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RESEARCH ARTICLE

Modules for double affine Lie algebras

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Abstract Imaginary Verma modules, parabolic imaginary Verma modules, and Verma modules at level zero for double affine Lie algebras are constructed using three different triangular decompositions. Their relations are investigated, and several results are generalized from the affine Lie algebras. In particular, imaginary highest weight modules, integrable modules, and irreducibility criterion are also studied.

Keywords Double affine Lie algebra, Verma module, irreducibility, Weyl module
MSC 17B67, 17B10, 17B65

1 Introduction

Let $\hat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated to a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. The highest weight irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ of the highest weight λ can be studied with the help of the Verma module $V(\lambda)$, which is an induced module of the one-dimensional module $\mathbb{C}1_\lambda$ of the Borel subalgebra $\hat{\mathfrak{b}} = \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+$ such that $\hat{\mathfrak{n}}_+ 1_\lambda = 0$. If one partitions the affine root system using the loop realization of $\hat{\mathfrak{g}}$, the associated imaginary Verma module [7] behaves quite differently. For example, the new Verma module can have infinite-dimensional weight subspaces.

Double affine Lie algebras are certain central extensions of maps from a 2-dimensional torus to the Lie algebra $\mathfrak{g}$. They are analogous to $\hat{\mathfrak{g}}$ but with two centers, and first appeared in Frenkel's work [6] on affinization of Kac-Moody algebras. Moody and Shi [12] have shown that the root systems have different properties from those of the affine root systems. For example, some roots cannot be spanned positively or negatively by the 'simple' ones. Thus, a usual highest weight module would blow up beyond control. In this paper, we generalize imaginary Verma modules (IVM) to double affine Lie algebras and

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use them to produce irreducible modules.

To study these imaginary Verma modules, we consider generalized imaginary Verma modules, which bare some similarity to the parabolic Verma modules in classical Lie theory. The structure of IVM's is characterized by the generalized IVM's, and we show that they are irreducible if the second center is nonzero. When the second center is zero, they are similar to the evaluation modules of the affine Lie algebras, for which we also generalize several results from the Tits-Kantor-Koecher algebras [1].

To further understand the situation of trivial centers, we adopt Chari and Pressley's technique [3,4] of Weyl modules to modules of double affine Lie algebras, which has played an important role for loop algebras (see [2] for a survey). We remark that the triangular decomposition employed in our case is different from that used in previous work on toroidal Lie algebras (cf. [8,13,14]). In our modules, the Borel subalgebras are defined by carving out the second imaginary part to control the growth of the IVM's.

2 Double affine Lie algebras

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of simply laced type with $\mathfrak{h}$ its Cartan subalgebra. Let Δ be the root system generated by the simple roots α_i $(i = 1, 2, \ldots, s)$, and let $\alpha_i^\vee \in \mathfrak{h}$ $(i = 1, 2, \ldots, s)$ be the corresponding simple coroots such that

$$\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij},$$

the entries of the Cartan matrix of $\mathfrak{g}$. Denote by

$$\theta = \sum_{i=1}^{s} k_i' \alpha_i$$

the longest root of $\mathfrak{g}$, and $\theta^\vee$ the corresponding dual element in $\mathfrak{h}$. For any positive root

$$\alpha = \sum_i c_i \alpha_i \in \Delta,$$

we denote its height by

$$ht(\alpha) = \sum_{i=1}^{s} c_i.$$

The double affine Lie algebra $\overline{\mathfrak{T}}$ is the central extension of the 2-loop algebra defined by

$$\overline{\mathfrak{T}} = \mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}] \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2,$$

where $\mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}]$ is the ring of Laurent polynomials in two commuting variables t_1 and t_2. Writing $x \otimes t_1^m t_2^n$ as $x(m, n)$, the Lie bracket is given by

$$\begin{aligned}
&[x(m_1, n_1), y(m_2, n_2)] \\
&\quad = [x, y](m_1 + m_2, n_1 + n_2) + (x_1 \mid x_2)\delta_{m_1,-n_1}\delta_{m_2,-n_2}(m_1 c_1 + m_2 c_2), \quad (1)
\end{aligned}$$

$$[c_1, x(m_1, n_1)] = [c_2, x(m_1, n_1)] = 0, \tag{2}$$

where $(m_1, m_2), (n_1, n_2) \in \mathbb{Z}^2$, $x_1, x_2, x \in \mathfrak{g}$, and $(x_1 \mid x_2)$ is the $\mathfrak{g}$-invariant bilinear form. Adjoining the derivations, we define the extended double affine Lie algebra $\mathfrak{T}$ as

$$\mathfrak{T} = \mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}] \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2.$$

The derivations act on $\mathfrak{T}$ via

$$[d_i, x(m_1, m_2)] = m_i x(m_1, m_2), \quad [d_i, c_j] = 0, \quad i, j = 1, 2. \tag{3}$$

Let

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$$

be the Cartan subalgebra of $\mathfrak{T}$, and let $\hat{\mathfrak{h}}^*$ be the dual space. For $\beta \in \hat{\mathfrak{h}}^*$, the root subspace

$$\mathfrak{T}_\beta = \{x \in \mathfrak{T} \mid [h, x] = \beta(h)x, \,\forall\, h \in \hat{\mathfrak{h}}\}$$

is defined in the usual way. We then define the root system $\Delta_{\mathfrak{T}}$ to be the set of all nonzero $\beta \in \hat{\mathfrak{h}}^*$ such that $\mathfrak{T}_\beta \neq 0$. It is known that [12] the root system is different from the usual root system in that a root is no longer a positive or negative sum of the 'simple' roots.

To be specific, we let $\delta_i \in \hat{\mathfrak{h}}^*$ such that

$$\delta_i(\mathfrak{h}) = \delta_i(c_j) = 0, \quad \delta_i(d_j) = \delta_{ij}, \quad i, j = 1, 2.$$

Then the extended root system

$$\Delta_{\mathfrak{T}} = \{\alpha + \mathbb{Z}\delta_1 + \mathbb{Z}\delta_2 \mid \alpha \in \Delta\} \cup (\{\mathbb{Z}\delta_1 + \mathbb{Z}\delta_2\} \setminus \{0\}),$$

where the first and second subsets form the real and imaginary roots, denoted by $\Delta_{\mathfrak{T}}^{\mathrm{re}}$ and $\Delta_{\mathfrak{T}}^{\mathrm{im}}$, respectively. The corresponding root subspaces are

$$\mathfrak{T}_{m\delta_1 + n\delta_2} = \mathfrak{h} \otimes t_1^m t_2^n, \quad (m, n) \neq (0, 0),$$
$$\mathfrak{T}_{\alpha + m\delta_1 + n\delta_2} = \mathfrak{g}_\alpha \otimes t_1^m t_2^n, \quad (m, n) \in \mathbb{Z} \times \mathbb{Z}.$$

Then one has the root space decomposition

$$\mathfrak{T} = \hat{\mathfrak{h}} \oplus \bigoplus_{\beta \in \Delta_{\mathfrak{T}}} \mathfrak{T}_\beta.$$

Obviously, the extended double affine Lie algebra $\mathfrak{T}$ contains two affine Lie algebras as subalgebras:

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}[t_i, t_i^{-1}] \oplus \mathbb{C}c_i \oplus \mathbb{C}d_i, \quad i = 1, 2.$$

Recall that [10] $\alpha_0 = \delta_1 - \theta, \alpha_1, \alpha_2, \ldots, \alpha_s$ are the simple roots of $\hat{\mathfrak{g}}_1$. Similarly, the roots $\alpha_{-1} = \delta_2 - \theta, \alpha_1, \alpha_2, \ldots, \alpha_s$ are simple roots for $\hat{\mathfrak{g}}_2$. Following [12], we call the elements $\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_s$ the 'simple' roots of $\mathfrak{T}$. However, not all roots can be represented as non-negative or non-positive linear combination of the 'simple' roots. The corresponding coroots are $\alpha_{-1}^\vee, \alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_s^\vee$, where

$$\alpha_{-1}^\vee = c_2 - \theta^\vee \otimes 1, \quad \alpha_0^\vee = c_1 - \theta^\vee \otimes 1.$$

Consequently, the Cartan subalgebra $\hat{\mathfrak{h}}$ is spanned by $\alpha_{-1}^\vee, \alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_s^\vee, d_1, d_2$.

