$ such that for all $t>t_0(p)$, $\widetilde{T}_t^V$ is a bounded operator from $L^2(\mathbb{R}^d;\phi_1^2(x)\mathrm{d}x)$ to $L^p(\mathbb{R}^d;\phi_1^2(x)\mathrm{d}x)$, then we say that $(\widetilde{T}_t^V)_{t\geqslant 0}$ is hypercontractive, and equivalently, $(T_t^V)_{t\geqslant 0}$ is intrinsically hypercontractive. If one can take $t_0(p)=0$, then we say that $(\widetilde{T}_t^V)_{t\geqslant 0}$ is supercontractive, and equivalently, $(T_t^V)_{t\geqslant 0}$ is intrinsically supercontractive. In particular, the intrinsic ultracontractivity is stronger than the intrinsic supercontractivity, which is in turn stronger than the intrinsic hypercontractivity.

The intrinsic ultracontractivity of $(T_t^V)_{t\geqslant 0}$ associated with pure jump symmetric Lévy process $(X_t)_{t\geqslant 0}$ has been investigated in [12–14]. The approach of all these cited papers is based on two-sided estimates for ground state ϕ_1 corresponding to the semigroup $(T_t^V)_{t\geqslant 0}$, for which some restrictions on the density function of Lévy measure and the potential function V are needed, see [13, Assumptions 2.1, 2.5] or assumptions (H1)–(H3) below. Recently, the authors make use of super Poincaré inequalities with respect to infinite measure developed in [16,17] and functional inequalities for non-local Dirichlet forms recently studied in [3,20,23] to investigate the intrinsic ultracontractivity of Feynman-Kac semigroups for symmetric jump processes in [4]. The main result [4, Theorem 1.3] applies to symmetric jump process such that associated jump kernel is given by

$$J(x,y)\asymp |x-y|^{-d-\alpha}\mathbb{1}_{\{|x-y|\leqslant 1\}}+\mathrm{e}^{-|x-y|^\gamma}\mathbb{1}_{\{|x-y|>1\}}$$

with $\alpha\in(0,2)$ and $\gamma\in(1,+\infty]$, for which the approach of [12–14] does not work. In particular, when $\gamma=+\infty$,

$$J(x,y)\asymp |x-y|^{-d-\alpha}\mathbb{1}_{\{|x-y|\leqslant 1\}},$$

which is associated with the truncated symmetric α-stable-like process.

As already mentioned in [4], in the model above, finite range jumps play an essential role in the behavior of the associated process. In the present setting, the argument of [4] may lead to obtain some sufficient conditions for intrinsic ultracontractivity of $(T_t^V)_{t\geqslant 0}$. However, as we will see from examples below, the conclusions yielded by the approach of [4] are far from optimality because of the large range jumps. This explains the motivation of our present paper.

The main purpose of this paper is to derive explicit and sharp criterion for intrinsic contractivity properties of Feynman-Kac semigroups for symmetric jump processes with infinite range jumps. We will use the intrinsic super Poincaré inequalities introduced in [16,17] which have been applied in [15,19] to investigate the intrinsic ultracontactivity for diffusion processes on Riemannian manifolds. Our method to establish the intrinsic super Poincaré inequality is efficient for a large class of jump processes. Indeed, our main results not only work for jump processes of infinite range jumps without technical restrictions used in [12–14], but also apply to space-inhomogeneous jump processes and the corresponding Feynman-Kac semigroup with potential $V(x)$ not necessarily going to infinity as $|x| \to +\infty$.

1.3 Main results

We assume that (1.1)–(1.4) hold in all results of this paper. To state our main result, we need some necessary assumptions and notations. For $x \in \mathbb{R}^d$, define

$$J^*(x) = \begin{cases} \inf_{y-z \in B(x,3/2)} J(y,z), & |x| \geqslant 3, \\ 1, & |x| < 3, \end{cases}$$

and

$$V^*(x) = \sup_{z \in B(x,1)} V(z), \quad \varphi(x) = \frac{J^*(x)}{1+V^*(x)}.$$

For any $s, r > 0$, set

$$\alpha(r,s) = \inf\Big\{\frac{2}{|B(0,t)| \inf_{x\in B(0,r+t)} \varphi^2(x)} : t \leqslant r, \frac{2\sup_{0<|x-y|\leqslant t} J(x,y)^{-1}}{|B(0,t)|} \leqslant s\Big\}.$$

In particular, by (1.1),

$$\lim_{t\downarrow 0} \frac{\sup_{0<|x-y|\leqslant t} J(x,y)^{-1}}{|B(0,t)|} = 0,$$

which implies that the set of infimum in the definition of $\alpha(r,s)$ is not empty for all $r, s > 0$.

1.3.1 Regular potential function: $\lim_{|x|\to+\infty} V(x) = +\infty$

Without loss of generality, we assume that in the result below

$$\inf_{x\in\mathbb{R}^d} V(x) = 0,$$

otherwise, one can take

$$\widetilde{V}(x) := V(x) - \inf_{z\in\mathbb{R}^d} V(z)$$

instead of V.

Theorem 1.1 *Suppose that*

$$\lim_{|x|\to+\infty} V(x) = +\infty.$$

For any $s, \delta_1, \delta_2 > 0$, define

$$\Phi(s) = \inf_{|x|\geqslant s} V(x)$$

and

$$\beta(s) = \beta(s;\delta_1,\delta_2) = \delta_1 \alpha\Big(\Phi^{-1}\Big(\frac{4}{s\wedge\delta_2}\Big), \frac{s\wedge\delta_2}{4}\Big), \tag{1.6}$$

where Φ^{-1} *is the generalized inverse of* Φ*, i.e.,*

$$\Phi^{-1}(r) = \inf\{s \geqslant 0\colon \Phi(s) \geqslant r\}.$$

(*The set* $\{s \geqslant 0\colon \Phi(s) \geqslant r\}$ *is not empty for every* $r > 0$ *since* $\lim_{|x|\to+\infty} V(x) = +\infty$ *and* $\inf_{x\in\mathbb{R}^d} V(x) = 0$.) *Let* $\beta^{-1}(s)$ *be the generalized inverse of* $\beta(s)$*. Then we have the following three statements.*

(1) *If for any constants* δ_1 *and* $\delta_2 > 0$,

$$\int_t^{+\infty} \frac{\beta^{-1}(s)}{s}\,\mathrm{d}s < +\infty, \quad t > \inf\beta,$$

then the semigroup $(T_t^V)_{t\geqslant 0}$ *is intrinsically ultracontractive.*

(2) *If for any constants* δ_1 *and* $\delta_2 > 0$,

$$\lim_{s\to 0} s\log\beta(s) = 0,$$

then the semigroup $(T_t^V)_{t\geqslant 0}$ *is intrinsically supercontractive.*

(3) *If for any constants* δ_1 *and* $\delta_2 > 0$,

$$\limsup_{s\to 0} s\log\beta(s) < +\infty,$$

then the semigroup $(T_t^V)_{t\geqslant 0}$ *is intrinsically hypercontractive.*

For symmetric Lévy process, due to the space-homogeneous property, it holds that

$$J(x,y) = J(0,x-y) = \rho(x-y), \quad x \neq y,$$

where ρ is the density function of the associated Lévy measure. Obviously, in this case, Theorem 1.1 excludes the following assumptions used in [13] (see Assumptions 2.1, 2.3, and 2.5 therein):

(B1) there are constants c_3 and $c_4 \geqslant 1$ such that

$$c_3^{-1} \sup_{B(x,1)} \rho(z) \leqslant \rho(x) \leqslant c_3 \inf_{z\in B(x,1)} \rho(z), \quad |x| > 2, \tag{H1}$$

and

$$\int_{\{|z-x|>1,\,|z-y|>1\}} \rho(x-z)\rho(z-y)\mathrm{d}z \leqslant c_4\rho(x-y), \quad |x-y| > 1; \tag{H2}$$

(B2) for all $0 < r_1 < r_2 < r \leqslant 1$,

$$\sup_{x\in B(0,r_1)} \sup_{y\in B^c(0,r_2)} G_{B(0,r)}(x,y) < +\infty,$$

where $B(0,r)$ denotes the ball with center 0 and radius r, and $G_{B(0,r)}(x,y)$ is the Green function for the killed process of $(X_t)_{t\geqslant 0}$ on domain $B(0,r)$;

(B3) there exists a constant $c_5 \geqslant 1$ such that

$$\sup_{z \in B(x,1)} V(z) \leqslant c_5 V(x). \tag{H3}$$

In particular, assumption (B2) is less explicit as it is given by the Green function rather than the jump rate.

