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RESEARCH ARTICLE

Super $\mathscr{O}$-operators of Jordan superalgebras and super Jordan Yang-Baxter equations

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Abstract In this paper, the super $\mathscr{O}$-operators of Jordan superalgebras are introduced and the solutions of super Jordan Yang-Baxter equation are discussed by super $\mathscr{O}$-operators. Then pre-Jordan superalgebras are studied as the algebraic structure behind the super $\mathscr{O}$-operators. Moreover, the relations among Jordan superalgebras, pre-Jordan superalgebras, and dendriform superalgebras are established.

Keywords Super $\mathscr{O}$-operator, dendriform superalgebra, pre-Jordan superalgebra
MSC 17C50, 17C55

1 Introduction

Jordan algebras were first introduced by Jordan [8] in the context of axiomatic quantum mechanics. They were originally called 'r-number systems', but were renamed as 'Jordan algebras' by Albert [1,2], who began the systematic study of general Jordan algebras. Jordan algebras have become by far an important branch of algebras and appeared in many areas of mathematics like differential geometry ([3,4,13]), Lie theory ([7,14]), and analysis ([5,16]). It is well known that associative, Jordan, and Lie algebras are three kinds of strongly related algebras. A Jordan algebra can be regarded as an 'opposite' of a Lie algebra in the sense that the commutator of an associative algebra is a Lie algebra and the anticommutator of an associative algebra is a Jordan algebra, although not every Jordan algebra is isomorphic to the anticommutator of an associative algebra (such a Jordan algebra is called special).

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The notion of Jordan D-bialgebras was introduced by Zhelyabin [17] as an analogue of Lie bialgebras. A kind of Jordan D-bialgebras (coboundary cases) is obtained from the solutions of an algebraic equation in Jordan algebras. This algebraic equation is called Jordan Yang-Baxter equation (JYBE), which is an analogue of the classical Yang-Baxter equation (CYBE) in Lie algebra ([18]). The original form of a JYBE is given in the tensor form, so it is natural to consider operator form of the JYBE which satisfies certain conditions ([6]). It was proved that a skew-symmetric solution of the JYBE was exactly a special operator, which was called $\mathscr{O}$-operator.

Now, Jordan superalgebras have been rapidly developed ([9–12,15]). Jordan superalgebras were first studied in [9] by classifying finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic 0. In this paper, super JYBE in Jordan superalgebras and super $\mathscr{O}$-operators are introduced. Moreover, we exploit pre-Jordan superalgebras which are the algebraic structures behind the super $\mathscr{O}$-operators and study their relations with Jordan superalgebras and dendriform superalgebras. Though the results we give are mainly generalizations of the results for Jordan algebras, it seems to us that it is still worth presenting them here.

This paper is organized as follows. In Section 2, we give some fundamental results on Jordan superalgebras and super JYBE. In Section 3, we introduce the notion of super $\mathscr{O}$-operators of Jordan superalgebras, and then constuct a direct relation between super $\mathscr{O}$-operator and super JYBE. In Section 4, we introduce pre-Jordan superalgebras and give their relations with super $\mathscr{O}$-operators, Jordan superalgebras, and dendriform superalgebras.

Throughout this paper, all algebras are finite-dimensional and over a field $\mathbb{F}$ of characteristic 0.

2 Preliminaries

In this section, we will give the definitions and some results for superspaces and Jordan superalgebras in order to be self-contained.

A vector space V is called a super vector space if

$$V = V_{\bar{0}} + V_{\bar{1}}$$

is $\mathbb{Z}_2$-graded. The elements in $V_{\bar{0}} \cup V_{\bar{1}}$ are called homogeneous. We use the expression $|x|$ to denote the parity index of the homogeneous element x, where

$$|x| = \begin{cases} 0, & x \in V_{\bar{0}}, \\ 1, & x \in V_{\bar{1}}. \end{cases}$$

It is assumed in this paper that all elements are the homogeneous of their corresponding super vector spaces.