3 Imaginary Verma modules of $\mathfrak{T}$

Verma modules of affine Lie algebras are defined with the help of a triangular decomposition, which is constructed by a choice of the base in the root system. Since the root system for the extended double affine algebras cannot be divided into positive and negative roots in the usual sense, we use a closed subset to partition the root system. For affine Lie algebras, Futorny [7] studied a new class of Verma modules called the imaginary Verma modules (IVM) [5,15] associated with a closed subset defined by a function. For finite-dimensional simple Lie algebras, the partition derived from the closed subset coincides with the usual partition of positive and negative roots. In this section, we introduce the imaginary Verma modules for the extended double affine Lie algebras and derive their important properties.

3.1 Imaginary Verma modules

Set

$$\varphi = \sum_{i=0}^{s} \alpha_i^* - ht(\theta)\alpha_{-1}^*,$$

where $\alpha_i^* \in \hat{\mathfrak{h}}$ such that $\alpha_i^*(\alpha_j) = \delta_{ij}$ $(i, j = -1, 0, \ldots, s)$. Clearly,

$$\varphi(\alpha_i) = 1, \ i = 0, 1, \ldots, s, \quad \varphi(\delta_2) = 0, \quad \varphi(\delta_1) = 1 + ht(\theta).$$

Let

$$\mathscr{I} = \{\alpha \in \Delta_{\mathfrak{T}} \mid \varphi(\alpha) > 0\} \cup \mathbb{N}\delta_2.$$

Then it is closed under the addition.

Recall [10] that the root system of $\hat{\mathfrak{g}}_1$ is

$$\Delta_{\hat{\mathfrak{g}}_1} = \{\alpha + \mathbb{Z}\delta_1 \mid \alpha \in \Delta\} \cup (\{\mathbb{Z}\delta_1\} \backslash \{0\}),$$

and its positive roots are

$$\Delta_{\hat{\mathfrak{g}}_1+} = \{\pm\alpha + \mathbb{N}\delta_1 \mid \alpha \in \Delta\} \cup \Delta_+ \cup \mathbb{N}\delta_1.$$

Note that for any $\alpha \in \Delta_{\hat{\mathfrak{g}}_1+}$, $\varphi(\alpha) = ht(\alpha)$. Then $\mathscr{I}$ can be written as

$$\mathscr{I} = \{\alpha + \mathbb{Z}\delta_2 \mid \alpha \in \Delta_{\hat{\mathfrak{g}}_1+}\} \cup \mathbb{N}\delta_2.$$

Subsequently,

$$\mathscr{I} \cup (-\mathscr{I}) = \Delta_{\mathfrak{T}}, \quad \mathscr{I} \cap (-\mathscr{I}) = \emptyset.$$

Denote by $\mathscr{Q}(\mathscr{I})$ the $\mathbb{Z}$-span of $\mathscr{I}$.

Let

$$\mathfrak{T}_{\mathscr{I}} = \bigoplus_{\beta \in \mathscr{I}} \mathfrak{T}_\beta, \quad \mathfrak{T}_{-\mathscr{I}} = \bigoplus_{\beta \in -\mathscr{I}} \mathfrak{T}_\beta.$$

Then

$$\mathfrak{T} = \mathfrak{T}_{-\mathscr{I}} \oplus \hat{\mathfrak{h}} \oplus \mathfrak{T}_{\mathscr{I}} \tag{4}$$

is a triangular decomposition of $\mathfrak{T}$ associated with the closed subset $\mathscr{I}$. The Poincaré-Birkhoff-Witt (PBW) theorem implies that the universal enveloping algebra

$$U(\mathfrak{T}) = U(\mathfrak{T}_{-\mathscr{I}}) \otimes U(\hat{\mathfrak{h}}) \otimes U(\mathfrak{T}_{\mathscr{I}}).$$

Let $\mathfrak{b}_{\mathscr{I}} = \hat{\mathfrak{h}} \oplus \mathfrak{T}_{\mathscr{I}}$ be the imaginary Borel subalgebra, which is a solvable Lie algebra. A vector space V is called a weight module of $\mathfrak{T}$ if

$$V = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} V_\mu,$$

where

$$V_\mu = \{v \in V \mid hv = \mu(h)v, \forall\, h \in \hat{\mathfrak{h}}\}.$$

Set

$$\mathscr{P}(V) = \{\mu \in \hat{\mathfrak{h}}^* \mid V_\mu \neq 0\}.$$

We say that $\lambda \geqslant \mu$ ($\lambda, \mu \in \hat{\mathfrak{h}}^*$) with respect to φ if $\lambda - \mu$ is a non-negative linear combination of roots in $\mathscr{I}$. For simplicity, we will omit the reference to φ if no confusion arises from the context. For example, both $\mathfrak{T}$ and $U(\mathfrak{T})$ are weight modules for $\mathfrak{T}$.

Definition 1 Let $\lambda \in \hat{\mathfrak{h}}^*$. A nonzero vector v is called an imaginary highest vector with weight λ if $\mathfrak{T}_{\mathscr{I}}.v = 0$ and $h.v = \lambda(h)v$ for all $h \in \hat{\mathfrak{h}}$. If $V = U(\mathfrak{T})v$, then V is called a highest weight module of highest weight λ.

If V is a highest weight module of weight λ, then

$$V = U(\mathfrak{T}_{-\mathscr{I}})v = \bigoplus_{\eta \in \mathscr{Q}(\mathscr{I})^+} V_{\lambda - \eta}, \tag{5}$$

where $\mathscr{Q}(\mathscr{I})^+ = \mathbb{Z}_+$-span of $\mathscr{I}$.

For $\lambda \in \hat{\mathfrak{h}}^*$, let $\mathbb{C}1_\lambda$ be the one-dimensional $\mathfrak{b}_{\mathscr{I}}$-module defined by

$$(x + h).1_\lambda = \lambda(h) \cdot 1_\lambda, \quad x \in \mathfrak{T}_{\mathscr{I}},\, h \in \hat{\mathfrak{h}}.$$

The imaginary Verma module is the induced module

$$\overline{M}(\lambda) = U(\mathfrak{T}) \otimes_{U(\mathfrak{b}_{\mathscr{I}})} \mathbb{C}1_\lambda.$$

3.2 Properties of IVM

Based on the theory of the standard Verma modules [10] and IVM's for the affine Lie algebras [7], the following result can be proved similarly.

Proposition 1 *For any $\lambda \in \hat{\mathfrak{h}}^*$, one has the following statements.*

(i) *$\overline{M}(\lambda)$ is a $U(\mathfrak{T}_{-\mathscr{I}})$-free module generated by the imaginary highest vector $1 \otimes 1_\lambda$ of weight λ.*

(ii) (a) $\dim \overline{M}(\lambda)_\lambda = 1$;

(b) $0 < \dim \overline{M}(\lambda)_{\lambda - k\delta_2} < +\infty$ *for every positive integer k;*

(c) *If $\overline{M}(\lambda)_\mu \neq 0$ and $\mu \neq \lambda - k\delta_2$ for any nonnegative integer k, then we have* $\dim \overline{M}(\lambda)_\mu = +\infty$.

(iii) *Any imaginary highest weight $\mathfrak{T}$-module of highest weight λ is a quotient of $\overline{M}(\lambda)$.*

(iv) *$\overline{M}(\lambda)$ has a unique maximal submodule $\mathscr{J}$.*

(v) *If $\mu \in \hat{\mathfrak{h}}^*$, then any nonzero homomorphism $\overline{M}(\lambda) \to \overline{M}(\mu)$ is injective.*

We denote by $\overline{L}(\lambda)$ the irreducible quotient $\overline{M}(\lambda)/\mathscr{J}$.