To illustrate the optimality of Theorem 1.1, we consider the following two examples.

Example 1.2 *Let*

$$J(x,y) \asymp |x-y|^{-d-\alpha} \mathbb{1}_{\{|x-y| \leqslant 1\}} + \mathrm{e}^{-|x-y|^\gamma} \mathbb{1}_{\{|x-y|>1\}},$$

where $\alpha \in (0,2)$ *and* $\gamma \in (0,1]$. *Let* $V(x) = |x|^\lambda$ *for some* $\lambda > 0$. *Then, there is a constant* $C_1 > 0$ *such that for all* $x \in \mathbb{R}^d$,

$$\phi_1(x) \geqslant \frac{C_1}{(1+|x|)^\lambda \mathrm{e}^{|x|^\gamma}},$$

and the associated semigroup $(T_t^V)_{t \geqslant 0}$ *is intrinsically ultracontractive if and only if* $\lambda > \gamma$. *Furthermore, if* $\lambda > \gamma$ *and for every* $x \in \mathbb{R}^d$,

$$\int_{\{|z| \leqslant 1\}} |z|\, |J(x,x+z) - J(x,x-z)| \mathrm{d}z < +\infty, \tag{1.7}$$

then there is a constant $C_2 > 0$ *such that for all* $x \in \mathbb{R}^d$,

$$\phi_1(x) \leqslant \frac{C_2}{(1+|x|)^\lambda \mathrm{e}^{|x|^\gamma}}.$$

Remark 1.3 (1) For symmetric Lévy process, (1.7) is automatically satisfied.

(2) When $V(x) = |x|^\lambda$ with $\lambda > 1$, one can also use the argument in [4] to prove the intrinsic ultracontractivity of $(T_t^V)_{t \geqslant 0}$. However, the condition $\lambda > 1$ is much stronger than $\lambda > \gamma \in (0,1]$ required by the first assertion in Example 1.2.

Example 1.4 *Let* $(X_t)_{t \geqslant 0}$ *be a symmetric* α*-stable process with some* $\alpha \in (0,2)$, *i.e.*,

$$J(x,y) = \rho(x-y) := c(d,\alpha)|x-y|^{-d-\alpha},$$

where $c(d,\alpha)$ *is a constant only depending on* d *and* α. *Let*

$$V(x) = \log^\lambda(1+|x|)$$

for some $\lambda > 0$. *Then,*

(1) *the semigroup* $(T_t^V)_{t \geqslant 0}$ *is intrinsically ultracontractive if and only if* $\lambda > 1$;

(2) *the semigroup* $(T_t^V)_{t\geqslant 0}$ *is intrinsically supercontractive if and only if* $\lambda > 1$;

(3) *the semigroup* $(T_t^V)_{t\geqslant 0}$ *is intrinsically hypercontractive if and only if* $\lambda \geqslant 1$.

1.3.2 Irregular potential function: $\liminf_{|x|\to+\infty} V(x) < +\infty$

We make the following assumption as in [4].

(A) There exists a constant $K > 0$ such that

$$\lim_{R\to+\infty} \Phi_K(R) = +\infty,$$

where

$$\Phi_K(R) = \inf_{|x|\geqslant R, V(x)>K} V(x), \quad R > 0.$$

Let

$$\Theta_K(R) = |\{x \in \mathbb{R}^d\colon |x| \geqslant R,\, V(x) \leqslant K\}|, \quad R > 0,$$

where K is the constant given in (A). Then, by (1.4),

$$\lim_{R\to+\infty} \Theta_K(R) = 0.$$

Similar to Theorem 1.1, in Theorem 1.5 below, we can assume that

$$\inf_{x\in\mathbb{R}^d,\, V(x)>K} V(x) = 0,$$

otherwise, V is replaced by

$$\widetilde{V}(x) := V(x) - \big(\inf_{z\in\mathbb{R}^d,\, V(z)>K} V(z)\big)\mathbb{1}_{\{z\in\mathbb{R}^d,\, V(z)>K\}}(x).$$

In particular, under such assumption and (A), for any $r > 0$, the Borel set $\{s \geqslant 0\colon \Phi_K(s) \geqslant r\}$ is not empty.

Theorem 1.5 *Suppose that assumption* (A) *holds, and that* $d > \alpha_1$, *where* $\alpha_1 \in (0, 2)$ *is given in* (1.1). *For any* $s, \delta_i > 0$ *with* $1 \leqslant i \leqslant 4$, *define*

$$\hat{\beta}(s) = \hat{\beta}(s; \delta_1, \delta_2, \delta_3, \delta_4) = \delta_1 \alpha\Big(\Psi_K^{-1}\Big(\frac{8}{s\wedge\delta_2}\Big) \wedge \delta_3, \frac{s\wedge\delta_2}{8}\Big), \tag{1.8}$$

where

$$\Psi_K(R) = \Big[\frac{1}{\Phi_K(R)} + \delta_4 \Theta_K(R)^{\alpha_1/d}\Big]^{-1}, \quad \Phi_K^{-1}(r) = \inf\{s \geqslant 0\colon \Phi_K(s) \geqslant r\},$$

and Φ_K^{-1} *denotes the generalized inverse of* Φ_K. *Then all assertions in Theorem* 1.1 *hold with* $\beta(s)$ *replaced by* $\hat{\beta}(s)$.

Note that, when

$$\lim_{|x|\to+\infty} V(x) = +\infty,$$

for any constant $K > 0$, there exists $R_0 > 0$ such that

$$\Theta_K(R) = 0, \quad \Psi_K(R) = \Phi_K(R), \quad R \geqslant R_0.$$

Therefore, by (1.8) and (1.6), in this case, Theorem 1.5 reduces to Theorem 1.1. To show that Theorem 1.5 is sharp, we reconsider symmetric α-stable process both with irregular potential function.

Example 1.6 *Let $(X_t)_{t\geqslant 0}$ be a symmetric α-stable process on $\mathbb{R}^d$ with $d > \alpha$, and let V be a nonnegative measurable function defined by*

$$V(x) = \begin{cases} \log^\lambda(1 + |x|), & x \notin A, \\ 1, & x \in A, \end{cases} \tag{1.9}$$

where $\lambda > 1$ and A is a unbounded set on $\mathbb{R}^d$ such that $\inf_{x\notin A} V(x) = 0$.

(1) *Suppose that*

$$|A \cap B(0, R)^c| \leqslant \frac{c_1}{\log^\theta R}, \quad R > 1,$$

holds with some constants $c_1, \theta > 0$. Then, the associated semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive (and also intrinsically supercontractive) if $\theta > d/\alpha$; $(T_t^V)_{t\geqslant 0}$ is intrinsically hypercontractive if $\theta \geqslant d/\alpha$.

