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RESEARCH ARTICLE

Some remarks on cotilting comodules

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Abstract We study cotilting comodules and f-cotilting comodules and give a description of localization of f-cotilting comodules and classical tilting comodules. First, we introduce T-cotilting injective comodules and their dimensions which are important for researching cotilting comodules. Then we characterize the localization in f-cotilting comodules, finitely copresented comodules, and classical tilting comodules. In particular, we obtain a localizing property about finitely copresented comodules.

Keywords Coalgebra, comodule, localization
MSC 16T15, 18E35

1 Introduction

Simson [11] defined the concepts of cotilting comodules and he hoped developing a (co)tilting theory for comodule categories. Wang [12] introduced the concepts of tilting comodules and partial tilting comodules over coalgebras. In particular, he proved that each tilting comodule induces a torsion theory. In this paper, we use Wang's viewpoint and give some properties of cotilting comodules.

It is well known that the localization plays an important role in the theory of algebras. It has been developed gradually from different aspects. The canonical process is actually the formulation of rings of fractions and the associated process of localization which are the most significant technical tools in commutative algebra. At the same time, Goodearl, Warfield, and others researched the localization in the noncommutative situation and obtain many nice results ([3,4]). Also, Gabriel [1] gave a description of the localization abstractly in Abelian and Grothendieck category. In the process of localization, one usually applies a functor onto the quotient category, which has a right adjoint, the section functor. That is to say, if $T\colon \mathscr{A} \to \mathscr{A}'$ is an

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exact functor between Abelian categories, and $S\colon \mathscr{A}' \to \mathscr{A}$ is a full and faithful right adjoint functor of T, then the dense subcategory $\ker T$, with object class $\{X \in \mathscr{A} \mid T(X) = 0\}$, is a localizing subcategory of $\mathscr{A}$, and the category $\mathscr{A}'$ is equivalent to $\mathscr{A}/\ker T$. In particular, the localization in which $\mathscr{A}$ is a Grothendieck category is the same as in Abelian categories. Starting from the localization of rings, some mathematicians developed a theory of localization for coalgebras. For example, Jara et al. [5–7] and Navarro [9,10] elaborated Gabriel's ideas in comodule categories (Grothendieck categories of finite type). The key point of the theory lies in the description that quotient category becomes a comodule category. That is to say, a quotient category $\mathscr{M}^C/\mathscr{T}$ is a category of comodules $\mathscr{M}^D$ for certain coalgebra D, where C is a coalgebra and $\mathscr{T}$ is a localizing subcategory of $\mathscr{M}^C$. Indeed, this is because that the category $\mathscr{M}^C$ of right comodules over a coalgebra C is a locally finite Grothendieck category in which the theory of localization can be applied. The advantage of this is that it is better understood than that the case of modules over an arbitrary algebra. It is worth mentioning that the key point in most of such applications is the behavior of simple and injective comodules through the action of the section functor. Therefore, by studying on a set of localized coalgebras of any coalgebra C, we can obtain some information about C or its category of comodules $\mathscr{M}^C$. Simson studied the localization from another point of view, and successfully showed that the localising embedding technique via some functors can be used in researching the Euler defect of coalgebras.

In this paper, we get some interesting properties of cotilting comodules and give a description of localization of f-cotilting comodules and classical tilting comodules. In Section 2, we list some notations and basic facts about coalgebras, cotilting comodules, and localization, in order to make the article self-contained. In Section 3, we give some properties of a cotilting comodule by T-cotilting injective comodules and their dimensions. In Section 4, we characterize the localization in f-cotilting comodules, finitely copresented comodules, and classical tilting comodules. In particular, we obtain a localizing property about finitely copresented comodules.

2 Preliminaries

Throughout this paper, K will be a field. Let C be a K-coalgebra, and denote by $\mathscr{M}^C$ and $\mathscr{M}^C_{\rm f}$ the categories of right C-comodules and right C-comodules of finite K-dimension, respectively.

Following [7], a full subcategory $\mathscr{T}$ of $\mathscr{M}^C$ is said to be *dense* (or a *Serre class*) if each exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathscr{M}^C$ satisfies that M belongs to $\mathscr{T}$ if and only if M_1 and M_2 belong to $\mathscr{T}$. For any dense subcategory $\mathscr{T}$ of $\mathscr{M}^C$, there exists an abelian category $\mathscr{M}^C/\mathscr{T}$ and an exact functor $T\colon \mathscr{M}^C \to \mathscr{M}^C/\mathscr{T}$, such that $T(M) = 0$ for each $M \in \mathscr{T}$, satisfying the following universal property: for any exact functor $F\colon \mathscr{M}^C \to \mathscr{C}$ such that $F(M) = 0$ for each $M \in \mathscr{T}$, there exists a unique functor $\overline{F}\colon \mathscr{M}^C/\mathscr{T} \to \mathscr{C}$ verifying that $F = \overline{F}T$, where $\mathscr{C}$ is an arbitrary abelian category. The category $\mathscr{M}^C/\mathscr{T}$ is called the *quotient category* of $\mathscr{M}^C$ with respect to $\mathscr{T}$, and T is known as the *quotient functor*. A dense subcategory $\mathscr{T}$ of $\mathscr{M}^C$ is said to be *localizing* if the quotient functor $T\colon \mathscr{M}^C \to \mathscr{M}^C/\mathscr{T}$ has a right adjoint functor, S, called the *section functor*. If the section functor is exact, $\mathscr{T}$ is called *perfect localizing*. $\mathscr{T}$ is said to be *colocalizing* if T has a left adjoint functor, H, called the *colocalizing functor*. $\mathscr{T}$ is called a *perfect colocalizing subcategory* if the colocalizing functor is exact.