3.3 Irreducibility criterion for IVM

Futorny [7] found that the affine imaginary Verma module is irreducible if and only if the center acts nontrivially. It turns out that a similar result can be obtained for $\overline{M}(\lambda)$ (see Theorem 1 below).

Lemma 1 *Let*

$$\overline{M} = \bigoplus_{j=0}^{+\infty} \overline{M}(\lambda)_{\lambda - j\delta_2}.$$

For any nonzero $v \in \overline{M}(\lambda)$,

$$U(\mathfrak{T})v \cap \overline{M} \neq 0.$$

Proof Write $\lambda = \overline{\lambda} - r\delta_2$, where $\overline{\lambda}$ is the component of $\Delta_{\hat{\mathfrak{g}}_1}$. We can assume that r is minimum so that $v_\lambda \neq 0$, otherwise we can replace λ by λ' and $\overline{M}(\lambda) = \overline{M}(\lambda')$, where $\lambda \equiv \lambda' \pmod{\mathbb{Z}\delta_2}$. Therefore, $h \otimes t_2^{-n} v_\lambda \neq 0$ for any $n \geqslant 0$. Assume that $v \in \overline{M}(\lambda)_{\lambda - \mu}$, where

$$\mu = \sum_{i=0}^{s} n_i \alpha_i + k\delta_2, \quad n_i \in \mathbb{Z}_+,\ k \in \mathbb{Z}.$$

Define the height

$$ht(\mu) = \sum_{i=0}^{s} n_i$$

and use induction on $ht(\mu)$. If $ht(\mu) = 0$, then the weight of v is $\lambda - k\delta_2$ for some $k \in \mathbb{Z}_+$ by assumption, so the result clearly holds.

Let $e_i, f_i, \alpha_i^\vee$ $(0 \leqslant i \leqslant s)$ be the Chevalley generators of the derived affine Lie algebra $\hat{\mathfrak{g}}_1$. If $ht(\mu) > 0$, then there exists $i_0 \in \{0, 1, \dots, s\}$ such that $e_{i_0} v \neq 0$. In fact, when $ht(\mu) = 1$, say $v = f_i \otimes t_2^{-k} v_\lambda$. If $k < 0$, then

$$\alpha_i^\vee \otimes t_2^{k-1} v = -2 f_i \otimes t_2^{-1} v_\lambda \neq 0.$$

Then let

$$v' = \alpha_i^\vee \otimes t_2^{k-1} v, \quad e_i v' = -2\alpha_i^\vee \otimes t_2^{-1} v_\lambda \neq 0.$$

The case of $ht(\mu) \geqslant 2$ is treated similarly. Moreover, $e_{i_0}(h \otimes t_2^{-m}).v' \neq 0$ for all $m \geqslant 0$. Hence, $\mathfrak{T}_{\alpha_{i_0} - m\delta_2}.v' \neq 0$, and any of its nonzero element has weight $\lambda - \mu + \alpha_{i_0} - (m+1)\delta_2$. As $ht(\mu - \alpha_{i_0}) = ht(\mu) - 1$, by inductive hypothesis, we have

$$U(\mathfrak{T})(\mathfrak{T}_{\alpha_{i_0} - m\delta_2}.v') \cap \overline{M} \neq 0.$$

Since $U(\mathfrak{T})(\mathfrak{T}_{\alpha_{i_0} - m\delta_2}.v') \subset U(\mathfrak{T})v$, it follows that $U(\mathfrak{T})v \cap \overline{M} \neq 0$. $\square$

Let

$$\overline{M}(\lambda)^+ = \{v \in \overline{M}(\lambda) \mid \mathfrak{T}_{\mathscr{I}}.v = 0\}$$

be the space of extremal vectors. Clearly, $\overline{M}(\lambda)^+$ is $\hat{\mathfrak{h}}$-invariant. For any nonzero element $v \in \overline{M}(\lambda)^+$, $U(\mathfrak{T}).v$ generates a submodule of $\overline{M}(\lambda)$. The following result describes the form of extremal vectors.

Corollary 1 $\overline{M}(\lambda)^+ \subset \overline{M}$.

Proof Suppose on the contrary that there exists a nonzero $v \in \overline{M}(\lambda)^+ \cap \overline{M}(\lambda)_{\lambda-\mu}$ such that

$$\mu = \sum_{i=0}^{s} n_i \alpha_i + k\delta_2, \quad ht(\mu) = \sum_{i=0}^{s} n_i > 0.$$

Note that $U(\mathfrak{T})v = U(\mathfrak{T}_{-\mathscr{I}})v$. Then the weight of any homogeneous vector in $U(\mathfrak{T})v$ is $\lambda - \mu - \nu$ for $\nu \in \mathscr{Q}(\mathscr{I})^+$. As $ht(\mu) > 0$, $\lambda - \mu - \nu \neq \lambda \pmod{\mathbb{Z}\delta_2}$. Hence, $U(\mathfrak{T})v \cap \overline{M} = 0$, which contradicts with Lemma 1 on $\overline{M}(\lambda)_{\lambda-\mu}$. $\square$

Define the Heisenberg subalgebra

$$\hat{\mathfrak{H}}_2 = \bigoplus_{n \in \mathbb{Z}^\times} \mathfrak{h} \otimes t_2^n + \mathbb{C}c_2.$$

Then the space

$$\overline{M} = \bigoplus_{j=0}^{+\infty} \overline{M}(\lambda)_{\lambda - j\delta_2}$$

is a Verma module for $\hat{\mathfrak{H}}_2$. The following result is well known from Stone-von Neumann's theorem.

Lemma 2 *$\overline{M}$ is irreducible as a Verma module for $\hat{\mathfrak{H}}_2$ if and only if $\lambda(c_2) \neq 0$.*

Theorem 1 *The imaginary Verma module $\overline{M}(\lambda)$ is irreducible if and only if $\lambda(c_2) \neq 0$.*

Proof Let v_λ be the highest weight vector of $\overline{M}(\lambda)$. By definition, $\overline{M}(\lambda)$ is irreducible if and only if the space of extremal vectors $\overline{M}(\lambda)^+ = \mathbb{C}v_\lambda$.

Suppose that $\overline{M}(\lambda)$ is irreducible but $\lambda(c_2) = 0$. Since $\overline{M}$ is reducible as an $\hat{\mathfrak{H}}_2$-module by Lemma 2, there exists $w \neq 0$ with weight $\lambda - k\delta_2$ $(k > 0)$ such that $\mathfrak{T}_{l\delta_2}.w = 0$ for any $l > 0$. It is clear that $\mathfrak{T}_\beta.w = 0$ for all $\beta \in \{\alpha \in \Delta_{\mathfrak{T}} \mid \varphi(\alpha) > 0\}$, because the weight of $\mathfrak{T}_\beta.w$ is larger than λ. Thus, $w \notin \mathbb{C}v_\lambda$ is also an imaginary highest vector in $\overline{M}(\lambda)$, which is a contradiction. So we must have $\lambda(c_2) \neq 0$.

When $\lambda(c_2) \neq 0$, Lemma 2 implies that the $\hat{\mathfrak{H}}_2$-module $\overline{M}$ is irreducible. Consider any nonzero submodule $U(\mathfrak{T})v$, where $v \in \overline{M}(\lambda)$. Lemma 1 says that $U(\mathfrak{T})v \cap \overline{M} \neq 0$. Note that $U(\mathfrak{T})v \cap \overline{M}$ is then a non-trivial $\hat{\mathfrak{H}}_2$-submodule of $\overline{M}$. Consequently, $U(\mathfrak{T})v \cap \overline{M} = \overline{M}$, and thus, $\overline{M} \subset U(\mathfrak{T})v$. Because $v_\lambda \in \overline{M}$, we have $U(\mathfrak{T})v = \overline{M}(\lambda)$, i.e., $\overline{M}(\lambda)$ is irreducible. □

4 Generalized IVM $M(\lambda, \mathscr{A})$ and highest weight modules

In this section, we give a new class of modules $M(\lambda, \mathscr{A})$ generalizing the previous IVM's to study the structure of IVM. They are similar to parabolic Verma modules.