(2) *For any $\varepsilon > 0$, let*

$$A = \bigcup_{m=1}^{+\infty} B(x_m, r_m),$$

where $x_m \in \mathbb{R}^d$ with $|x_m| = \mathrm{e}^{m^{k_0}}$, and $r_m = m^{-\frac{k_0}{\alpha}+\frac{1}{d}}$ for some $k_0 > 2/\varepsilon$. Then

$$|A \cap B(0, R)^c| \leqslant \frac{c_2}{\log^{\frac{d}{\alpha}-\varepsilon} R}, \quad R > 1, \tag{1.10}$$

holds for some constant $c_2 > 0$; however, the semigroup $(T_t^V)_{t\geqslant 0}$ is not intrinsically ultracontractive.

The reminder of this paper is arranged as follows. In the next section, we will present some preliminary results, including lower bound estimate for the ground state and intrinsic local super Poincaré inequalities for non-local Dirichlet forms with infinite range jumps. Section 3 is devoted to the proofs of all the theorems and examples.

2 Some technical estimates

2.1 Lower bound for ground state

In this subsection, we consider lower bound estimate for the ground state ϕ_1.

Recall that for $x \in \mathbb{R}^d$,

$$J^*(x) = \begin{cases} \inf_{y-z \in B(x,3/2)} J(y,z), & |x| \geqslant 3, \\ 1, & |x| < 3, \end{cases}$$

and

$$V^*(x) = \sup_{z \in B(x,1)} V(z), \quad \varphi(x) = \frac{J^*(x)}{1+V^*(x)}.$$

Proposition 2.1 *Let φ be the function defined above. Then there exists a constant $C_0 > 0$ such that for all $x \in \mathbb{R}^d$,*

$$C_0 \phi_1(x) \geqslant \varphi(x). \tag{2.1}$$

The proof of Proposition 2.1 is mainly based on the argument of [14, Theorem 1.6] (in particular, see [14, pp. 5054, 5055]). For the sake of completeness, we present the details here.

First, for any Borel set $D \subseteq \mathbb{R}^d$, let $\tau_D := \inf\{t > 0 \colon X_t \notin D\}$ be the first exit time from D of the process $(X_t)_{t \geqslant 0}$. The following result is a consequence of [2, Theorem 2.1], and the reader can refer to [4, Lemma 3.1] for the proof of it.

Lemma 2.2 *There exist constants $c_0 := c_0(\kappa) > 0$ and $r_0 := r_0(\kappa) \in (0,1]$ such that for every $r \in (0, r_0]$ and $x \in \mathbb{R}^d$,*

$$\mathbb{P}^x\big(\tau_{B(x,r)} \geqslant c_0 r^{\alpha_2 + \frac{(\alpha_2-\alpha_1)d}{\alpha_1}}\big) \geqslant \frac{1}{2}.$$

In the following, we will fix r_0, c_0 in Lemma 2.2 and set

$$t_0 = c_0 r_0^{\alpha_2 + \frac{(\alpha_2-\alpha_1)d}{\alpha_1}}.$$

Lemma 2.3 *Let $0 \leqslant t_1 < t_2 \leqslant t_0$, $x \in \mathbb{R}^d$ with $|x| \geqslant 3$, $D = B(0, r_0)$, and $B = B(x, r_0)$. Then we have*

$$\mathbb{P}^x(X_{\tau_B} \in D/2,\ t_1 \leqslant \tau_B < t_2) \geqslant c_1(t_2 - t_1) J^*(x) \tag{2.2}$$

for some constant $c_1 > 0$.

Proof Denote by $p_B(t,x,y)$ the density of the process $(X_t)_{t \geqslant 0}$ killed on exiting the set B, i.e.,

$$p_B(t,x,y) = p(t,x,y) - \mathbb{E}^x(\tau_B \leqslant t;\ p(t-\tau_B, X(\tau_B), y)).$$

According to the Ikeda-Watanabe formula for $(X_t)_{t \geqslant 0}$ (see, e.g., [14, Proposition

2.5]), we have

$$\begin{aligned}
&\mathbb{P}^x(X(\tau_B)\in D/2,\, t_1\leqslant \tau_B<t_2)\\
&\quad=\int_B\int_{t_1}^{t_2} p_B(s,x,y)\mathrm{d}s\int_{D/2}J(y,z)\mathrm{d}z\mathrm{d}y\\
&\quad\geqslant |D/2|\inf_{y-z\in B(x,3r_0/2)}J(y,z)\int_{t_1}^{t_2}\int_B p_B(s,x,y)\mathrm{d}y\mathrm{d}s\\
&\quad\geqslant c_2\inf_{y-z\in B(x,3r_0/2)}J(y,z)\int_{t_1}^{t_2}\mathbb{P}^x(\tau_B\geqslant s)\mathrm{d}s\\
&\quad\geqslant c_2\big(\inf_{|z|\geqslant 3}\mathbb{P}^z(\tau_{B(z,r_0)}\geqslant t_0)\big)(t_2-t_1)\inf_{y-z\in B(x,3r_0/2)}J(y,z)\\
&\quad\geqslant \frac{c_2}{2}(t_2-t_1)\inf_{y-z\in B(x,3/2)}J(y,z)\\
&\quad\geqslant \frac{c_2}{2}(t_2-t_1)J^*(x),
\end{aligned}$$

where, in the forth inequality, we have used Lemma 2.2 and the fact that $r_0\leqslant 1$. This completes the proof. □

Now, we are in a position to present the proof of Proposition 2.1.

Proof of Proposition 2.1 We only need to consider $x\in\mathbb{R}^d$ with $|x|\geqslant 3$. Still let $B=B(x,r_0)$ and $D=B(0,r_0)$. First, we have

$$\begin{aligned}
\phi_1(x)&=\mathrm{e}^{\lambda_1 t_0}T_{t_0}^V(\phi_1)(x)\\
&\geqslant \mathrm{e}^{\lambda_1 t_0}T_{t_0}^V(\mathbb{1}_D\phi_1)(x)\\
&\geqslant \mathrm{e}^{\lambda_1 t_0}\big(\inf_{x\in D}\phi_1(x)\big)T_{t_0}^V(\mathbb{1}_D)(x)\\
&\geqslant c_2T_{t_0}^V(\mathbb{1}_D)(x),
\end{aligned}$$

where, in the last inequality, we have used the fact that ϕ_1 is strictly positive and continuous.

Second, by the strong Markov property, it holds that

$$\begin{aligned}
&T_{t_0}^V(\mathbb{1}_D)(x)\\
&=\mathbb{E}^x\big(X_{t_0}\in D;\,\mathrm{e}^{-\int_0^{t_0}V(X_s)\mathrm{d}s}\big)\\
&\geqslant \mathbb{E}^x\big(X_{\tau_B}\in D/2,\,\tau_B<t_0,\,X_s\in D,\,\forall\, s\in[\tau_B,t_0];\,\mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s-\int_{\tau_B}^{t_0}V(X_s)\mathrm{d}s}\big)\\
&\geqslant \mathrm{e}^{-t_0\sup_{z\in D}V(z)}\mathbb{E}^x\big(X_{\tau_B}\in D/2,\,\tau_B<t_0,\,X_s\in D,\,\forall\, s\in[\tau_B,t_0];\mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\big)\\
&\geqslant \mathrm{e}^{-t_0\sup_{z\in D}V(z)}\mathbb{E}^x\big(X_{\tau_B}\in D/2,\,\tau_B<t_0;\,\mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\cdot\mathbb{P}^{X_{\tau_B}}(\tau_D>t_0)\big)\\
&\geqslant \mathrm{e}^{-t_0\sup_{z\in D}V(z)}\big(\inf_{|z|\leqslant r_0/2}\mathbb{P}^z(\tau_{B(z,r_0/2)}>t_0)\big)\\
&\qquad\cdot\mathbb{E}^x\big(X_{\tau_B}\in D/2,\,\tau_B<t_0;\,\mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\big)\\
&\geqslant c_3\mathbb{E}^x\big(X_{\tau_B}\in D/2,\,\tau_B<t_0;\,\mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\big),
\end{aligned}$$

where, in the last inequality, we have used Lemma 2.2.