Let V^* be the dual space of V, and let its $\mathbb{Z}_2$-gradation be given by

$$(V^*)_a = \{f \in V^* \mid f(V_{a+\bar{1}}) = 0,\ a \in \mathbb{Z}_2\}.$$

Let

$$\langle \cdot, \cdot \rangle \colon V^* \times V \to \mathbb{F}$$

be the canonical pairing. Then we identify V with V^* by the pairing

$$f(v) = \langle f, v \rangle = (-1)^{|v|\,|f|} \langle v, f \rangle, \quad \forall\, v \in V,\ \forall\, f \in V^*.$$

Let U and V be two super vector spaces. Then the tensor product $U \otimes V$ has a natural $\mathbb{Z}_2$-gradation defined as

$$(U \otimes V)_c = \bigoplus_{a+b=c} (U_a \otimes V_b), \quad a, b, c \in \mathbb{Z}_2.$$

Next, we define a linear operator

$$\sigma \colon V \otimes V \to V \otimes V$$

by

$$\sigma(v \otimes w) = (-1)^{|v|\,|w|} w \otimes v, \quad \forall\, v, w \in V.$$

The element $r \in V \otimes V$ is called skew-supersymmetric if

$$r + \sigma(r) = 0.$$

Definition 1 A *Jordan superalgebra* is a super vector space

$$J = J_{\overline{0}} \oplus J_{\overline{1}}$$

equipped with a bilinear product: $J \times J \to J$ satisfying

$$J_\alpha J_\beta \subseteq J_{\alpha+\beta}, \quad \forall\, \alpha, \beta \in \mathbb{Z}_2,$$

and the following conditions:

$$xy = (-1)^{|x|\,|y|} yx, \quad \forall\, x, y \in J, \tag{1}$$

$$\begin{aligned} &((xy)z)t + (-1)^{|y|(|z|+|t|)+|z|\,|t|}((xt)z)y + (-1)^{|x|(|z|+|t|+|y|)+|z|\,|t|}((yt)z)x \\ &= (xy)(zt) + (-1)^{|y|\,|z|}(xz)(yt) + (-1)^{|t|(|y|+|z|)}(xt)(yz), \quad \forall\, x, y, z, t \in J. \end{aligned} \tag{2}$$

Let J be a Jordan superalgebra, and let V be a super vector space. The space $\mathrm{gl}(V)$ consisting of all linear transformations on V has a natural $\mathbb{Z}_2$-gradation as

$$\mathrm{gl}(V)_a = \{f \in \mathrm{gl}(V) \mid f(V_b) \subseteq V_{a+b},\ a, b \in \mathbb{Z}_2\}.$$

A J-representation is a vector space V with an even linear map $\rho \colon J \to \mathrm{gl}(V)$ (i.e., $|\rho| = 0$) such that for any $x, y, z \in J$,

$$\begin{aligned} &\rho(xy)\rho(z) + (-1)^{|y|\,|z|}\rho(xz)\rho(y) + (-1)^{|x|(|y|+|z|)}\rho(yz)\rho(x) \\ &\quad = \rho(x)\rho(y)\rho(z) + (-1)^{|y|\,|z|}\rho((xz)y) + (-1)^{|x|(|y|+|z|)+|y|\,|z|}\rho(z)\rho(y)\rho(x), \end{aligned}$$

$$[\rho(x), \rho(yz)] + (-1)^{|x|\,|y|}[\rho(y), \rho(xz)] + (-1)^{|z|(|x|+|y|)}[\rho(z), \rho(xy)] = 0,$$

where $[\cdot,\cdot]$ is the commutator. We denote it by (ρ, V). Furthermore, there is a Jordan superalgebra structure on the direct sum $J \oplus V$ of the underlying vector space of J and V given by

$$(x+u)(y+v) = xy + \rho(x)v + (-1)^{|x|\,|y|}\rho(y)u, \quad \forall\ x, y \in J,\ \forall\ u, v \in V,$$

and denoted by $J \ltimes_\rho V$.