Let us list some properties of the (co)localizing functors (see [1,8]).

Lemma 1 [1,8] *Let $\mathscr{T}$ be a dense subcategory of the category of right comodules $\mathscr{M}^C$ over a coalgebra C. Then the following statements hold.*

(a) *T is exact.*

(b) *If $\mathscr{T}$ is localizing, then the section functor S is left exact and the equivalence $TS \simeq 1_{\mathscr{M}^C/\mathscr{T}}$ holds.*

(c) *If $\mathscr{T}$ is colocalizing, then the colocalizing functor H is right exact and the equivalence $TH \simeq 1_{\mathscr{M}^C/\mathscr{T}}$ holds.*

In [7,13], localizing subcategories are described by means of idempotents in the dual algebra C^*. In particular, it is proved that the quotient category is the category of right comodules over the coalgebra eCe, where e is the idempotent associated to the localizing subcategory. The coalgebra structure of eCe is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e, \quad \varepsilon_{eCe}(exe) = e(x),$$

where

$$\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}, \quad \forall\, x \in C.$$

If M is a right C-comodule, then eM has a natural structure of right eCe-comodule given by

$$\rho(ex) = \sum_{(x)} ex_{(0)} \otimes ex_{(1)}e,$$

where

$$\rho_M(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}, \quad \forall\, x \in M.$$

Lemma 2 [7] *Let C be a coalgebra, and let e be an idempotent in C^*. Then the following statements hold.*

(a) *The quotient functor $T\colon \mathscr{M}^C \to \mathscr{M}^{eCe}$ is naturally equivalent to the functor $e(-)$. T is also equivalent to the cotensor functor $-\square_C eC$.*

(b) *The section functor* $S\colon \mathscr{M}^{eCe} \to \mathscr{M}^C$ *is naturally equivalent to the cotensor functor* $-\square_{eCe}Ce$.

(c) *If* $\mathscr{T}$ *is a colocalizing subcategory of* $\mathscr{M}^C$, *then the colocalizing functor* $H\colon \mathscr{M}^{eCe} \to \mathscr{M}^C$ *is naturally equivalent to the functor* $\mathrm{Cohom}_{eCe}(eC, -)$.

To any coalgebra C, denote by $\{S_j\}_{j\in I_C}$ and $\{E_j\}_{j\in I_C}$ a complete set of pairwise nonisomorphic simple and indecomposable injective right C-comodules, respectively. From now on, we fix an idempotent element $e \in C^*$. We also denote by $\mathscr{T}_e$ the localizing subcategory associated to e and by $\{S_j\}_{j\in I_e}$ the set of simple comodules of the quotient category, where I_e is a subset of I_C. In what follows, we will denote by $\{\overline{E}_j\}_{j\in I_e}$ a complete set of pairwise nonisomorphic indecomposable injective right eCe-comodules, and assume that $\overline{E}_j$ is the injective envelope of the simple right eCe-comodule S_j for each $j \in I_e$.

Unless otherwise stated, we always assume $T(M) \neq 0$ for a right C-comodule M.

3 Cotilting comodules

Following [11], given a right C-comodule M, we denote by $\mathrm{Add}\, M$ the full subcategory of $\mathscr{M}^C$, the objects of which are the comodules isomorphic to direct summands of an arbitrary direct sum of the comodule M; by $\mathrm{add}\, M$ the full subcategory of $\mathscr{M}^C$, the objects of which are the comodules isomorphic to finite direct sums of direct summands of the comodule M; and by $\mathrm{Prod}\, M$ the subcategory of $\mathscr{M}^C$, the objects of which are the comodules isomorphic to direct summands of an arbitrary product of the comodule M. Given a set L, we denote by M^L the product of L-copies of M, and by $M^{(L)}$ the direct sum of L-copies of M.

Definition 1 [11] Let M be a right C-comodule for a coalgebra C. We define M to be a cotilting comodule if M satisfies the following four conditions:

(c_1) M is quasi-finite;

(c_2) $\mathrm{inj.dim}(M) \leqslant 1$;

(c_3) $\mathrm{Ext}^1_C(M^L, M) = 0$ for all sets L;

(c_4) there exists an exact sequence

$$0 \to M_1 \to M_0 \to C \to 0, \quad M_1, M_0 \in \mathrm{Prod}\, M.$$

Definition 2 [11] Let M be a right C-comodule for a coalgebra C. We define M to be an f-cotilting comodule if M satisfies the following four conditions:

(c_1) M is quasi-finite;

(c_2') M admits an injective resolution

$$0 \to M \to E_x \to E_y \to 0,$$

where the comodules E_x and E_y are injective, quasi-finite, and lie in $\mathrm{add}\, C$;

(c_3') $\mathrm{Ext}^1_C(M, M) = 0$;

(c_4') there exists an exact sequence

$$0 \to M_1 \to M_0 \to C \to 0, \quad M_1, M_0 \in \operatorname{add} M.$$

Remark 1 Comparing the two definitions, we know that condition (c_3) implies (c_3'), but condition (c_3') does not imply (c_3).