Third, according to (2.2),

$$
\begin{aligned}
&\mathbb{E}^x\big(X_{\tau_B}\in D/2,\ \tau_B<t_0;\ \mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\big)\\
&\quad\geqslant \sum_{j=1}^{+\infty}\mathbb{E}^x\Big(X_{\tau_B}\in D/2,\ \frac{t_0}{j+1}\leqslant\tau_B<\frac{t_0}{j};\ \mathrm{e}^{-\int_0^{\tau_B}V(X_s)\mathrm{d}s}\Big)\\
&\quad\geqslant \sum_{j=1}^{+\infty}\mathrm{e}^{-\frac{t_0}{j}\sup_{z\in B(x,r_0)}V(z)}\mathbb{E}^x\Big(X_{\tau_B}\in D/2,\ \frac{t_0}{j+1}\leqslant\tau_B<\frac{t_0}{j}\Big)\\
&\quad\geqslant c_1J^*(x)\sum_{j=1}^{+\infty}\frac{t_0}{j(j+1)}\,\mathrm{e}^{-\frac{t_0}{j}\sup_{z\in B(x,r_0)}V(z)}\\
&\quad\geqslant \frac{c_4J^*(x)}{1+\sup_{z\in B(x,r_0)}V(z)}\\
&\quad\geqslant \frac{c_4J^*(x)}{1+\sup_{z\in B(x,1)}V(z)},
\end{aligned}
$$

where the forth inequality follows from [14, Lemma 5.2], i.e.,

$$\sum_{j=1}^{+\infty}\frac{\mathrm{e}^{-r/j}}{j(j+1)}\geqslant\frac{\mathrm{e}^{-1}}{r+1},\quad r\geqslant 0.$$

Combining all the conclusions above, we prove the desired assertion. □

2.2 Intrinsic local super Poincaré inequality

In this subsection, we are concerned with the local intrinsic super Poincaré inequality for $D^V(f,f)$.

Proposition 2.4 *Let φ be a strictly positive measurable function on $\mathbb{R}^d$. Then for any $s,r>0$ and any $f\in C_c^2(\mathbb{R}^d)$,*

$$\int_{B(0,r)}f^2(x)\mathrm{d}x\leqslant sD^V(f,f)+\alpha(r,s)\Big(\int|f|(x)\varphi(x)\mathrm{d}x\Big)^2,\tag{2.3}$$

where

$$\alpha(r,s)=\inf\Big\{\frac{2}{|B(0,t)|\inf_{x\in B(0,r+t)}\varphi^2(x)}:t\leqslant r,\ \frac{2\sup_{0<|x-y|\leqslant t}J(x,y)^{-1}}{|B(0,t)|}\leqslant s\Big\}.$$

Proof Since $V\geqslant 0$,

$$D(f,f)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(x)-f(y))^2J(x,y)\mathrm{d}x\mathrm{d}y\leqslant D^V(f,f),\quad f\in C_c^2(\mathbb{R}^d),$$

it suffices to prove (2.3) with $D^V(f,f)$ replaced by $D(f,f)$.

We can follow step (1) of the proof of [6, Theorem 3.1] or [23, Lemma 2.1] to verify that for any $0 < s \leqslant r$ and $f \in C_c^2(\mathbb{R}^d)$,

$$\int_{B(0,r)} f^2(x)\mathrm{d}x \leqslant \Big(\frac{2\sup_{0<|x-y|\leqslant s} J(x,y)^{-1}}{|B(0,s)|}\Big) \iint_{\{|x-y|\leqslant s\}} (f(x)-f(y))^2 \cdot J(x,y)\mathrm{d}x\mathrm{d}y + \frac{2}{|B(0,s)|}\Big(\int_{B(0,r+s)} |f(x)|\mathrm{d}x\Big)^2. \tag{2.4}$$

Note that, if (2.4) holds, then for any $0 < s \leqslant r$ and $f \in C_c^2(\mathbb{R}^d)$,

$$\int_{B(0,r)} f^2(x)\mathrm{d}x \leqslant \Big(\frac{2\sup_{0<|x-y|\leqslant s} J(x,y)^{-1}}{|B(0,s)|}\Big) D(f,f) + \frac{2}{|B(0,s)|\inf_{x\in B(0,r+s)} \varphi^2(x)}\Big(\int_{B(0,r+s)} |f(x)|\varphi(x)\mathrm{d}x\Big)^2.$$

This immediately yields (2.3) by the definition of $\alpha(s,r)$.

Next, we turn to the proof of (2.4). For any $0 < s \leqslant r$ and $f \in C_c^2(\mathbb{R}^d)$, define

$$f_s(x) = \frac{1}{|B(0,s)|}\int_{B(x,s)} f(z)\mathrm{d}z, \quad x \in B(0,r).$$

We have

$$\sup_{x\in B(0,r)} |f_s(x)| \leqslant \frac{1}{|B(0,s)|}\int_{B(0,r+s)} |f(z)|\mathrm{d}z,$$

and

$$\begin{aligned}\int_{B(0,r)} |f_s(x)|\mathrm{d}x &\leqslant \int_{B(0,r)} \frac{1}{|B(0,s)|}\int_{B(x,s)} |f(z)|\mathrm{d}z\mathrm{d}x \\ &\leqslant \int_{B(0,r+s)} \Big(\frac{1}{|B(0,s)|}\int_{B(z,s)} \mathrm{d}x\Big)|f(z)|\mathrm{d}z \\ &\leqslant \int_{B(0,r+s)} |f(z)|\mathrm{d}z.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{B(0,r)} f_s^2(x)\mathrm{d}x &\leqslant \big(\sup_{x\in B(0,r)} |f_s(x)|\big)\int_{B(0,r)} |f_s(x)|\mathrm{d}x \\ &\leqslant \frac{1}{|B(0,s)|}\Big(\int_{B(0,r+s)} |f(z)|\mathrm{d}z\Big)^2.\end{aligned}$$

Therefore, for any $f \in C_c^2(\mathbb{R}^d)$ and $0 < s \leqslant r$,

$$\begin{aligned}
&\int_{B(0,r)} f^2(x)\mathrm{d}x \\
&\leqslant 2\int_{B(0,r)} (f(x)-f_s(x))^2\mathrm{d}x + 2\int_{B(0,r)} f_s^2(x)\mathrm{d}x \\
&\leqslant 2\int_{B(0,r)} \frac{1}{|B(0,s)|}\int_{B(x,s)} (f(x)-f(y))^2\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{2}{|B(0,s)|}\Big(\int_{B(0,r+s)} |f(z)|\mathrm{d}z\Big)^2 \\
&\leqslant \frac{2\sup_{0<|x-y|\leqslant s} J(x,y)^{-1}}{|B(0,s)|}\iint_{\{|x-y|\leqslant s\}} (f(x)-f(y))^2 J(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{2}{|B(0,s)|}\Big(\int_{B(0,r+s)} |f(z)|\mathrm{d}z\Big)^2 \\
&\leqslant \frac{2\sup_{0<|x-y|\leqslant s} J(x,y)^{-1}}{|B(0,s)|}\iint_{\{|x-y|\leqslant s\}} (f(x)-f(y))^2 J(x,y)\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{2}{|B(0,s)|}\Big(\int_{B(0,r+s)} |f(z)|\mathrm{d}z\Big)^2.
\end{aligned}$$

This proves the desired assertion (2.4). □

3 Proofs of theorems and examples

We begin with proofs of Theorems 1.1 and 1.5.