For $x \in J$, define the left and right actions $L(x), R(x) \colon J \to J$ by

$$L(x)y = xy, \quad R(x)y = (-1)^{|x|\,|y|}yx, \quad \forall\ y \in J,$$

respectively. From (1), it is obvious that

$$L(x)y = R(x)y.$$

In addition, we define

$$\begin{aligned} L\colon J &\to \mathrm{gl}(J), \\ x &\mapsto L(x). \end{aligned}$$

Then (L, J) is a representation of J, which is called the regular representation of J.

Let (ρ, V) be a J-representation, and let V^* denote the dual space of V. Define a linear map

$$\rho^*\colon J \to \mathrm{gl}(V^*)$$

by

$$\langle \rho^*(x)u^*, v\rangle = (-1)^{|x|\,|u|}\langle u^*, \rho(x)v\rangle, \quad \forall\ x \in J,\ \forall\ v \in V,\ \forall\ u^* \in V^*.$$

Then (ρ^*, V^*) is also a J-representation, which is called the dual representation of (ρ, V).

For a Jordan superalgebra J and $r \in J \otimes J$, the standard form of the super Jordan Yang-Baxter equation (super JYBE) is given as follows:

$$r_{12}r_{13} + r_{13}r_{23} - r_{12}r_{23} = 0, \tag{3}$$

where r is called a solution of the super JYBE. For

$$r = \sum_i a_i \otimes b_i \in J \otimes J,$$

note that

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

and

$$r_{12}r_{13} = \sum_{i,j}(-1)^{|a_j|\,|b_i|}a_i a_j \otimes b_i \otimes b_j,$$
$$r_{13}r_{23} = \sum_{i,j}(-1)^{|a_j|\,|b_i|}a_i \otimes a_j \otimes b_i b_j,$$
$$r_{12}r_{23} = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j,$$

where 1 is a unit element in J.

3 Super $\mathscr{O}$-operators and super Jordan Yang-Baxter equations

Let V be a super vector space. For any $r \in V \otimes V$, r can be regarded as a map from V^* to V in the following way:

$$\langle v^* \otimes u^*, r\rangle = (-1)^{|r|\,|u^*|}\langle v^*, r(u^*)\rangle, \quad \forall\ u^*, v^* \in V^*. \tag{4}$$

Equation (3) gives the tensor form of super JYBE. What we will do next is to replace the tensor form by a linear operator satisfying some conditions.

Theorem 1 *Let J be a Jordan superalgebra and $r \in J \otimes J$ be skew-supersymmetric with $|r| = 0$. Then r is a solution of the super JYBE in J if and only if*

$$r(x^*)r(y^*) = r(L^*(r(x^*))y^* + (-1)^{|x^*|\,|y^*|}L^*(r(y^*))x^*), \quad \forall\ x^*, y^* \in J^*. \tag{5}$$

Proof Let $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ be a basis of J, where $e_i \in J_{\bar{0}}$ $(1 \leqslant i \leqslant m)$ and $f_k \in J_{\bar{1}}$ $(1 \leqslant k \leqslant n)$. Denote by $\{e_1^*, \dots, e_m^*, f_1^*, \dots, f_n^*\}$ the dual basis with $e_i^* \in J_{\bar{0}}^*$ $(1 \leqslant i \leqslant m)$ and $f_k^* \in J_{\bar{1}}^*$ $(1 \leqslant k \leqslant n)$. Since r is skew-supersymmetric and $|r| = 0$, we can set

$$r = \sum_{1 \leqslant i,j \leqslant m} a_{ij} e_i \otimes e_j + \sum_{1 \leqslant k,l \leqslant n} b_{kl} f_k \otimes f_l,$$