(a) If $\dim_K M$ is finite, then condition (c_3') implies (c_3). More generally, one can show that condition (c_3') implies (c_3) if the algebra $\Lambda_M = \mathrm{End}_C M$ is left coherent and right perfect, and M viewed as a left module over the algebra Λ_M is finitely presented (see [11]).

(b) For a comodule M, we obtain the following conclusions from [2]:

(1) $\mathrm{Ext}(-, M)$ preserves products if and only if M is injective;

(2) $\mathrm{Ext}(-, M)$ preserves products from torsion-free group if and only if M is cotorsion.

Therefore, from above discussion, we get that some f-cotilting comodules can be cotilting comodules under some conditions.

We consider the following full subcategories of $\mathscr{M}^C$ and $\mathscr{M}^C_{\mathrm{f}}$:

$$\mathscr{T}_C(M) = \{X \in \mathscr{M}^C;\ \mathrm{Hom}_C(X, M) = 0\} \subseteq \mathscr{M}^C,$$
$$\mathscr{T}^{\mathrm{f}}_C(M) = \{X \in \mathscr{M}^C_{\mathrm{f}};\ \mathrm{Hom}_C(X, M) = 0\} \subseteq \mathscr{M}^C_{\mathrm{f}},$$
$$\mathscr{F}_C(M) = \{Y \in \mathscr{M}^C;\ \mathrm{Ext}^1_C(Y, M) = 0\} \subseteq \mathscr{M}^C,$$
$$\mathscr{F}^{\mathrm{f}}_C(M) = \{Y \in \mathscr{M}^C_{\mathrm{f}};\ \mathrm{Ext}^1_C(Y, M) = 0\} \subseteq \mathscr{M}^C_{\mathrm{f}}.$$

Denote by $\operatorname{Cogen} M$ (resp. $\operatorname{cogen} M$) the full subcategory of $\mathscr{M}^C$ consisting of all comodules N such that there is a monomorphism $N \to M^L$ for some set L (resp. finite set L).

For a comodule M, we have a category of comodules $\operatorname{Cogen} M$, which cogenerates a torsion theory (torsion pair) $(\mathscr{T}, \mathscr{F})$, where $\mathscr{F}$ is the smallest torsion-free class containing $\operatorname{Cogen} M$.

Proposition 1 [11] *Assume that C is a basic coalgebra.*

(a) *If M is a cotilting right C-comodule, then*

(a_1) $\mathscr{F}_C(M) = \operatorname{Cogen} M$;

(a_2) *$(\mathscr{T}_C(M), \mathscr{F}_C(M))$ is a torsion pair in $\mathscr{M}^C$ and $(\mathscr{T}^{\mathrm{f}}_C(M), \mathscr{F}^{\mathrm{f}}_C(M))$ is a torsion pair in $\mathscr{M}^C_{\mathrm{f}}$.*

(b) *If M is an f-cotilting right C-comodule, then*

(b_1) $\mathscr{F}^{\mathrm{f}}_C(M) = \operatorname{cogen} M$;

(b_2) *$(\mathscr{T}^{\mathrm{f}}_C(M), \mathscr{F}^{\mathrm{f}}_C(M))$ is a torsion pair in $\mathscr{M}^C_{\mathrm{f}}$.*

Proposition 2 *Let C be a right semi-perfect coalgebra, and let*

$$\mathscr{F}_C(M) = \operatorname{Cogen} M.$$

Then

(i) $\operatorname{Cogen} M$ *is closed under taking extensions, direct sums, direct products, and subcomodules. In particular,* $\operatorname{Cogen} M$ *is a torsion-free class.*

(ii) $\operatorname{Ext}^1_C(M^L, M) = 0$ *for any set* L *and* $\operatorname{inj.dim}(M) \leqslant 1$.

Proof (i) It follows from the definition of $\operatorname{Cogen} M$ and [12].

(ii) Since

$$M^L \in \operatorname{Cogen} M = \mathscr{F}_C(M)$$

for any set L, we have

$$\operatorname{Ext}^1_C(M^L, M) = 0.$$

For any $Y \in \mathscr{M}^C$, there exists an exact sequence

$$0 \to N \to P \to Y \to 0,$$

where P is projective since C is right semi-perfect. Applying the functor $\operatorname{Ext}_C(-, M)$, we have the exact sequence

$$\operatorname{Ext}^1_C(P, M) = 0 \to \operatorname{Ext}^1_C(N, M) \to \operatorname{Ext}^2_C(Y, M) \to \operatorname{Ext}^2_C(P, M) = 0.$$

Since

$$P \in \operatorname{Cogen} M = \mathscr{F}_C(M)$$

and $\operatorname{Cogen} M$ is closed under taking subcomodules, we have

$$\operatorname{Ext}^1_C(N, M) = 0,$$

and then,

$$\operatorname{Ext}^2_C(Y, M) = 0, \quad \operatorname{inj.dim}(M) \leqslant 1.$$

□

Definition 3 Let T be a cotilting comodule. A comodule M is called T-cotilting injective if $\operatorname{Hom}_C(-, M)$ preserves the exactness of sequence in $\mathscr{F}_C(T)$.