4.1 Definition of $M(\lambda, \mathscr{A})$

Let $\mathscr{A} \subset \mathscr{B} = \{0, 1, 2, \dots, s\}$, the index set of $\hat{\mathfrak{g}}_1$. We denote $\mathscr{A}^* = \mathscr{A}\backslash\{0\}$ and $\mathscr{B}^* = \mathscr{B}\backslash\{0\}$. Define

$$f_{\mathscr{A}} = \begin{cases} \displaystyle\sum_{i\in\mathscr{B}\backslash\mathscr{A}} \alpha_i^* - \Big(\sum_{i\in\mathscr{B}^*\backslash\mathscr{A}^*} k_i'\Big)\alpha_{-1}^*, & \mathscr{A} \neq \mathscr{B}, \\ 0, & \mathscr{A} = \mathscr{B}. \end{cases}$$

Then $f_{\mathscr{A}}(\delta_2) = 0$. Set

$$Q(\mathscr{A}) = \{\alpha \in \Delta_{\mathfrak{T}} \mid f_{\mathscr{A}}(\alpha) \geqslant 0\},$$

which is closed under addition. Then

$$Q(\mathscr{A}) = \mathscr{I} \cup \Big\{ -\sum_{i\in\mathscr{A}} l_i\alpha_i + \mathbb{Z}\delta_2 \,\Big|\, l_i \geqslant 0, \prod_i l_i \neq 0 \Big\} \cup \{-\mathbb{N}\delta_2\}.$$

Note that $\mathscr{I} \subsetneqq Q(\mathscr{A})$.

The following result is clear.

Proposition 2

$$Q(\mathscr{A}) \cap (-Q(\mathscr{A})) = \Big\{ \sum_{i\in\mathscr{A}} \mathbb{Z}\alpha_i + \mathbb{Z}\delta_2 \Big\} \cap \Delta_{\mathfrak{T}}.$$

Recall that the Cartan matrix is given by

$$\langle \alpha_j^\vee, \alpha_i \rangle = a_{ij}, \quad i = -1, 0, 1, \ldots, s,\ j = 0, 1, \ldots, s.$$

Let $\hat{\mathfrak{h}}_{\mathscr{A}} \subset \hat{\mathfrak{h}}$ be the space spanned by $\alpha_i^\vee$ $(i \in \mathscr{A})$. Consider the subspaces $\mathfrak{T}_{Q(\mathscr{A})}$ and $\mathfrak{T}_{-\overline{Q(\mathscr{A})}}$, where

$$\overline{Q(\mathscr{A})} = Q(\mathscr{A}) \backslash (-Q(\mathscr{A})).$$

Then $\mathfrak{T}$ decomposes itself as

$$\mathfrak{T} = \mathfrak{T}_{-\overline{Q(\mathscr{A})}} \oplus \hat{\mathfrak{h}} \oplus \mathfrak{T}_{Q(\mathscr{A})}.$$

Let $\lambda \in \hat{\mathfrak{h}}^*$ such that

$$\lambda(\hat{\mathfrak{h}}_{\mathscr{A}} \oplus \mathbb{C}c_2) = 0.$$

Let $\mathbb{C}1_\lambda$ be the one-dimensional $\hat{\mathfrak{h}} \oplus \mathfrak{T}_{Q(\mathscr{A})}$-module such that

$$(x+h).1_\lambda = \lambda(h) \cdot 1_\lambda, \quad x \in \mathfrak{T}_{Q(\mathscr{A})},\ h \in \hat{\mathfrak{h}}.$$

Define the induced $\mathfrak{T}$-module associated with $\mathscr{I}$, $\mathscr{A}$, and λ as follows:

$$M(\lambda, \mathscr{A}) = U(\mathfrak{T}) \otimes_{U(\hat{\mathfrak{h}} \oplus \mathfrak{T}_{Q(\mathscr{A})})} \mathbb{C}1_\lambda.$$

4.2 Properties of $M(\lambda, \mathscr{A})$

The following result is similar to Proposition 1.

Proposition 3 *For any $\lambda \in \hat{\mathfrak{h}}^*$ such that $\lambda(\hat{\mathfrak{h}}_{\mathscr{A}} \oplus \mathbb{C}c_2) = 0$, one has the following statements.*

(i) *$M(\lambda, \mathscr{A})$ is a $U(\mathfrak{T}_{-\overline{Q(\mathscr{A})}})$-free module generated by $1 \otimes 1_\lambda$.*

(ii) $\dim M(\lambda, \mathscr{A})_\mu = 0, 1$ *for*

$$\mu = \lambda - \sum_{i \in \mathscr{A}} k_i \alpha_i - \alpha_j + k\delta_2, \quad k_i \in \mathbb{Z}_+,\ k \in \mathbb{Z},\ j \in \mathscr{B} \backslash \mathscr{A}.$$

Otherwise, $\dim M(\lambda, \mathscr{A})_\mu = +\infty$.

(iii) *The $\mathfrak{T}$-module $M(\lambda, \mathscr{A})$ has a unique irreducible quotient $L(\lambda, \mathscr{A})$.*

(iv) *Let $\mathscr{A}'' \subset \mathscr{A}$. Then there exists a chain of surjective homomorphisms*:

$$\overline{M}(\lambda) \to M(\lambda, \mathscr{A}'') \to M(\lambda, \mathscr{A}).$$

(v) *Let $\lambda, \mu \in \hat{\mathfrak{h}}^*$. Then every nonzero map in* $\mathrm{Hom}_{\mathfrak{T}}(M(\lambda, \mathscr{A}), M(\mu, \mathscr{A}))$ *is injective.*

(vi) *Let $\mathscr{A} \subset \mathscr{B}$. Then the module $M(\lambda, \mathscr{A})$ is irreducible if and only if $\lambda(\alpha_i^\vee) \neq 0$ for any $i \in \mathscr{A}' \setminus \mathscr{A}$, $\mathscr{A} \subsetneqq \mathscr{A}'$, i.e., $\mathscr{A}$ is the maximal set such that $\lambda(\alpha_i^\vee) = 0$.*

Remark 1 If $\mathscr{A} = \mathscr{B}$, then $M(\lambda, \mathscr{A}) = L(\lambda, \mathscr{A})$ is a trivial one-dimensional module.

Corollary 2 *Let $\lambda \in \hat{\mathfrak{h}}^*$, $\mathscr{A} \subset \mathscr{B}$, and $\hat{\mathfrak{h}}_{\mathscr{A}} \oplus \mathbb{C}c_2 \subset \ker \lambda$. Also assume that $\lambda(\alpha_i^\vee) \neq 0$ for any $i \in \mathscr{A}' \setminus \mathscr{A}$, $\mathscr{A} \subsetneq \mathscr{A}'$. Then*

$$\overline{L}(\lambda) \cong M(\lambda, \mathscr{A}) = L(\lambda, \mathscr{A}) \cong L(\lambda, \mathscr{A}''), \quad \forall\, \mathscr{A}'' \subset \mathscr{A}.$$

Proof Proposition 3 (vi) implies $M(\lambda, \mathscr{A}) = L(\lambda, \mathscr{A})$. Meanwhile, Proposition 3 (iv) implies

$$L(\lambda, \mathscr{A}) \cong \overline{L}(\lambda), \quad L(\lambda, \mathscr{A}'') \cong L(\lambda, \mathscr{A}), \quad \forall\, \mathscr{A}'' \subset \mathscr{A}.$$

This completes the proof. $\square$

Corollary 3 *Let $\lambda \in \hat{\mathfrak{h}}^*$ and $\lambda(c_2) = 0$. If $\lambda(\alpha_i^\vee) \neq 0$ for any $i \in \mathscr{B}$, then $\overline{M}(\lambda)^+ = \overline{M}$.*

Proof By Corollary 1, it suffices to show $\overline{M} \subset \overline{M}(\lambda)^+$. If $\lambda(\alpha_i^\vee) \neq 0$ for any $i \in \mathscr{B}$, then

$$\overline{L}(\lambda) \cong M(\lambda, \emptyset) = L(\lambda, \emptyset).$$

The result follows by comparing the definition of $\overline{L}(\lambda)$ and that of $M(\lambda, \emptyset)$. $\square$

The following result is a consequence of Corollary 3.