where

$$a_{ij} = -a_{ji}, \quad b_{kl} = b_{lk}.$$

Now, we have

$$\begin{aligned} r_{12}r_{13} &= \Bigg(\sum_{1 \leqslant i,j \leqslant m} a_{ij} e_i \otimes e_j \otimes 1 + \sum_{1 \leqslant k,l \leqslant n} b_{kl} f_k \otimes f_l \otimes 1\Bigg) \\ &\quad \cdot \Bigg(\sum_{1 \leqslant p,q \leqslant m} a_{pq} e_p \otimes 1 \otimes e_q + \sum_{1 \leqslant s,t \leqslant n} b_{st} f_s \otimes 1 \otimes f_t\Bigg) \\ &= \sum_{1 \leqslant i,j,p,q,a \leqslant m} C_{ip}^a a_{ij} a_{pq} e_a \otimes e_j \otimes e_q \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{1\leqslant i,j\leqslant m}\ \sum_{1\leqslant s,t,b\leqslant n} C_{is}^{b} a_{ij} b_{st} f_b \otimes e_j \otimes f_t \\
&+ \sum_{1\leqslant p,q\leqslant m}\ \sum_{1\leqslant k,l,c\leqslant n} C_{kp}^{c} a_{pq} b_{kl} f_c \otimes f_l \otimes e_q \\
&- \sum_{1\leqslant k,l,s,t\leqslant n}\ \sum_{1\leqslant d\leqslant m} C_{ks}^{d} b_{kl} b_{st} e_d \otimes f_l \otimes f_t,
\end{aligned}
$$

$$
\begin{aligned}
r_{13}r_{23} &= \Big(\sum_{1\leqslant i,j\leqslant m} a_{ij} e_i \otimes 1 \otimes e_j + \sum_{1\leqslant k,l\leqslant n} b_{kl} f_k \otimes 1 \otimes f_l\Big) \\
&\quad \cdot \Big(\sum_{1\leqslant p,q\leqslant m} a_{pq} 1 \otimes e_p \otimes e_q + \sum_{1\leqslant s,t\leqslant n} b_{st} 1 \otimes f_s \otimes f_t\Big) \\
&= \sum_{1\leqslant i,j,p,q,a\leqslant m} C_{jq}^{a} a_{ij} a_{pq} e_i \otimes e_p \otimes e_a \\
&\quad + \sum_{1\leqslant i,j\leqslant m}\ \sum_{1\leqslant s,t,b\leqslant n} C_{jt}^{b} a_{ij} b_{st} e_i \otimes f_s \otimes f_b \\
&\quad + \sum_{1\leqslant p,q\leqslant m}\ \sum_{1\leqslant k,l,c\leqslant n} C_{lq}^{c} a_{pq} b_{kl} f_k \otimes e_p \otimes f_c \\
&\quad - \sum_{1\leqslant k,l,s,t\leqslant n}\ \sum_{1\leqslant d\leqslant m} C_{lt}^{d} b_{kl} b_{st} f_k \otimes f_s \otimes e_d,
\end{aligned}
$$

$$
\begin{aligned}
r_{12}r_{23} &= \Big(\sum_{1\leqslant i,j\leqslant m} a_{ij} e_i \otimes e_j \otimes 1 + \sum_{1\leqslant k,l\leqslant n} b_{kl} f_k \otimes f_l \otimes 1\Big) \\
&\quad \cdot \Big(\sum_{1\leqslant p,q\leqslant m} a_{pq} 1 \otimes e_p \otimes e_q + \sum_{1\leqslant s,t\leqslant n} b_{st} 1 \otimes f_s \otimes f_t\Big) \\
&= \sum_{1\leqslant i,j,p,q,a\leqslant m} C_{jp}^{a} a_{ij} a_{pq} e_i \otimes e_a \otimes e_q \\
&\quad + \sum_{1\leqslant i,j\leqslant m}\ \sum_{1\leqslant s,t,b\leqslant n} C_{js}^{b} a_{ij} b_{st} e_i \otimes f_b \otimes f_t \\
&\quad + \sum_{1\leqslant p,q\leqslant m}\ \sum_{1\leqslant k,l,c\leqslant n} C_{lp}^{c} a_{pq} b_{kl} f_k \otimes f_c \otimes e_q \\
&\quad + \sum_{1\leqslant k,l,s,t\leqslant n}\ \sum_{1\leqslant d\leqslant m} C_{ls}^{d} b_{kl} b_{st} f_k \otimes e_d \otimes f_t,
\end{aligned}
$$