Proof of Theorem 1.1 (1) For all $r > 0$ and $f \in C_c^2(\mathbb{R}^d)$,

$$\int_{B(0,r)^c} f^2(x)\mathrm{d}x \leqslant \frac{1}{\Phi(r)}\int_{B(0,r)^c} f^2(x)V(x)\mathrm{d}x \leqslant \frac{1}{\Phi(r)} D^V(f,f).$$

This, along with (2.3) and (2.1), gives us that for any $r, \widetilde{s} > 0$,

$$\int f^2(x)\mathrm{d}x \leqslant \Big(\frac{1}{\Phi(r)} + \widetilde{s}\Big)D^V(f,f) + C_0^2\alpha(r,\widetilde{s})\Big(\int |f|(x)\phi_1(x)\mathrm{d}x\Big)^2.$$

For any $s > 0$, taking $r = \Phi^{-1}(2/s)$ and $\widetilde{s} = s/2$ in the inequality above, we arrive at

$$\int f^2(x)\mathrm{d}x \leqslant sD^V(f,f) + C_0^2\alpha\Big(\Phi^{-1}\Big(\frac{2}{s}\Big), \frac{s}{2}\Big)\Big(\int |f|(x)\phi_1(x)\mathrm{d}x\Big)^2. \quad (3.1)$$

(2) Let $(\widetilde{T}_t^V)_{t\geqslant 0}$ be the strongly continuous Markov semigroup defined by (1.5). Due to the fact that $L_V\phi_1 = -\lambda_1\phi_1$, the (regular) Dirichlet form

$(D_{\phi_1}, \mathscr{D}(D_{\phi_1}))$ associated with $(\widetilde{T}_t^V)_{t\geqslant 0}$ enjoys the properties that, $C_c^2(\mathbb{R}^d)$ is a core for $(D_{\phi_1}, \mathscr{D}(D_{\phi_1}))$, and for any $f \in C_c^2(\mathbb{R}^d)$,

$$D_{\phi_1}(f,f) = D^V(f\phi_1, f\phi_1) - \lambda_1 \int_{\mathbb{R}^d} f^2(x)\phi_1^2(x)\mathrm{d}x. \tag{3.2}$$

Let

$$\mu_{\phi_1}(\mathrm{d}x) = \phi_1^2(x)\mathrm{d}x.$$

Combining (3.2) with (3.1) gives us the following intrinsic super Poincaré inequality:

$$\mu_{\phi_1}(f^2) \leqslant s(D_{\phi_1}(f,f) + \lambda_1\mu_{\phi_1}(f^2)) + C_0^2\alpha\Big(\Phi^{-1}\Big(\frac{2}{s}\Big), \frac{s}{2}\Big)\mu_{\phi_1}^2(|f|).$$

In particular, for any $s \in (0, 1/(2\lambda_1))$,

$$\mu_{\phi_1}(f^2) \leqslant 2sD_{\phi_1}(f,f) + 2C_0^2\alpha\Big(\Phi^{-1}\Big(\frac{2}{s}\Big), \frac{s}{2}\Big)\mu_{\phi_1}(|f|)^2, \quad f \in C_c^2(\mathbb{R}^d),$$

which implies that

$$\mu_{\phi_1}(f^2) \leqslant sD_{\phi_1}(f,f) + \beta(s)\mu_{\phi_1}(|f|)^2, \quad f \in C_c^2(\mathbb{R}^d),\ s > 0,$$

where $\beta(s)$ is the rate function defined by (1.6) with some proper constants $\delta_1, \delta_2 > 0$.

Therefore, the desired assertions for the ultracontractivity, supercontractivity, and hypercontractivity of the semigroup $(\widetilde{T}_t^V)_{t\geqslant 0}$ (or, equivalently, the intrinsic ultracontractivity, intrinsic supercontractivity, and intrinsic hypercontractivity of the semigroup $(T_t^V)_{t\geqslant 0}$) follow from [18, Theorem 3.3.13] and [16, Theorem 3.1]. □

Proof of Theorem 1.5 By (1.1) and $d > \alpha_1$, there is a constant $c_1 := c_1(\kappa) > 0$ such that the following Sobolev inequality holds:

$$\|f\|^2_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;\mathrm{d}x)} \leqslant c_1(D(f,f) + \|f\|^2_{L^2(\mathbb{R}^d;\mathrm{d}x)}), \quad f \in C_c^\infty(\mathbb{R}^d), \tag{3.3}$$

see [4, Proposition 3.7].

For the constant K in (A), let

$$A_1 := \{x \in \mathbb{R}^d \colon V(x) > K\}, \quad A_2 := \mathbb{R}^d \setminus A_1.$$

Then, for any $R > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{B(0,R)^c} f^2(x)\mathrm{d}x &= \int_{B(0,R)^c\cap A_1} f^2(x)\mathrm{d}x + \int_{B(0,R)^c\cap A_2} f^2(x)\mathrm{d}x\\ &\leqslant \frac{1}{\Phi_K(R)}\int_{B(0,R)^c\cap A_1} f^2(x)V(x)\mathrm{d}x\\ &\quad + |B(0,R)^c\cap A_2|^{\alpha_1/d}\,\|f\|^2_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;\mathrm{d}x)}\\ &\leqslant \frac{1}{\Phi_K(R)}\,D^V(f,f) + \Theta_K(R)^{\alpha_1/d}\|f\|^2_{L^{2d/(d-\alpha_1)}(\mathbb{R}^d;\mathrm{d}x)}.\end{aligned}$$

This, along with (2.1), (2.3), and (3.3), gives us that for any $R, \widetilde{s} > 0$,

$$\begin{aligned}\int f^2(x)\mathrm{d}x &\leqslant \Big(\frac{1}{\Phi_K(R)} + \widetilde{s} + c_1\Theta_K(R)^{\alpha_1/d}\Big)D^V(f,f) \\ &\quad + C_0^2\alpha(R,\widetilde{s})\Big(\int |f|(x)\phi_1(x)\mathrm{d}x\Big)^2 + c_1\Theta_K(R)^{\alpha_1/d}\int f^2(x)\mathrm{d}x \\ &\leqslant (\Psi_K(R)^{-1} + \widetilde{s})D^V(f,f) + C_0^2\alpha(R,\widetilde{s})\Big(\int |f|(x)\phi_1(x)\mathrm{d}x\Big)^2 \\ &\quad + \Psi_K(R)^{-1}\int f^2(x)\mathrm{d}x,\end{aligned}$$

where Ψ_K is defined in the theorem with $\delta_4 = c_1$.

For any $s > 0$, taking

$$R = \Psi_K^{-1}\Big(\frac{4}{s}\Big) \wedge \Psi_K^{-1}(2)$$

and $\widetilde{s} = s/4$ in the inequality above, we arrive at

$$\begin{aligned}\int f^2(x)\mathrm{d}x \leqslant sD^V(f,f) + 2C_0^2\alpha\Big(\Psi_K^{-1}\Big(\frac{4}{s}\Big) \wedge \Psi_K^{-1}(2), \frac{s}{4}\Big) \\ \cdot\Big(\int |f|(x)\phi_1(x)\mathrm{d}x\Big)^2.\end{aligned} \tag{3.4}$$

According to the intrinsic super Poincaré inequality (3.4) and the argument of Theorem 1.1 (2), we can obtain the desired conclusions. □

Finally, we present the proofs of Examples 1.2, 1.4, and 1.6.