Remark 2 Any injective comodule is T-cotilting injective.

Proposition 3 *Let* C *be a right semi-perfect basic coalgebra, and let* T *be a cotilting comodule. The following statements are equivalent for a comodule* M:

(i) M *is* T*-cotilting injective*;

(ii) $\operatorname{Ext}^1_C(U, M) = 0$ *for any* $U \in \mathscr{F}_C(T)$;

(iii) *for any exact sequence*

$$0 \to N_1 \to^h N \to N_2 \to 0, \quad N_2 \in \mathscr{F}_C(T),$$

and any comodule map $f\colon N_1 \to M$, *there exists a comodule map* $g\colon N \to M$ *such that* $g \circ h = f$.

Proof (i) $\Rightarrow$ (ii). For any $U \in \mathscr{F}_C(T)$, there is an exact sequence

$$0 \to N \to P \to U \to 0,$$

where P is projective. Since

$$\mathrm{Ext}^1_C(P,T)=0,\quad P\in\mathscr{F}_C(T)=\mathrm{Cogen}\,T,$$

we have $N\in\mathscr{F}_C(T)$ because $\mathrm{Cogen}\,T$ is closed under taking subcomodules. Therefore, this sequence must be in $\mathscr{F}_C(T)$ and (ii) follows easily by the definition.

(ii) $\Rightarrow$ (iii). Applying the functor $\mathrm{Hom}_C(-,M)$ to the exact sequence

$$0\to N_1\to^h N\to N_2\to 0,\quad N_2\in\mathscr{F}_C(T),$$

we obtain

$$0\to\mathrm{Hom}_C(N_2,M)\to\mathrm{Hom}_C(N,M)\to\mathrm{Hom}_C(N_1,M)\to\mathrm{Ext}^1_C(N_2,M)=0.$$

Therefore, for any comodule map $f\colon N_1\to M$, there exists a comodule map $g\colon N\to M$ such that $g\circ h=f$.

(iii) $\Rightarrow$ (i). By the condition, $\mathrm{Hom}_C(-,M)$ preserves the exactness of sequence

$$0\to N_1\to^h N\to N_2\to 0,\quad N_2\in\mathscr{F}_C(T).$$

Therefore, $\mathrm{Hom}_C(-,M)$ preserves the exactness of sequence in $\mathscr{F}_C(T)$. □

Since $\mathscr{M}^C$ is a locally finite abelian category, every comodule has an injective envelope. It follows that every comodule M has a T-cotilting injective resolution. Like the usual injective dimension of M, we define ct-inj.dim(M) as the least number n such that there is a T-cotilting injective resolution

$$0\to M\to F_0\to F_1\to\cdots\to F_n\to 0,$$

where all F_i are T-cotilting injective. If there exists no such n, we say that the T-cotilting injective dimension of M is infinity, denoted by

$$\text{ct-inj.dim}(M)=\infty.$$

Lemma 3 *Let C be a right semi-perfect basic coalgebra, and let T be a cotilting comodule. Then the following conditions are equivalent for a right C-comodule M:*

(i) *M is T-cotilting injective*;

(ii) $\mathrm{Ext}^n_C(A,M)=0$ *for any right C-comodule $A\in\mathscr{F}_C(T)$ and $n>0$.*

Proof (i) $\Rightarrow$ (ii). Assume that $\{P_n,d_n\}$ is the projective resolution of A. Let $A_n=\mathrm{Im}\,d_n$. Then we have the short exact sequence

$$0\to A_n\to^\eta P_{n-1}\to^\pi A_{n-1}\to 0.$$

We conclude that this exact sequence is in $\mathscr{F}_C(T)$, because $P_n\in\mathscr{F}_C(T)$ for $n=0,1,\ldots$ and $\mathscr{F}_C(T)$ is closed under taking subcomodules. Since M is T-cotilting injective, we have

$$0\to\mathrm{Hom}_C(A_{n-1},M)\to\mathrm{Hom}_C(P_{n-1},M)\to\mathrm{Hom}_C(A_n,M)\to 0.$$

Then

$$\mathrm{Ext}_C^n(A,M)=\mathrm{Hom}_C(A_n,M)/\mathrm{ImHom}_C(\eta,M)=0,\quad n>0.$$

(ii) ⇒ (i). For any exact sequence

$$0\to A_1\to A\to A_2\to 0$$

in $\mathscr{F}_C(T)$, since $\mathrm{Ext}_C^1(A_2,M)=0$, we have

$$0\to \mathrm{Hom}_C(A_2,M)\to \mathrm{Hom}_C(A,M)\to \mathrm{Hom}_C(A_1,M)\to 0$$

is exact, that is to say, $\mathrm{Hom}_C(-,M)$ is an exact functor in $\mathscr{F}_C(T)$. By the definition, M is T-cotilting injective. □

Theorem 1 *Let C be a right semi-perfect basic coalgebra, and let T be a cotilting comodule. Then the following conditions are equivalent for a right C-comodule N:*

(i) ct-inj.dim$(N)\leqslant n$;

(ii) *if there is an exact sequence*

$$0\to N\to F_0\to F_1\to\cdots\to F_{n-1}\to X\to 0,$$

where all F_i are T-cotilting injective, then X is T-cotilting injective;

(iii) $\mathrm{Ext}_C^{n+1}(A,N)=0$ *for any comodule $A\in\mathscr{F}_C(T)$*;