Theorem 2 *Suppose that $\lambda \in \hat{\mathfrak{h}}^*$, $\lambda(c_2) = 0$, and $\lambda(\alpha_i^\vee) \neq 0$ for any $i \in \mathscr{B}$. Then one has the following statements.*

(i) *$\overline{M}(\lambda)$ has infinitely many proper submodules:*

$$\overline{M}(\lambda) \supset \overline{M}(\lambda - \delta_2) \supset \overline{M}(\lambda - 2\delta_2) \supset \cdots,$$

where

$$\dim \overline{M}(\lambda - k\delta_2)_{\lambda - k\delta_2} = \dim \overline{M}(\lambda)_{\lambda - k\delta_2} = m_k$$

are finite. Moreover,

$$\overline{L}(\lambda) = \overline{M}(\lambda)/\overline{M}(\lambda - \delta_2).$$

(ii) *The root multiplicities $\dim \overline{M}(\lambda)_{\lambda - k\delta_2} = m_k$ for all $k \geqslant 0$, and $M(\lambda - k\delta_2, \emptyset)$ exhaust all irreducible subquotients of $\overline{M}(\lambda)$.*

(iii) *For any integer $k \geqslant 0$,*

$$\dim \mathrm{Hom}_{\mathfrak{T}}(\overline{M}(\lambda - k\delta_2), \overline{M}(\lambda)) = m_k.$$

One can describe general highest weight modules as follows.

Corollary 4 *Let V be a highest weight $\mathfrak{T}$-module of highest weight λ. If c_2 acts trivially and $\lambda(\alpha_i^\vee) \neq 0$ for $i = 0, 1, \ldots, s$, then $V \simeq \overline{M}(\lambda)/\overline{M}(\lambda - k\delta_2)$ for some k.*

5 Highest weight modules of $\mathfrak{T}$

In this section, we construct another class of highest weight $\mathfrak{T}$-modules by slightly modifying the triangular decomposition for IVM. Using the method of [1], we generalize some results of TKK modules to the highest weight $\mathfrak{T}$-modules under the condition that $\lambda(c_1) = \lambda(c_2) = 0$.

These centerless modules are introduced to understand our earlier IVMs and parabolic IVMs. We remark that our construction differs from [13,14] in that the Cartan subalgebra is purely generated by the imaginary root δ_2.

5.1 Definition of $M(\lambda)$

Let

$$\Phi_+ = \mathscr{I}\backslash\mathbb{N}\delta_2, \quad \Phi_- = -\Phi_+, \quad \Phi_0 = \mathbb{Z}\delta_2.$$

Correspondingly, the root spaces are

$$\mathfrak{T}_0 = \bigoplus_{\alpha\in\Phi_0} \mathfrak{T}_\alpha, \quad \mathfrak{T}_\pm = \bigoplus_{\alpha\in\Phi_+} \mathfrak{T}_{\pm\alpha}.$$

Obviously,

$$\mathfrak{T} = \mathfrak{T}_+ \oplus \mathfrak{T}_- \oplus \mathfrak{T}_0, \quad \Delta_{\mathfrak{T}} = \Phi_+ \cup \Phi_- \cup (\Phi_0\backslash\{0\}).$$

We define a new module structure on $\mathbb{C}1_\lambda$ such that

$$h.1_\lambda = \lambda(h)\cdot 1_\lambda \ (h\in\mathfrak{T}_0), \quad \mathfrak{T}_+.1_\lambda = 0.$$

Similarly, we define the induced module $M(\lambda)$ of $\mathfrak{T}$:

$$M(\lambda) = U(\mathfrak{T}) \otimes_{U(\mathfrak{T}_0\oplus\mathfrak{T}_+)} \mathbb{C}1_\lambda.$$

5.2 Properties of $M(\lambda)$

The following result describes the relations among IVM's, generalized IVM's, and highest weight modules.

Proposition 4 *For any $\lambda\in\hat{\mathfrak{h}}^*$, one has the following statements.*

(i) *$M(\lambda)$ is a $U(\mathfrak{T}_-)$-free module generated by $1\otimes 1_\lambda =: v_\lambda$.*

(ii) *$M(\lambda)$ has a unique irreducible quotient $L(\lambda)$.*

(iii) *Let $\lambda\in\hat{\mathfrak{h}}^*$, $\hat{\mathfrak{h}}_{\mathscr{A}} \oplus \mathbb{C}c_2 \subset \ker\lambda$, and $\mathscr{A}'' \subset \mathscr{A}$. Then there exists a chain of surjective homomorphisms*:

$$\overline{M}(\lambda) \to M(\lambda) \to M(\lambda,\mathscr{A}'') \to M(\lambda,\mathscr{A}).$$

(iv) *Let $\mu\in\hat{\mathfrak{h}}^*$. Then every nonzero element of $\mathrm{Hom}_{\mathfrak{T}}(M(\lambda),M(\mu))$ is injective.*

(v) *Let $\mathscr{A}\subset\mathscr{B}$ and $\hat{\mathfrak{h}}_{\mathscr{A}}\oplus\mathbb{C}c_2\subset\ker\lambda$. If*

$$\lambda(\alpha_i^\vee)\neq 0, \quad \forall\, i\in\mathscr{A}'\setminus\mathscr{A},\ \mathscr{A}\subsetneqq\mathscr{A}',$$

then

$$\overline{L}(\lambda) \cong L(\lambda) \cong L(\lambda, \mathscr{A}).$$

Remark 2 (i) If $\mathscr{A} = \emptyset$, then $M(\lambda) = M(\lambda, \mathscr{A})$.

(ii) $L(0)$ is one-dimensional.

5.3 Irreducible module $L(\lambda)$

In this subsection, we study the integrability of $L(\lambda)$ and its weight subspaces.

Definition 2 A $\mathfrak{T}$-module M is called integrable if M is a weight module and $x_\alpha(m,n) \in \mathfrak{T}$ ($\alpha \in \Delta$; $m, n \in \mathbb{Z}$) are locally nilpotent on every nonzero $v \in M$, i.e., there exists a positive integer $N = N(\alpha, m, n)$ such that $x_\alpha(m,n)^N.v = 0$.

We write $x(\alpha, n) = x_\alpha \otimes t_2^n$ for $\alpha \in \Delta_{\hat{\mathfrak{g}}_1}$. The following result is clear.

Lemma 3 *$\mathfrak{T}_\pm$ is generated by $\{x(\pm\alpha_i, n) \mid i = 0, 1, \ldots, s;\ n \in \mathbb{Z}\}$, respectively.*

For an arbitrary Lie algebra $\mathfrak{g}$, we recall the following results.

Proposition 5 [10] *Let $v_1, v_2, \ldots$ be a system of generators of a $\mathfrak{g}$-module V, and suppose that each $x \in \mathfrak{g}$ is locally* ad*-nilpotent on $\mathfrak{g}$ and $x^{N_i}(v_i) = 0$ for some positive integers N_i ($i = 1, 2, \ldots$). Then x is locally nilpotent on V.*

Proposition 6 [11] *Let $\pi\colon \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$. If both $\operatorname{ad} x$ and $\pi(x)$ are locally nilpotent for any $x \in \mathfrak{g}$, then*

$$\pi(\exp(\operatorname{ad} x)(y)) = (\exp \pi(x))\pi(y)(\exp \pi(x))^{-1}, \quad \forall\, y \in \mathfrak{g}. \tag{6}$$

For a real root $\gamma \in \Delta^{\mathrm{re}}_{\hat{\mathfrak{g}}_1} = \{\alpha + n\delta_1 \mid \alpha \neq 0,\ n \in \mathbb{Z}\}$, define the reflection r_γ on $\hat{\mathfrak{h}}^*$ by

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma,$$

where $\gamma^\vee = \alpha^\vee + nc_1$ if $\gamma = \alpha + n\delta_1$. Let W_a be the affine Weyl group of $\hat{\mathfrak{g}}_1$ generated by the reflections r_γ, $\gamma \in \Delta^{\mathrm{re}}_{\hat{\mathfrak{g}}_1}$. Then W_a is a Coxeter group.