Proof of Example 1.2 Let $V(x) = (1 + |x|)^\lambda$ for some $\lambda > 0$. Then, according to Theorem 1.1, the rate function β given by (1.6) satisfies that

$$\beta(s) = c_1 \exp(c_2(1 + s^{-\gamma/\lambda})).$$

Therefore, by Theorem 1.1 (1), the semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically ultracontractive for any $\lambda > \gamma$. To verify that the semigroup $(T_t^V)_{t\geqslant 0}$ is not intrinsically ultracontractive for $\lambda \in (0, \gamma]$, we can follow the proof of Example 1.4 (1) below, by using [8, (1.18)] instead. We omit the details here.

The lower bound estimate for ϕ_1 follows from Proposition 2.1. Now, we turn to the upper bound estimate. It is easy to check that for any $r > 0$ large enough,

$$x \mapsto \mathbb{1}_{B(0,2r)^c}\int_{\{|x+z|\leqslant r\}} J(x, x+z)\mathrm{d}z \in L^2(\mathbb{R}^d; \mathrm{d}x), \tag{3.5}$$

According to [22, Theorem 1.1], (1.7), and (3.5), $C_c^2(\mathbb{R}^d) \subset \mathscr{D}(L^V)$ and for any

$f \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}L^V f(x) = & \int_{\mathbb{R}^d} (f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leqslant 1\}}) J(x, x+z) \mathrm{d}z \\ & + \frac{1}{2} \int_{\{|z|\leqslant 1\}} \langle \nabla f(x), z\rangle (J(x, x+z) - J(x, x-z)) \mathrm{d}z - V(x) f(x) \\ =: & \, Lf(x) - V(x) f(x).\end{aligned}$$

Let

$$\psi(x) = \frac{\mathrm{e}^{-(1+|x|^2)^{\gamma/2}}}{C_0 + (1+|x|^2)^{\lambda/2}},$$

where $C_0 \geqslant 1$ is a constant to be determined later. By the approximation argument, it is easy to verify that $\psi \in \mathscr{D}(L^V)$. Next, we set

$$\rho(z) = |z|^{-d-\alpha} \mathbb{1}_{\{|z|\leqslant 1\}} + \mathrm{e}^{-|z|^\gamma} \mathbb{1}_{\{|z|>1\}}.$$

Then, for any $x \in \mathbb{R}^d$ with $|x| > 3$,

$$\begin{aligned}L\psi(x) = & \int_{\{|z|\leqslant 1\}} (\psi(x+z) - \psi(x) - \langle \nabla \psi(x), z\rangle) J(x, x+z) \mathrm{d}z \\ & + \int_{\{|z|>1\}} (\psi(x+z) - \psi(x)) J(x, x+z) \mathrm{d}z \\ & + \frac{1}{2} \int_{\{|z|\leqslant 1\}} \langle \nabla \psi(x), z\rangle (J(x, x+z) - J(x, x-z)) \mathrm{d}z, \\ \leqslant & \, c_3 \psi(x) + \int_{\{|z|>1\}} \frac{c_4 \mathrm{e}^{-(1+|x+z|^2)^{\gamma/2}}}{C_0 + V(x+z)} \rho(z) \mathrm{d}z \\ \leqslant & \, c_3 \psi(x) + \frac{c_4}{C_0} \int_{\{|x+z|\leqslant 1\}} \rho(z) \mathrm{d}z + \frac{c_4}{C_0} \int_{\{|z|>1, \, |x+z|>1\}} \rho(z) \rho(x+z) \mathrm{d}z \\ \leqslant & \, c_3 \psi(x) + \frac{c_5}{C_0} \sup_{z \in B(x,1)} \rho(z) + \frac{c_6}{C_0} \rho(x) \\ \leqslant & \, c_3 \psi(x) + \frac{c_7}{C_0} \rho(x),\end{aligned}$$

where the constants c_i ($i = 3, 4, \ldots, 7$) are independent of the choice of C_0. Here, in the first inequality, we have used (1.7) and the fact that there exists a constant $c_0 > 0$ such that for all $x \in \mathbb{R}^d$ with $|x| \geqslant 3$,

$$\sup_{z \in B(x,1)} (|\nabla \psi(x)| + |\nabla^2 \psi(x)|) \leqslant c_0 \psi(x),$$

and the third and the forth inequalities follow from (H1) and (H2) (they have been verified in [13, Example 4.1]). Thus, for any $x \in \mathbb{R}^d$ with $|x|$ large enough,

$$L^V \psi(x) \leqslant c_3 \psi(x) + \frac{c_7}{C_0} \rho(x) - \frac{V(x) \mathrm{e}^{-(1+|x|^2)^{\gamma/2}}}{C_0 + (1+|x|^2)^{\lambda/2}}.$$

In particular, taking $C_0 \geqslant 1 + 2c_7$ large enough in the inequality above, by the fact that

$$V(x) = |x|^\lambda \to +\infty, \quad |x| \to +\infty,$$

we get

$$L^V \psi(x) \leqslant 0$$

for $|x|$ large enough. On the other hand, since $\psi \in C_b^2(\mathbb{R}^d)$, it is easy to check that the function $x \mapsto L^V \psi(x)$ is locally bounded. Therefore, there exists $\lambda > 0$ such that for any $x \in \mathbb{R}^d$,

$$L^V \psi(x) \leqslant \lambda \psi(x),$$

which implies that

$$T_t^V \psi(x) \leqslant \mathrm{e}^{\lambda t} \psi(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

Furthermore, according to [11, Theorem 3.2], the intrinsic ultracontractivity of $(T_t^V)_{t \geqslant 0}$ implies that for every $t > 0$, there is a constant $c_t > 0$ such that

$$p^V(t, x, y) \geqslant c_t \phi_1(x) \phi_1(y), \quad x, y \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned} \psi(x) &\geqslant \mathrm{e}^{-\lambda} T_1^V \psi(x) \\ &= \mathrm{e}^{-\lambda} \int p^V(1, x, y) \psi(y) \mathrm{d}y \\ &\geqslant c_8 \mathrm{e}^{-\lambda} \int \psi(y) \phi_1(y) \mathrm{d}y \phi_1(x) \\ &= c_9 \phi_1(x), \end{aligned}$$

which yields the required upper bound for the ground state ϕ_1. □

Proof of Example 1.4 (1) Let $V(x) = \log^\lambda(1 + |x|)$ for some $\lambda > 0$. Then, according to Theorem 1.1, the rate function β given by (1.6) satisfies

$$\beta(s) = c_1 \exp(c_2(1 + s^{-1/\lambda})). \tag{3.6}$$

Therefore, by Theorem 1.1 (1), the semigroup $(T_t^V)_{t \geqslant 0}$ is intrinsically ultracontractive for any $\lambda > 1$.

To prove that for any $\lambda \in (0, 1]$, the semigroup $(T_t^V)_{t \geqslant 0}$ is not intrinsically ultracontractive, we mainly follow the proof of [14, Theorem 1.6] (see [14, pp. 5055, 5056]). Let $p(t, x, y)$ be the heat kernel for the symmetric α-stable process $(X)_{t \geqslant 0}$. It is well known that for any fixed $t \in (0, 1]$ and $|x - y|$ large enough,

$$p(t, x, y) \leqslant \frac{c_3 t}{|x - y|^{d+\alpha}}.$$

Set $D = B(0, 1)$. For $|x|$ large enough,

$$T_t^V(\mathbb{1}_D)(x) \leqslant \int_D p(t, x, y) \mathrm{d}y \leqslant \frac{c_4 t}{|x|^{d+\alpha}}. \tag{3.7}$$

On the other hand, since $\lambda \in (0,1]$, for $|x|$ large enough and $t \in (0,1]$,

$$\begin{aligned} T_t^V(\mathbb{1}_{B(x,1)})(x) &\geqslant \mathbb{E}^x\bigg(\tau_{B(x,1)} > t; \exp\bigg(-\int_0^t V(X_s)\mathrm{d}s\bigg)\bigg) \\ &\geqslant c_5\mathbb{P}^x(\tau_{B(x,1)} > t)\mathrm{e}^{-t\log^\lambda |x|} \\ &\geqslant c_5\mathbb{P}^x(\tau_{B(x,1)} > 1)\mathrm{e}^{-t\log^\lambda |x|} \\ &\geqslant c_6\mathbb{P}^x(\tau_{B(x,1)} > 1)\mathrm{e}^{-t\log |x|} \\ &\geqslant \frac{c_7}{|x|^t}. \end{aligned}$$

Combining both conclusions above, we get that for any fixed $t \in (0, d+\alpha)$, there is not a constant $C_t > 0$ such that for $|x|$ large enough,

$$T_t^V(\mathbb{1}_D)(x) \geqslant C_t T_t^V(\mathbb{1}_{B(x,1)})(x),$$

which contradicts with [14, Condition 1.3]. Hence, according to the remark below [14, Condition 1.3], the semigroup $(T_t^V)_{t\geqslant 0}$ is not intrinsically ultracontractive.