(iv) $\mathrm{Ext}_C^{n+j}(A,N)=0$ *for any comodule $A\in\mathscr{F}_C(T)$ and $j=1,2,\ldots$.*

Proof (i) ⇒ (iii). We use induction on n. If

$$\text{ct-inj.dim}(N)\leqslant 1,$$

then we have an exact sequence

$$0\to N\to F_0\to F_1\to 0,$$

where F_0 and F_1 are T-cotilting injective. Hence, (iii) holds by Lemma 3. Inductively, suppose that the result holds for

$$\text{ct-inj.dim}(N)\leqslant n-1.$$

Since C is right semi-perfect, we have an exact sequence

$$0\to A'\to P\to A\to 0$$

with P projective. Since T is a cotilting comodule, $P\in\mathscr{F}_C(T)$ and so $A'\in\mathscr{F}_C(T)$. We obtain

$$\mathrm{Ext}_C^{n+1}(A,N)\simeq \mathrm{Ext}_C^n(A',N)=0$$

from the sequence.

(iii) ⇒ (ii). It is easy to get from the isomorphism

$$\mathrm{Ext}_C^{n+1}(A,N) \simeq \mathrm{Ext}_C^1(A,X)$$

and Proposition 3.

(ii) ⇒ (i). It holds clearly by the definition of ct-inj.dim(N).

(iii) ⇒ (iv). It follows from the isomorphism

$$\mathrm{Ext}_C^{n+j}(A,N) \simeq \mathrm{Ext}_C^{n+j-1}(A',N), \quad j = 1, 2, \ldots.$$

(iv) ⇒ (iii). It is obvious. □

Theorem 2 *Let C be a right semi-perfect basic coalgebra, and let T be a cotilting comodule. Then, for an exact sequence*

$$0 \to A \to B \to C \to 0,$$

we have following results.

(1) *If* ct-inj.dim(A) = ct-inj.dim(C), *then* ct-inj.dim(B) = ct-inj.dim(C);
if ct-inj.dim(A) < ct-inj.dim(C), *then* ct-inj.dim(B) $\leqslant$ ct-inj.dim(C);
if ct-inj.dim(A) > ct-inj.dim(C), *then* ct-inj.dim(B) $\leqslant$ ct-inj.dim(A).

(2) *If* ct-inj.dim(A) = ct-inj.dim(B), *then* ct-inj.dim(C) = ct-inj.dim(B);
if ct-inj.dim(A) < ct-inj.dim(B), *then* ct-inj.dim(C) $\leqslant$ ct-inj.dim(B);
if ct-inj.dim(A) > ct-inj.dim(B), *then* ct-inj.dim(C) $\leqslant$ ct-inj.dim(A).

(3) *If* ct-inj.dim(B) = ct-inj.dim(C), *then* ct-inj.dim(A) = ct-inj.dim(B) + 1;
if ct-inj.dim(B) < ct-inj.dim(C), *then* ct-inj.dim(A) $\leqslant$ ct-inj.dim(C) + 1;
if ct-inj.dim(B) > ct-inj.dim(C), *then* ct-inj.dim(A) $\leqslant$ ct-inj.dim(B) + 1.

Proof For any $N \in \mathscr{F}_C(T)$, we have

$$\cdots \to \mathrm{Ext}_C^n(N,C) \to \mathrm{Ext}_C^{n+1}(N,A) \to \mathrm{Ext}_C^{n+1}(N,B) \to \mathrm{Ext}_C^{n+1}(N,C) \to \cdots$$

(1) Suppose

$$\text{ct-inj.dim}(A) \leqslant n, \quad \text{ct-inj.dim}(C) \leqslant m.$$

If

$$m = n, \quad \mathrm{Ext}_C^{n+1}(N,A) \simeq \mathrm{Ext}_C^{n+1}(N,C) = 0,$$

then

$$\mathrm{Ext}_C^{n+1}(N,B) = 0, \quad \text{ct-inj.dim}(B) \leqslant n.$$

If

$$m > n, \quad \mathrm{Ext}_C^{m+1}(N,A) \simeq \mathrm{Ext}_C^{m+1}(N,C) = 0,$$

then

$$\mathrm{Ext}_C^{m+1}(N,B)=0,\quad \text{ct-inj.dim}(B)\leqslant m.$$

If

$$m<n,\quad \mathrm{Ext}_C^{n+1}(N,A)\simeq \mathrm{Ext}_C^{n+1}(N,C)=0,$$

then

$$\mathrm{Ext}_C^{n+1}(N,B)=0,\quad \text{ct-inj.dim}(B)\leqslant n.$$

(2) Suppose

$$\text{ct-inj.dim}(A)\leqslant n,\quad \text{ct-inj.dim}(B)\leqslant l.$$

If

$$l=n,\quad \mathrm{Ext}_C^{n+j}(N,A)\simeq \mathrm{Ext}_C^{n+j}(N,B)=0,$$

then

$$\mathrm{Ext}_C^{n+j}(N,C)=0,\ j=1,2,\dots,\quad \text{ct-inj.dim}(C)\leqslant n.$$