Lemma 4 *Suppose that $x(-\alpha_i, n)$ ($i = 0, 1, \ldots, s$, $n \in \mathbb{Z}$) are locally nilpotent on the highest weight vector v_λ in $L(\lambda)$. Then all $x(m,n)$ ($m \in \mathbb{Z}$) are locally nilpotent on $L(\lambda)$.*

Proof Since $x(-\alpha_i, n)$ ($i = 0, 1, \ldots, s$) are locally nilpotent on v_λ, they act locally nilpotent on any $x(m,n)$ via the adjoint representation. By Proposition 5, the elements $x(-\alpha_i, n)$ are locally nilpotent on $L(\lambda)$. So $L(\lambda)$ is integrable for each of the sl_2-algebras: $\langle x(\alpha_i, n), x(-\alpha_i, -n), \alpha_i^\vee \rangle$, $i = 1, 2, \ldots, s$, as well as $\langle x(\alpha_0, n), x(-\alpha_0, -n), \alpha_0^\vee = -\theta^\vee \rangle$ (note that we assume that c_1, c_2 act as zero).

Suppose that

$$\beta = \alpha + m\delta_1 + n\delta_2 \in \Delta^{\mathrm{re}}_{\mathfrak{T}}$$

is the root of $x(m,n)$. Let $\gamma = \alpha + m\delta_1$. Then

$$\gamma = \beta - n\delta_2 \in \Delta^{\mathrm{re}}_{\hat{\mathfrak{g}}_1}.$$

For any $i \in \{0, 1, \ldots, s\}$, there exists a $w \in W_a$ such that $w(\gamma) = \alpha_i$ [10]. Since $w(\delta_2) = \delta_2$, we have $w(\beta) = \alpha_i + n\delta_2$. Let s_w be the linear automorphism of $\mathfrak{T}$ associated with w. Up to a nonzero constant, we have $s_w(x(m, n)) = Y$ for $Y \in \mathfrak{T}_{\alpha_i + n\delta_2}$. It follows from Proposition 6 that all $x(m, n)$ are locally nilpotent on $L(\lambda)$. □

We now recall Weyl modules [4] for the loop algebra

$$\hat{\mathfrak{sl}}_2(\mathbb{C}) = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}].$$

Let $a_1, a_2, \ldots, a_n \in \mathbb{C}^\times$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{Z}_+$ with $|\lambda| = \sum_i \lambda_i$. We define $B(a, \lambda)$ to be the cyclic $\hat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by w such that

$$e(m)w = f(0)^{|\lambda|+1} w = 0, \quad \forall\, m,$$

$$h(m)w = \sum_{j=1}^{n} \lambda_j a_j^m w, \quad \forall\, m.$$

The following result was proved by Chari and Pressley.

Proposition 7 [4] *The $\hat{\mathfrak{sl}}_2(\mathbb{C})$-module $B(a, \lambda)$ is finite dimensional. If B' is a finite-dimensional $\hat{\mathfrak{sl}}_2(\mathbb{C})$-module generated by w' such that*

$$\dim U(\alpha^\vee \otimes \mathbb{C}[t, t^{-1}])w' = 1,$$

then B' is a quotient of some $B(a, \lambda)$ constructed above.

We also need the following remarkable formula proved by Garland.

Lemma 5 [9] *Let*

$$\beta = \alpha + r_1\delta_1 \in \Delta_{\hat{\mathfrak{g}}_1 +}, \quad r_1 \in \mathbb{Z}.$$

Then, for any $t \geqslant 1$, we have

$$x(\beta, \pm 1)^t x(-\beta, 0)^{t+1} = \sum_{m=0}^{t} x(-\beta, \pm m)\Lambda^{\pm}(\beta^\vee, t - m) + X,$$

$$x(\beta, \pm 1)^{t+1} x(-\beta, 0)^{t+1} = \Lambda^{\pm}(\beta^\vee, t + 1) + Y,$$

where X and Y are in the left ideal of $\mathfrak{T}$ generated by the subalgebra $\mathfrak{T}_+$ and $\Lambda^{\pm}(\beta^\vee, j)$ is the coefficient of u^j in

$$\Lambda^{\pm}(\beta^\vee, u) = \exp\left(- \sum_{j=1}^{+\infty} \frac{\beta^\vee t_2^{\pm j} u^j}{j} \right).$$

Remark 3 By definition of $\Lambda^{\pm}(\beta^\vee, u)$, it follows that every element $h \otimes t_2^m$ ($h \in \mathfrak{h}$, $m \in \mathbb{Z}^\times$) is a polynomial in the variables $\Lambda^{\pm}(\alpha_i^\vee, j)$ ($i = 1, 2, \ldots, s$; $j \in \mathbb{N}$).

Theorem 3 *For each* $p = 1, 2, \ldots, s$, *let* $\lambda_{p,i} \in \mathbb{Z}_+$, $a_{p,i} \in \mathbb{C}^\times$, $i = 1, 2, \ldots, k_p$. *If* λ *satisfies*

$$\lambda(\alpha_p^\vee(0, n)) = \sum_{i=1}^{k_p} \lambda_{p,i} a_{p,i}^n \tag{7}$$

and $\lambda(c_1) = \lambda(c_2) = 0$, *then*

(i) $L(\lambda)$ *is integrable*;

(ii) $L(\lambda) = U(\hat{\mathfrak{n}}_{1-} \otimes \mathbb{C}[t_2]).v_\lambda$, *where* $\hat{\mathfrak{n}}_{1-}$ *is the negative nilpotent subalgebra of* $\hat{\mathfrak{g}}_1$.

Moreover, if $L(\lambda)$ *and* $L(\lambda')$ *are nonzero irreducible modules, then*

$$L(\lambda) \cong L(\lambda') \iff \lambda = \lambda'. \tag{8}$$

Proof (i) By Lemma 4, it suffices to show that $x(-\alpha_i, n)$ are locally nilpotent on $L(\lambda)$ for $i = 0, 1, \ldots, s$ and $n \in \mathbb{Z}$. Since $L(\lambda)$ is irreducible, one only needs to prove that there exists $N \geqslant 0$ such that

$$x(\alpha_j, m)x(-\alpha_i, n)^N.v_\lambda = 0, \quad i, j = 0, 1, \ldots, s,\ m, n \in \mathbb{Z}. \tag{9}$$

If $j \neq i$, then $\alpha_j - \alpha_i + (m+n)\delta_2$ is not a root. So (9) holds.

When $j = i$, denote

$$x_n = x(\alpha_i, n), \quad y_n = x(-\alpha_i, n), \quad h_n = \alpha_i^\vee(0, n).$$

Then we have

$$[x_m, y_n] = h_{m+n}, \quad [h_p, x_m] = 2x_{p+m}, \quad [h_p, y_n] = -2y_{p+n}.$$

So $\{x_n, y_n, h_n \colon n \in \mathbb{Z}\}$ is a basis of the loop algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We consider the subspace $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))v_\lambda$ inside $M(\lambda)$. It follows from Proposition 7 that $x(-\alpha_i, n)^N v_\lambda$ belongs to a proper submodule of $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))v_\lambda$ for some $N \geqslant 0$. In fact, if $x(-\alpha_i, n)^N v_\lambda$ does not belong to the proper maximal submodule M of $U(\hat{\mathfrak{sl}}_2(\mathbb{C}))v_\lambda$ for any $N \geqslant 0$, then each $x(-\alpha_i, n)^N v_\lambda + M$ ($N \in \mathbb{N}$) is nonzero in the irreducible quotient $U' = U(\hat{\mathfrak{sl}}_2(\mathbb{C}))v_\lambda/M$. Therefore, U' is infinite dimensional, but it is also isomorphic to some $B(a, \lambda)$. This is a contradiction by Proposition 7. Applying PBW theorem to $M(\lambda)$, we get Eq. (9), which finishes the proof.