(2) According to (3.6) and Theorem 1.1 (2), we know that if $\lambda > 1$, then the semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically supercontractive. Now, suppose that the semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically supercontractive for some $\lambda \in (0,1]$, which is equivalently saying that the semigroup $(\widetilde{T}_t^V)_{t\geqslant 0}$ defined by (1.5) is supercontractive for some $\lambda \in (0,1]$. Then, by [18, Theorem 3.3.13 (2)], we know that the super Poincaré inequality

$$\int f(x)^2\phi_1^2(x)\mathrm{d}x \leqslant rD_{\phi_1}(f,f) + \beta(r)\bigg(\int |f|(x)\phi_1^2(x)\mathrm{d}x\bigg)^2, \qquad r > 0,\ f \in C_c^2(\mathbb{R}^d), \tag{3.8}$$

holds with some rate function β such that

$$\lim_{r\to 0} r\log\beta(r) = 0,$$

where the bilinear form D_{ϕ_1} is given by (3.2).

For a fixed strictly positive $\phi \in C_b^2(\mathbb{R}^d)$ and any $\varepsilon > 0$, define

$$\hat{L}_\varepsilon f(x) = \frac{1}{\phi(x)}\int_{\{|x-y|\geqslant\varepsilon\}} (f(y)-f(x))\phi(y)\frac{c(d,\alpha)}{|x-y|^{d+\alpha}}\,\mathrm{d}y, \quad f \in C_c^2(\mathbb{R}^d).$$

Then

$$\begin{aligned} L^V(\phi f)(x) &= c(d,\alpha)\ \text{p.v.}\int ((\phi f)(y) - (\phi f)(x))\frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}y - V(x)(\phi f)(x) \\ &= \phi(x)\lim_{\varepsilon\to 0}\hat{L}_\varepsilon f(x) \\ &\quad + f(x)\bigg[c(d,\alpha)\,\text{p.v.}\int(\phi(y)-\phi(x))\frac{1}{|x-y|^{d+\alpha}}\mathrm{d}y - V(x)\phi(x)\bigg] \\ &= \phi(x)\lim_{\varepsilon\to 0}\hat{L}_\varepsilon f(x) + f(x)L^V\phi(x), \end{aligned}$$

where p.v. denotes the principal value integral. Therefore, for the probability measure $\mu(\mathrm{d}x)=\phi^2(x)\mathrm{d}x$, we get

$$\begin{aligned}
D^V(\phi f,\phi f) &= -\langle \phi f, L^V(\phi f)\rangle_{L^2(\mathbb{R}^d;\mathrm{d}x)}\\
&= -\Big\langle f, \frac{1}{\phi} L^V(\phi f)\Big\rangle_{L^2(\mathbb{R}^d;\mu)}\\
&= -\lim_{\varepsilon\to 0}\langle f, \hat{L}_\varepsilon f\rangle_{L^2(\mathbb{R}^d;\mu)} - \Big\langle f, \frac{f}{\phi} L^V\phi\Big\rangle_{L^2(\mathbb{R}^d;\mu)}\\
&= -\lim_{\varepsilon\to 0}\iint_{\{|x-y|\geqslant\varepsilon\}} \frac{c(d,\alpha)(f(y)-f(x))f(x)}{|x-y|^{d+\alpha}}\phi(y)\phi(x)\mathrm{d}x\mathrm{d}y\\
&\quad - \Big\langle f, \frac{f}{\phi} L^V\phi\Big\rangle_{L^2(\mathbb{R}^d;\mu)}\\
&= \frac{c(d,\alpha)}{2}\iint \frac{(f(y)-f(x))^2}{|x-y|^{d+\alpha}}\phi(y)\phi(x)\mathrm{d}x\mathrm{d}y - \Big\langle f, \frac{f}{\phi} L^V\phi\Big\rangle_{L^2(\mathbb{R}^d;\mu)},
\end{aligned}$$

where, in the third equality, we have used the dominated convergence theorem, and the last equality follows from the symmetry of kernel $c(d,\alpha)/|x-y|^{d+\alpha}$. Whence, if $\phi_1\in C_b^2(\mathbb{R}^d)$, then we have

$$\begin{aligned}
D_{\phi_1}(f,f) &= \frac{c(d,\alpha)}{2}\iint \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\phi_1(x)\phi_1(y)\mathrm{d}x\mathrm{d}y\\
&\quad - \Big\langle f, \frac{f}{\phi_1} L^V\phi_1\Big\rangle_{L^2(\mathbb{R}^d;\mu)} + \lambda_1\int_{\mathbb{R}^d} f^2(x)\phi_1^2(x)\mathrm{d}x\\
&= \frac{c(d,\alpha)}{2}\iint \frac{(f(x)-f(y))^2}{|x-y|^{d+\alpha}}\phi_1(x)\phi_1(y)\mathrm{d}x\mathrm{d}y. \qquad (3.9)
\end{aligned}$$

Since $C_c^2(\mathbb{R}^d)$ is a core for $(L^V,\mathscr{D}(L^V))$ and $\phi_1\in\mathscr{D}(L^V)$, by the standard approximation argument, we get that (3.9) is still true for ground state ϕ_1 without the assumption that $\phi_1\in C_b^2(\mathbb{R}^d)$.

Second, according to [13, Corollary 2.2] (in this case, [13, Assumption 2.3] holds true and so [13, Corollary 2.2] applies), there exists a constant $c_1>1$ such that

$$\frac{c_1^{-1}}{(1+|x|)^{d+\alpha}\log^\lambda(1+|x|)} \leqslant \phi_1(x) \leqslant \frac{c_1}{(1+|x|)^{d+\alpha}\log^\lambda(1+|x|)}. \qquad (3.10)$$

Third, we consider the following reference function $g_n\in C_b^2(\mathbb{R}^d)$ for $n\geqslant 1$ such that

$$g_n(x)\begin{cases} =0, & |x|\leqslant n,\\ \in[0,1], & n\leqslant|x|\leqslant 2n,\\ =1, & |x|\geqslant 2n,\end{cases}$$

and $|\nabla g_n(x)|\leqslant 2/n$ for all $x\in\mathbb{R}^d$. It is easy to see that

$$\int g_n(x)^2\phi_1^2(x)\mathrm{d}x \geqslant \frac{c_2}{n^{d+2\alpha}\log^{2\lambda}(1+n)}$$

and

$$\left(\int |g_n|(x)\phi_1^2(x)\mathrm{d}x\right)^2 \leqslant \frac{c_3}{n^{2d+4\alpha}\log^{4\lambda}(1+n)}$$

hold for some constants $c_2, c_3 > 0$. On the other hand,

$$\begin{aligned} D_{\phi_1}(g_n, g_n) &= c(d,\alpha)\iint_{\{|x|\leqslant n,\,|y|\geqslant n\}} \frac{(g_n(x)-g_n(y))^2}{|x-y|^{d+\alpha}}\,\phi_1(x)\phi_1(y)\mathrm{d}x\mathrm{d}y \\ &\quad + c(d,\alpha)\iint_{\{|x|>n\}} \frac{(g_n(x)-g_n(y))^2}{|x-y|^{d+\alpha}}\,\phi_1(x)\phi_1(y)\mathrm{d}x\mathrm{d}y \\ &=: I_1 + I_2. \end{aligned}$$