If

$$l>n,\quad \mathrm{Ext}_C^{l+j}(N,A)\simeq \mathrm{Ext}_C^{l+j}(N,B)=0,$$

then

$$\mathrm{Ext}_C^{l+j}(N,C)=0,\ j=1,2,\dots,\quad \text{ct-inj.dim}(C)\leqslant l.$$

If

$$l<n,\quad \mathrm{Ext}_C^{n+j}(N,A)\simeq \mathrm{Ext}_C^{n+j}(N,B)=0,$$

then

$$\mathrm{Ext}_C^{n+j}(N,C)=0,\ j=1,2,\dots,\quad \text{ct-inj.dim}(C)\leqslant n.$$

(3) Suppose

$$\text{ct-inj.dim}(B)\leqslant l,\quad \text{ct-inj.dim}(C)\leqslant m.$$

If

$$m=l,\quad \mathrm{Ext}_C^{l+j}(N,B)\simeq \mathrm{Ext}_C^{l+j}(N,C)=0,\ j=1,2,\dots,$$

then

$$\mathrm{Ext}_C^{l+2}(N,A)=0,\quad \text{ct-inj.dim}(A)\leqslant l+1.$$

If

$$m>l,\quad \mathrm{Ext}_C^{m+j}(N,B)\simeq \mathrm{Ext}_C^{m+j}(N,C)=0,\ j=1,2,\dots,$$

then

$$\mathrm{Ext}_C^{m+2}(N,A)=0,\quad \text{ct-inj.dim}(A)\leqslant m+1.$$

If

$$m<l,\quad \mathrm{Ext}_C^{l+j}(N,B)\simeq \mathrm{Ext}_C^{l+j}(N,C)=0,\ j=1,2,\dots,$$

then

$$\mathrm{Ext}_C^{l+2}(N,A)=0,\quad \text{ct-inj.dim}(A)\leqslant l+1.$$

□

Corollary 1 *Let C be a right semi-perfect basic coalgebra, and let T be a cotilting comodule. For an exact sequence*

$$0 \to A \to B \to C \to 0,$$

if the T-cotilting injective dimension of two comodules in A, B, and C are finite, then the T-cotilting injective dimension of the third comodule is also finite.

4 Localization

Theorem 3 *Assume that C is a coalgebra and $e \in C^*$ is the idempotent associated to the set I_e.*

(a) *If M is an f-cotilting right C-comodule and $X \simeq Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is an f-cotilting right eCe-comodule.*

(b) *If N is an f-cotilting right eCe-comodule and $e \in C^*$ defines a perfect localization, then $S(N)$ is an f-cotilting right C-comodule.*

Proof (a) Following [13, Theorem 1.13], we obtain the fact that the functor T is an equivalence if and only if $X \simeq Ce$ is a quasi-finite injective cogenerator. Now, we prove that $T(M)$ satisfies the four conditions of f-cotilting comodule.

(c_1). If M is quasi-finite, then, since the functor $T\colon \mathscr{M}^C \to \mathscr{M}^{eCe}$ restricts to a functor between the categories of quasi-finite which also denoted by $T\colon \mathscr{M}^C_{\mathrm{qf}} \to \mathscr{M}^{eCe}_{\mathrm{qf}}$ (see [10]), we know that $T(M)$ is quasi-finite.

(c'_2). Since T is an equivalence, we get $T(M)$ admits an injective resolution

$$0 \to T(M) \to T(E_x) \to T(E_y) \to 0,$$

where the comodules $T(E_x)$ and $T(E_y)$ are injective, quasi-finite, and lie in $\operatorname{add} T(C)$. In fact, since E_x lies in $\operatorname{add} C$, that is,

$$E_x = \bigoplus_{i\in I} C_i,$$

where I is a finite set and C_i is the direct summand of C, we have

$$T(E_x) \simeq T\biggl(\bigoplus_{i\in I} C_i\biggr) = \bigoplus_{i\in I} T(C_i)$$

because the quotient functor T preserves direct sum. Thus, $T(E_x)$ lies in $\operatorname{add} T(C)$. Similarly, $T(E_y)$ lies in $\operatorname{add} T(C)$.

(c'_3). If $\operatorname{Ext}^1_C(M,M) = 0$, we have the following diagrams (denote $T(\cdot)$ by $T\cdot$ for convenience):

$$0 \to M \to E \xrightarrow{f} E/M \to 0,$$

$$0 \to TM \to TE \overset{Tf}{\to} TE/TM \to 0,$$

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_C(M,M) & \to & \mathrm{Hom}_C(M,E) & \overset{f^*}{\to} & \mathrm{Hom}_C(M,E/M) & \to & \mathrm{Ext}^1_C(M,M) & =0 \\ & & & \downarrow T & & \downarrow T & & & \\ 0 \to & \mathrm{Hom}_{eCe}(TM,TM) & \to & \mathrm{Hom}_{eCe}(TM,TE) & \overset{(Tf)^*}{\to} & \mathrm{Hom}_{eCe}(TM,TE/TM) & \to & \mathrm{Ext}^1_{eCe}(TM,TM) & \to 0 \end{array}$$

Since T is an equivalence, it follows that

$$\mathrm{Ext}^1_{eCe}(T(M),T(M)) = 0.$$

(c_4'). If there is a short exact sequence

$$0 \to M_1 \to M_0 \to C \to 0, \quad M_1, M_0 \in \mathrm{add}\, M,$$

then we get

$$0 \to T(M_1) \to T(M_0) \to T(C) \to 0, \quad T(M_1), T(M_0) \in \mathrm{add}\, T(M),$$

since T is exact and preserves direct sum.