(ii) We consider the action of real root vectors $x(-\alpha_i, -r)$, where $i = 0, 1, \ldots, s$ and $r \in \mathbb{Z}_+$. By (i), there exists a (minimal) positive integer N_i such that

$$x(-\alpha_i, 0)^{N_i+1} v_\lambda = 0. \tag{10}$$

Let $x(\alpha_i, 1)^{N_i}$ act on Eq. (10) and by Lemma 5, we get

$$\sum_{m=0}^{N_i} x(-\alpha_i, m)\Lambda^+(\alpha_i^\vee, N_i - m).v_\lambda = 0. \tag{11}$$

Applying $\alpha_i^\vee \otimes t_2^{-r}$ to Eq. (11), we get

$$\sum_{m=0}^{N_i} x(-\alpha_i, m-r)\Lambda^+(\alpha_i^\vee, N_i - m).v_\lambda = 0. \tag{12}$$

Therefore, $x(-\alpha_i, -r).v_\lambda$ is written as a linear combination of the elements in the set $\{x(-\alpha_i, m).v_\lambda,\ m > -r\}$. We claim that

$$\Lambda^+(\alpha_i^\vee, N_i).v_\lambda \neq 0.$$

In fact, applying $\alpha_i^\vee(0, -1)$ to Eq. (10), one gets

$$x(-\alpha_i, -1)x(-\alpha_i, 0)^{N_i} v_\lambda = 0.$$

Note that the subalgebra $\{x(\alpha_i, 1), x(-\alpha_i, -1), \alpha_i^\vee\}$ is isomorphic to sl_2. Then

$$x(\alpha_i, 1)^q x(-\alpha_i, 0)^{N_i} v_\lambda \neq 0, \quad 0 \leqslant q \leqslant N_i,$$

by properties of sl_2-modules. When $N_i = 0$,

$$\Lambda^+(\alpha_i^\vee, 0) = 1.$$

When $N_i = 1$, note that

$$x(\alpha_i, 1)x(-\alpha_i, 0)v_\lambda \neq 0.$$

Then

$$\Lambda^+(\alpha_i^\vee, 1).v_\lambda = -\alpha_i^\vee(0, i).v_\lambda \neq 0.$$

When $N_i > 1$, we choose $q = N_i$ in the previous equation. Then Lemma 5 implies that

$$\Lambda^+(\alpha_i^\vee, N_i).v_\lambda \neq 0.$$

Using induction on r, one shows that for arbitrary $r > 0$, the element $x(-\alpha_i, -r).v_\lambda$ can be represented by the elements of the form

$$\{x(-\alpha_i, m).v_\lambda,\ m \geqslant 0\}.$$

This completes the proof by Lemma 3. The last statement is easily seen. □

IVM's have both finite- and infinite-dimensional weight subspaces. In the following, we study the weight spaces of an irreducible $L(\lambda)$ as a module for

$$\hat{\mathfrak{h}}' = \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1.$$

Let

$$\mathfrak{T}' = \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1.$$

Set

$$\hat{\mathfrak{h}}'[t_2^{\pm}] = \mathrm{span}\{\mathfrak{h}(0, n), c_1, c_2, d_1;\ n \in \mathbb{Z}\},$$

which is abelian. Note that $\mathfrak{T}_0 = \hat{\mathfrak{h}}'[t_2^{\pm}] \oplus \mathbb{C}d_2$. Hence,

$$\mathfrak{T}' = \mathfrak{T}_+ \oplus \mathfrak{T}_- \oplus \hat{\mathfrak{h}}'[t_2^{\pm}].$$

Theorem 4 *If λ satisfies the conditions of Theorem 3 (cf. (7)), then the weight spaces of $L(\lambda)$ are finite dimensional as $\mathfrak{T}'$-modules with respect to*

$$\hat{\mathfrak{h}}' = \mathfrak{h} \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1.$$

Proof Let $\lambda|_{\hat{\mathfrak{h}}'} = \lambda_1$. Since d_2 is removed, the root δ_2 can be viewed as nullified, and thus, the weight set

$$P(L(\lambda)) \subset \lambda_1 - \Big(\sum_{i=0}^{s} \mathbb{Z}_+\alpha_i\Big).$$

Consider the weight space $L(\lambda)_{\lambda_1-\varepsilon}$, where $\varepsilon \in \sum_{i=0}^{s}\mathbb{Z}_+\alpha_i$. By PBW theorem, $L(\lambda)_{\lambda_1-\varepsilon}$ is spanned by

$$x(\beta_1, n_1)x(\beta_2, n_2)\cdots x(\beta_k, n_k).v_\lambda, \tag{13}$$

where $\beta_1, \beta_2, \ldots, \beta_k$ are negative roots of the affine Lie algebra $\hat{\mathfrak{g}}_1$ such that $\varepsilon = -\sum_{i=0}^{k}\beta_i$ and $n_i \in \mathbb{Z}$. There are only finitely many β_i for a given ε.

For fixed β_i's, $x(\beta_1, n_1)x(\beta_2, n_2)\cdots x(\beta_k, n_k).v_\lambda$ ($n_i \in \mathbb{Z}$) generate a finite-dimensional subspace. In fact, define

$$e_p(t_2) = \prod_{j=1}^{k_p}(t_2 - a_{p,j}) = \sum_{i=0}^{k_p} \epsilon_{p,i} t_2^i.$$

Let

$$E_p = e_p(t_2)\mathbb{C}[t_2, t_2^{-1}].$$

By $e_p(a_{p,j}) = 0$, it is easy to check that

$$\lambda(\alpha_p^\vee \otimes E_p) = 0, \quad p = 1, 2, \ldots, s.$$

Let

$$e(t_2) = \prod_{p=1}^{s} e_p(t_2) = \sum_{i=0}^{k} \epsilon_i t_2^i, \quad k = \sum_{p=1}^{s} k_p.$$

Set

$$E = e(t_2)\mathbb{C}[t_2, t_2^{-1}] \subset E_p.$$

Then

$$\lambda(\alpha_p^\vee \otimes E) = 0. \tag{14}$$

First, we show that, for any negative affine root β of $\hat{\mathfrak{g}}_1$,

$$\sum_{i=0}^{k} \epsilon_i x(\beta, m+i).v_\lambda = 0, \quad \forall\, m \in \mathbb{Z}, \tag{15}$$

in $L(\lambda)$. Since $L(\lambda)$ is irreducible, it is enough to check that $\mathfrak{T}_+$ annihilates the left-hand side (LHS) of (15). By Lemma 3, this means that $x(\alpha_j, n)$ kills the LHS of (15) for any $j \in \{0, 1, \ldots, s\}$. We use induction on $ht(-\beta)$.

First of all, let us consider the case of $ht(-\beta) = 1$ say $\beta = -\alpha_p$. If $p \neq j$, then, clearly, $x(\alpha_j, n)$ annihilates the LHS of (15). Also

$$x(\alpha_p, n)\left(\sum_{i=0}^{k} \epsilon_i x(-\alpha_p, m+i).v_\lambda\right) = \sum_{i=0}^{k} \epsilon_i \alpha_p^\vee(0, m+n+i).v_\lambda = 0$$

by (14), where $\alpha_0^\vee = -\theta^\vee$ as c_1 acts as 0. Hence, Eq. (15) holds for $ht(-\beta) = 1$.

Now, consider general β of $ht(-\beta) > 1$. There exists a simple root α_j such that $\alpha_j + \beta$ is a negative affine root and $ht(-\alpha_j - \beta) < ht(-\beta)$. Therefore,

$$x(\alpha_j, n)\left(\sum_{i=0}^{k} \epsilon_i x(\beta, m+i).v_\lambda\right) = \sum_{i=0}^{k} \epsilon_i x(\alpha_j + \beta, n+m+i).v_\lambda = 0$$

by the induction hypothesis.