Then, by (3.10),

$$\begin{aligned} I_1 &\leqslant \frac{c_1^*}{n^{d+\alpha}\log^{\lambda}(1+n)}\bigg[\frac{1}{n^2}\iint_{\{|x-y|\leqslant n\}} \frac{|x-y|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\phi_1(x)\mathrm{d}x \\ &\quad + \iint_{\{|x-y|\geqslant n\}} \frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}y\phi_1(x)\mathrm{d}x\bigg] \\ &\leqslant \frac{c_2^*}{n^{d+2\alpha}\log^{\lambda}(1+n)}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &\leqslant \frac{c_3^*}{n^{d+\alpha}\log^{\lambda}(1+n)}\bigg[\frac{1}{n^2}\iint_{\{|x-y|\leqslant n\}} \frac{|x-y|^2}{|x-y|^{d+\alpha}}\,\mathrm{d}x\phi_1(y)\mathrm{d}y \\ &\quad + \iint_{\{|x-y|\geqslant n\}} \frac{1}{|x-y|^{d+\alpha}}\,\mathrm{d}x\phi_1(y)\mathrm{d}y\bigg] \\ &\leqslant \frac{c_4^*}{n^{d+2\alpha}\log^{\lambda}(1+n)}. \end{aligned}$$

Combining all the conclusions above, we obtain

$$\frac{c_2}{\log^{\lambda}(1+n)} \leqslant c_4 r + \frac{c_3\beta(r)}{n^{d+2\alpha}\log^{3\lambda}(1+n)}$$

for some constant $c_4 > 0$. Taking

$$r = r_n := \frac{c_2}{2c_4\log^{\lambda}(1+n)},$$

we get

$$\beta(r_n) \geqslant \frac{c_2}{2c_3}\,n^{d+2\alpha}\log^{2\lambda}(1+n).$$

In particular, due to $\lambda \in (0,1]$,

$$\limsup_{r\to 0} r\log\beta(r) \geqslant \limsup_{r\to 0} r^{1/\lambda}\log\beta(r) \geqslant \liminf_{n\to+\infty} r_n^{1/\lambda}\log\beta(r_n) > 0,$$

which contradicts with

$$\lim_{r\to 0} r\log\beta(r)=0.$$

This proves the second desired assertion.

(3) By (3.6) and Theorem 1.1 (3), the semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically hypercontractive for $\lambda\geqslant 1$. Assume that the semigroup $(T_t^V)_{t\geqslant 0}$ is intrinsically hypercontractive for some $\lambda\in(0,1)$. Then, by [18, Theorem 3.3.13 (1)], the super Poincaré inequality (3.8) holds with

$$\beta(r)\leqslant \exp(c(1+r^{-1})),\quad r>0. \tag{3.11}$$

Now, we can follow the proof of part (2) above, and obtain

$$\liminf_{n\to+\infty} r_n^{1/\lambda}\log\beta(r_n)>0,$$

where r_n is the same sequence as that in (2). In particular, $r_n\to 0$ as $n\to+\infty$, and

$$\beta(r_n)\geqslant \exp(c_1 r_n^{-1/\lambda})$$

for n large enough and some constant $c_1>0$. This is a contradiction with (3.11), also thanks to the fact that $\lambda\in(0,1)$. Hence, we complete the proof. □

Proof of Example 1.6 (1) Take $K=1$ in assumption (A). Then

$$\Phi_K(r)=\log^{\lambda}(1+r),\quad \Theta_K(r)=c_1\log^{-\theta}(1+r)$$

for $r\geqslant 1$ large enough. Thus, according to Theorem 1.5, the rate function $\hat\beta$ given by (1.8) satisfies

$$\hat\beta(s)\leqslant c_2\exp\big(c_3\big(1+s^{-\max(\frac{1}{\lambda},\frac{d}{\theta\alpha})}\big)\big).$$

This, along with Theorem 1.5 again, yields the first desired assertion.

(2) For any $R>0$ with $\mathrm{e}^{m^{k_0}}\leqslant R\leqslant \mathrm{e}^{(m+1)^{k_0}}$ for some $m\geqslant 1$, we have

$$\begin{aligned}|A\cap B(0,R)^c|&\leqslant \sum_{k=m}^{+\infty}|B(x_k,r_k)|\\ &=c_0\sum_{k=m}^{+\infty}k^{-\frac{dk_0}{\alpha}+1}\\ &\leqslant c_1 m^{-\frac{dk_0}{\alpha}+2}\\ &\leqslant c_2(m+1)^{k_0(-\frac{d}{\alpha}+\frac{2}{k_0})}\\ &\leqslant \frac{c_2}{\log^{\frac{d}{\alpha}-\varepsilon}R}.\end{aligned}$$

This proves (1.10).

Let $D = B(0,1)$ and $t = 1$. According to (3.7), for all m large enough,

$$T_1^V(\mathbb{1}_D)(x_m) \leqslant \frac{c_3}{|x_m|^{d+\alpha}} = c_3 \exp(-(d+\alpha)m^{k_0}). \tag{3.12}$$

On the other hand, by the definition of V and the space-homogeneous property and scaling property of symmetric α-stable process, for m large enough,

$$\begin{aligned} T_1^V(\mathbb{1}_{B(x_m,1)})(x_m) &\geqslant T_1^V(\mathbb{1}_{B(x_m,r_m)})(x_m) \\ &\geqslant \mathbb{E}^{x_m}\bigg(\tau_{B(x_m,r_m)} > 1; \exp\bigg(-\int_0^1 V(X_s)\mathrm{d}s\bigg)\bigg) \\ &= \mathrm{e}^{-1}\mathbb{P}^{x_m}(\tau_{B(x_m,r_m)} > 1) \\ &= \mathrm{e}^{-1}\mathbb{P}^0(\tau_{B(0,r_m)} > 1) \\ &= \mathrm{e}^{-1}\mathbb{P}^0(\tau_{B(0,1)} > r_m^{-\alpha}). \end{aligned}$$

Let $p_B(t,x,y)$ be the Dirichlet heat kernel of symmetric α-stable process killed on exiting B. We find that the right-hand side of the inequality above is just

$$\int_{B(0,1)} p_{B(0,1)}(r_m^{-\alpha}, 0, z)\mathrm{d}z \geqslant c_4 \mathrm{e}^{-\lambda r_m^{-\alpha}} = c_4 \mathrm{e}^{-\lambda m^{k_0 - \frac{\alpha}{d}}}$$

for some positive constants c_4 and λ, where the inequality above follows from [9, Theorem 1.1 (ii)]. Hence, we have

$$T_1^V(\mathbb{1}_{B(x_m,1)})(x_m) \geqslant c_4 \mathrm{e}^{-\lambda m^{k_0 - \frac{\alpha}{d}}}. \tag{3.13}$$

According to (3.12) and (3.13), we know that for any constant $C > 0$, the inequality

$$T_1^V(\mathbb{1}_{B(x,1)})(x) \leqslant C T_1^V(\mathbb{1}_D)(x).$$

does not hold for $x = x_m$ with m large enough. In particular, [14, Condition 1.3] is not satisfied, and so the semigroup $(T_t^V)_{t\geqslant 0}$ is not intrinsically ultracontractive. □

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