(b) (c_1). Since the section functor S preserves quasi-finite comodules, the result follows.

(c_2'). If N admits an injective resolution

$$0 \to N \to \overline{E}_x \to \overline{E}_y \to 0,$$

where the comodules $\overline{E}_x$ and $\overline{E}_y$ are injective, quasi-finite, and lie in $\mathrm{add}\, eCe$. Since S is exact and it preserves injective comodules and direct sum, we obtain

$$0 \to S(N) \to S(\overline{E}_x) \to S(\overline{E}_y) \to 0,$$

where the comodules $S(\overline{E}_x)$ and $S(\overline{E}_y)$ are injective, quasi-finite, and lie in $\mathrm{add}\, C$.

(c_3'). Assume $\mathrm{Ext}^1_{eCe}(N,N) = 0$. Since the injective envelope of N lies in $\mathrm{add}\, eCe$, there is an exact sequence

$$0 \to N \to eCe^{(m)} \to F \to 0,$$

and we have the following diagram (denote $S(\cdot)$ by $S\cdot$ for convenience):

$$\begin{array}{cccccccc} 0 \to & \mathrm{Hom}_{eCe}(N,N) & \to & \mathrm{Hom}_{eCe}(N,eCe^{(m)}) & \to & \mathrm{Hom}_{eCe}(N,F) & \to & 0 \\ & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\ 0 \to & \mathrm{Hom}_C(SN,SN) & \to & \mathrm{Hom}_C(SN,C^{(m)}) & \to & \mathrm{Hom}_C(SN,SF) & \to & \mathrm{Ext}^1_C(SN,SN) \to 0 \end{array}$$

The diagram is communicate because S is fully faithful. Therefore,

$$\mathrm{Ext}^1_C(S(N),S(N)) = 0.$$

(c_4'). If there is an exact sequence

$$0 \to N_1 \to N_0 \to eCe \to 0, \quad N_1, N_0 \in \mathrm{add}\, N,$$

then

$$0 \to S(N_1) \to S(N_0) \to C \to 0, \quad S(N_1), S(N_0) \in \operatorname{add} S(N),$$

because S is exact and preserves direct sum of comodules. □

Proposition 4 *Assume that C is a basic coalgebra. If M is an f-cotilting right C-comodule, $e \in C^*$ is the idempotent which associates to the set I_e, and $X \simeq Ce$ is a quasi-finite injective cogenerator, then*

(a) $\mathscr{F}^{\mathrm{f}}_{eCe}(T(M)) = \operatorname{cogen}(T(M))$;

(b) $(\mathscr{T}^{\mathrm{f}}_{eCe}(T(M)), \mathscr{F}^{\mathrm{f}}_{eCe}(T(M)))$ *is a torsion pair in* $\mathscr{M}^{eCe}_{\mathrm{f}}$.

Proof (a) Assume that Z is a comodule in $\operatorname{cogen}(T(M))$, and let

$$u\colon Z \to (T(M))^L$$

be a monomorphism with a finite set L. Since T is equivalence, there exists a C-comodule X such that $T(X) = Z$ and

$$u\colon T(X) \to (T(M))^L \simeq T(M^L).$$

u is injective if and only if $X \to M^L$ is injective, if and only if

$$\operatorname{Ext}^1_C(X, M) = 0,$$

if and only if

$$\operatorname{Ext}^1_{eCe}(T(X), T(M)) = 0,$$

that is, Z lies in $\mathscr{F}^{\mathrm{f}}_{eCe}(T(M))$.

(b) We show that $(\mathscr{T}^{\mathrm{f}}_{eCe}(T(M)), \mathscr{F}^{\mathrm{f}}_{eCe}(T(M)))$ is a torsion pair in $\mathscr{M}^{eCe}_{\mathrm{f}}$ by checking that the following three conditions are satisfied:

(t_1) $\operatorname{Hom}_{eCe}(Y, Z) = 0$ for all $Y \in \mathscr{T}^{\mathrm{f}}_{eCe}(T(M))$ and $Z \in \mathscr{F}^{\mathrm{f}}_{eCe}(T(M))$;

(t_2) $\operatorname{Hom}_{eCe}(Y, -)|_{\mathscr{F}^{\mathrm{f}}_{eCe}(T(M))} = 0$ implies $Y \in \mathscr{T}^{\mathrm{f}}_{eCe}(T(M))$; and

(t_3) $\operatorname{Hom}_{eCe}(-, Z)|_{\mathscr{T}^{\mathrm{f}}_{eCe}(T(M))} = 0$ implies $Z \in \mathscr{F}^{\mathrm{f}}_{eCe}(T(M))$.

It is easy to see that (b) holds because T is equivalence. □

Next, we consider the localization of classical tilting comodule which was mentioned in [12].

Definition 4 We call M finitely copresented if there is an exact sequence

$$0 \to M \to E_x \to E_y$$

in $\mathscr{M}^C$, where E_x and E_y are finite direct sum of indecomposable injective comodule (or socle-finite).