Next, we show that

$$\sum_{i=0}^{k} \epsilon_i x(\gamma_1, n_1) \cdots x(\gamma_j, m+i) x(\gamma_{j+1}, n_{j+1}) \cdots x(\gamma_l, n_l).v_\lambda = 0 \tag{16}$$

for any fixed $\gamma_1, \gamma_2, \ldots, \gamma_l$ in $\Delta_{\hat{\mathfrak{g}}_1 -}$. This is again proved by another induction on $ht(-\gamma_{j+1} - \cdots - \gamma_l)$.

If the height of $-(\gamma_{j+1} + \cdots + \gamma_l)$ is 0, Eq. (16) is clear. Then

$$\begin{aligned}
&\sum_{i=0}^{k} \epsilon_i x(\gamma_1, n_1) \cdots x(\gamma_j, m+i) x(\gamma_{j+1}, n_{j+1}) \cdots x(\gamma_l, n_l).v_\lambda \\
&\quad = \sum_{i=0}^{k} \epsilon_i x(\gamma_1, n_1) \cdots [x(\gamma_j, m+i), x(\gamma_{j+1}, n_{j+1})] \cdots x(\gamma_l, n_l).v_\lambda \\
&\qquad + \sum_{i=0}^{k} \epsilon_i x(\gamma_1, n_1) \cdots x(\gamma_{j+1}, n_{j+1}) x(\gamma_j, m+i) \cdots x(\gamma_l, n_l).v_\lambda.
\end{aligned}$$

Each term of the right-hand side is zero by induction hypothesis. Therefore, Eq. (16) holds.

For fixed $\beta_1, \beta_2, \ldots, \beta_k$, the vectors of form (13) generate a finite-dimensional weight space due to the fact that

$$\dim \mathbb{C}[t_2, t_2^{-1}]/E < +\infty.$$

This finishes the proof. □

Theorem 5 *If $L(\lambda)$ is irreducible as a $\mathfrak{T}'$-module with finite-dimensional weight spaces and the action of c_1 and c_2 are zero, then λ satisfies the conditions of Theorem* 3 (*cf.* (7)).

Proof For $p \neq 0$, the algebra $\mathfrak{L}$ generated by $\{x_{\alpha_p}, x_{-\alpha_p}, \alpha_p^\vee\}$ is isomorphic to $\mathfrak{sl}_2$. Let V be the irreducible quotient of $U(\hat{\mathfrak{L}}).v_\lambda$, where

$$\hat{\mathfrak{L}} = \mathfrak{L} \otimes \mathbb{C}[t_2, t_2^{-1}] \oplus \mathbb{C}c_2.$$

Since the weight spaces of $L(\lambda)$ are finite dimensional, the set

$$\{x(-\alpha_p, n).v_\lambda,\ n \in \mathbb{Z}\}$$

is linearly dependent. Thus, there exists a nonzero polynomial $g = \sum_i g_i t_2^i$ such that

$$x_{-\alpha_p} \otimes g.v_\lambda = 0.$$

Let

$$G(t_2) = g\mathbb{C}[t_2, t_2^{-1}].$$

Then

$$x_{-\alpha_p} \otimes G.v_\lambda = 0.$$

In fact,

$$0 = (\alpha_p^\vee \otimes t_2^m)x_{-\alpha_p} \otimes g.v_\lambda = (x_{-\alpha_p} \otimes g)\alpha_p^\vee \otimes t_2^m.v_\lambda - 2x_{-\alpha_p} \otimes t_2^m g.v_\lambda$$

and

$$\alpha_p^\vee \otimes t_2^m.v_\lambda = \lambda(\alpha_p^\vee \otimes t_2^m).v_\lambda.$$

Naturally,

$$\alpha_p^\vee \otimes G.v_\lambda = 0.$$

Subsequently,

$$(\mathfrak{L} \otimes G \oplus \mathbb{C}c_2).v_\lambda = 0.$$

Set

$$W = \{v \in V \mid (\mathfrak{L} \otimes G \oplus \mathbb{C}c_2).v = 0\}.$$

Clearly, W is a nonzero submodule of V. Thus, $V = W$ due to irreducibility of V. Then V is an $\hat{\mathfrak{L}}/\mathfrak{L} \otimes G \oplus \mathbb{C}c_2$-module, and thus, $\dim V < +\infty$. By Proposition 7, λ satisfies the conditions of Theorem 3. □

Remark 4 The above proof also shows that when c_1 and c_2 act trivially, the irreducible $\mathfrak{T}'$-module $L(\lambda)$ has finite-dimensional weight spaces if and only if there is an ideal $\mathscr{S}$ of $\mathbb{C}[t_2, t_2^{-1}]$ such that

$$\lambda(\alpha_p^\vee \otimes \mathscr{S}) = 0, \quad p = 1, 2, \ldots, s.$$

Corollary 5 *Let $L(\lambda)$ be an irreducible $\mathfrak{T}'$-module with finite-dimensional weight spaces and suppose that the centers c_1 and c_2 act trivially. Then there exists an ideal $\mathscr{S}$ of $\mathbb{C}[t_2, t_2^{-1}]$ such that*

$$\widetilde{\mathfrak{g}}_1 \otimes \mathscr{S}.L(\lambda) = 0,$$

where

$$\widetilde{\mathfrak{g}}_1 = \mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}].$$

Proof First, there exists an ideal $\mathscr{S}$ of $\mathbb{C}[t_2, t_2^{-1}]$ such that

$$\lambda(\alpha_i^\vee \otimes \mathscr{S}) = 0, \quad i = 1, 2, \ldots, s.$$

By the definition of $L(\lambda)$, we have

$$x_{\alpha_i} \otimes \mathscr{S}.v_\lambda = 0, \quad i = 0, 1, 2, \ldots, s.$$

The next step is to show

$$y_{\alpha_i} \otimes \mathscr{S}.v_\lambda = 0, \quad i = 0, 1, 2, \ldots, s.$$

Since

$$x_{\alpha_j} \otimes t_2^m.y_{\alpha_i} \otimes \mathscr{S}.v_\lambda = \delta_{ji}\alpha_i^\vee \otimes \mathscr{S}.v_\lambda = 0, \quad j, i = 0, 1, 2, \ldots, s, \ m \in \mathbb{Z},$$

where we set $\alpha_0^\vee = -\theta^\vee$, and $L(\lambda)$ is irreducible, one sees that

$$y_{\alpha_i} \otimes \mathscr{S}.v_\lambda = 0, \quad i = 0, 1, 2, \ldots, s.$$

Hence, $\widetilde{\mathfrak{g}}_1 \otimes \mathscr{S}.v_\lambda = 0$ by induction.

Now, consider

$$\overline{W} = \{w \in L(\lambda), \widetilde{\mathfrak{g}}_1 \otimes \mathscr{S}.w = 0\},$$

which is a submodule of $L(\lambda)$. Then $L(\lambda) = \overline{W}$ by the irreducibility of $L(\lambda)$. Therefore,

$$\widetilde{\mathfrak{g}}_1 \otimes \mathscr{S}.L(\lambda) = 0. \qquad \square$$

Since $\lambda(c_1) = \lambda(c_2) = 0$, $L(\lambda)$ can be viewed as a module for the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t_1, t_1^{-1}, t_2, t_2^{-1}]$. The following proposition is easily derived from Corollary 5 and [15, Proposition 3.8].

Proposition 8 *Let $\mathscr{S}_1, \mathscr{S}_2$ be co-prime and co-finite ideals of $\mathbb{C}[t_2, t_2^{-1}]$, and suppose that λ and μ satisfy the conditions in Theorem* 3. *Then*

$$L(\lambda + \mu) \cong L(\lambda) \otimes L(\mu).$$

Rao [14] and Chang-Tan [1] have shown, respectively, that irreducible integrable modules for toroidal and TKK modules with finite-dimensional weight spaces and $c_1 > 0$, $c_2 = 0$ are highest weight modules. In general, our modules do not seem to be of highest weight type when $c_1 = c_2 = 0$.

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