Definition 5 [12] A finitely copresented comodule M is called a classical tilting comodule if

(1) there is an exact sequence

$$0 \to M_0 \to M_1 \to C \to 0, \quad M_i \in \operatorname{add} M,$$

(2) $\operatorname{Ext}_C^1(M, M) = 0$,

(3) $\operatorname{inj.dim}(M) \leqslant 1$.

Proposition 5 *Let C be a coalgebra, and let T be a quotient functor. If M is a finitely copresented right C-comodule, then $T(M)$ is a finitely copresented right eCe-comodule.*

Proof By the hypothesis, there is an exact sequence

$$0 \to M \to E_x \to E_y$$

in $\mathscr{M}^C$, where E_x and E_y are finite direct sum of indecomposable injective comodule. Since T is an exact functor, we get an exact sequence

$$0 \to T(M) \to T(E_x) \to T(E_y).$$

If $T(E_x)$ and $T(E_y)$ are both injective right eCe-comodule, then $T(M)$ is a finitely copresented right eCe-comodule since T preserves finite dimensional comodule. Otherwise, we can construct a minimal injective copresentation of $T(M)$:

$$0 \to T(M) \to \overline{E}_x \to \overline{E}_y.$$

Since M is socle-finite, $T(M)$ is socle-finite, that is,

$$\operatorname{soc} T(M) = \coprod_{i=1}^{r} S_i \quad (r < \infty).$$

We have

$$S_i \hookrightarrow \overline{E}_i, \quad \operatorname{soc} T(M) \hookrightarrow \coprod_{i=1}^{r} \overline{E}_i.$$

Since $\coprod_{i=1}^{r} \overline{E}_i$ is injective, the injective envelope $\overline{E}_x$ of $\operatorname{soc} T(M)$ is socle-finite. By the same way, $\overline{E}_y$ is socle-finite. Thus, $T(M)$ is a finitely copresented right eCe-comodule. □

Corollary 2 *Let C be a coalgebra, and let T be a quotient functor which has an exact left adjoint H. If M is a finitely copresented right C-comodule, then $T(M)$ is a finitely copresented right eCe-comodule.*

Proof Since H is exact, T carries injective comodules to injective comodules. By Proposition 5, $T(M)$ is a finitely copresented right eCe-comodule. □

Proposition 6 *Let C be a coalgebra, and let S be a section functor. If N is a finitely copresented right eCe-comodule, then $S(N)$ is a finitely copresented right C-comodule.*

Proof According to the hypothesis, there is an exact sequence

$$0 \to N \to \overline{E}_x \to \overline{E}_y,$$

where $\overline{E}_x$ and $\overline{E}_y$ are socle-finite injective comodules. Since S is left exact, $\operatorname{soc} S(S_i) = S_i$, and S preserves injective envelopes, we have

$$0 \to S(N) \to S(\overline{E}_x) \to S(\overline{E}_y),$$

where $S(\overline{E}_x)$ and $S(\overline{E}_y)$ are socle-finite injective comodules. Therefore, $S(N)$ is a finitely copresented right C-comodule. □

From Propositions 5 and 6, we obtain the following localization property about finitely copresented comodules.

Theorem 4 *Let C be a coalgebra, and let $e \in C^*$ be an idempotent. Then M is a finitely copresented right C-comodule if and only if $T(M)$ is a finitely copresented right eCe-comodule.*

Proof If M is a finitely copresented right C-comodule, then $T(M)$ is a finitely copresented right eCe-comodule by Proposition 5. Conversely, if $T(M)$ is a finitely copresented right eCe-comodule, then $ST(M)$ is a finitely copresented right C-comodule by Proposition 6. It only needs to prove that M is finitely copresented. Since M is an essential subcomodule of $ST(M)$, we construct the minimal injective copresentations of $ST(M)$ and M:

$$0 \to ST(M) \to E' \to E'', \quad 0 \to M \to E_x \to E_y,$$

where $E_x = E'$. Since E_x/M is socle-finite, the injective envelope E_y is socle-finite. Therefore, M is finitely copresented. □

Proposition 7 *Let C be a coalgebra, and let $e \in C^*$ be an idempotent defining a perfect localization. If N is a classical tilting right eCe-comodule, then $S(N)$ is a classical tilting right C-comodule.*

Proof By Proposition 6, $S(N)$ is a finitely copresented right C-comodule. It is sufficient to prove the three conditions in Definition 5.

(1) By the hypothesis, there exists an exact sequence

$$0 \to N_2 \to N_1 \to eCe \to 0, \quad N_i \in \operatorname{add} N.$$

Since S is an exact functor and preserves direct sum of comodules, we know that

$$0 \to S(N_2) \to S(N_1) \to C \to 0$$

is exact and $S(N_i) \in \operatorname{add} S(N)$.

(2) The result follows from Theorem 3.

(3) Suppose inj.dim$(N) \leqslant 1$, that is, there is an exact sequence

$$0 \to N \to \overline{E}_x \to \overline{E}_y \to 0,$$

where $\overline{E}_x$ and $\overline{E}_y$ are injective right eCe-comodules. Since S is exact and preserves injective comodule, we know that

$$0 \to S(N) \to S(\overline{E}_x) \to S(\overline{E}_y) \to 0$$

is exact, where $S(\overline{E}_x)$ and $S(\overline{E}_y)$ are injective right C-comodules. Thus, we get

$$\text{inj.dim}(S(N)) \leqslant 1. \qquad \square$$

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