where C_{ij}^{k}'s are the structure coefficients of Jordan superalgebra J on the basis $\{e_1, \dots, e_m, f_1, \dots, f_n\}$. Then r is a solution of super JYBE in J if and only if

$$
\begin{aligned}
&\sum_{1\leqslant i,p\leqslant m} (C_{ip}^{a} a_{ij} a_{pq} + C_{pi}^{q} a_{ap} a_{ji} - C_{ip}^{j} a_{ai} a_{pq}) e_a \otimes e_j \otimes e_q \\
+&\Big(\sum_{i=1}^{m}\sum_{s=1}^{n} C_{is}^{b} a_{ij} b_{st} + \sum_{i=1}^{m}\sum_{s=1}^{n} C_{si}^{t} a_{ji} b_{bs} - \sum_{1\leqslant l,s\leqslant n} C_{ls}^{j} b_{bl} b_{st}\Big) f_b \otimes e_j \otimes f_t
\end{aligned}
$$

$$+\Big(\sum_{p=1}^{m}\sum_{k=1}^{n} C_{kp}^{c} a_{pq} b_{kl} - \sum_{1\leqslant k,t\leqslant n} C_{kt}^{q} b_{ck} b_{lt} - \sum_{p=1}^{m}\sum_{k=1}^{n} C_{kp}^{l} a_{pq} b_{ck}\Big) f_c \otimes f_l \otimes e_q$$

$$+\Big(-\sum_{1\leqslant k,s\leqslant n} C_{ks}^{d} b_{kl} b_{st} + \sum_{j=1}^{m}\sum_{s=1}^{n} C_{js}^{t} a_{dj} b_{ls} - \sum_{j=1}^{m}\sum_{s=1}^{n} C_{js}^{l} a_{dj} b_{st}\Big) e_d \otimes f_l \otimes f_t = 0.$$

On the other hand, by (4), we get

$$r(e_j^*) = -\sum_{i=1}^{m} a_{ji} e_i, \quad r(f_k^*) = -\sum_{s=1}^{n} b_{ks} f_s, \quad 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n.$$

We prove the conclusion in the following four cases.

Case 1 $x^* = e_j^*$ and $y^* = e_q^*$.

By (5), we have

$$\sum_{1\leqslant i,p\leqslant m} (C_{ip}^{a} a_{ij} a_{pq} + C_{pi}^{q} a_{ap} a_{ji} - C_{ip}^{j} a_{ai} a_{pq}) e_a = 0.$$

Case 2 $x^* = e_q^*$ and $y^* = f_c^*$.

By (5), we have

$$\Big(\sum_{p=1}^{m}\sum_{k=1}^{n} C_{kp}^{c} a_{pq} b_{kl} - \sum_{1\leqslant k,t\leqslant n} C_{kt}^{q} b_{ck} b_{lt} - \sum_{p=1}^{m}\sum_{k=1}^{n} C_{kp}^{l} a_{pq} b_{ck}\Big) f_l = 0.$$

Case 3 $x^* = f_b^*$ and $y^* = e_j^*$.

By (5), we have

$$\Big(\sum_{i=1}^{m}\sum_{s=1}^{n} C_{is}^{b} a_{ij} b_{st} + \sum_{i=1}^{m}\sum_{s=1}^{n} C_{si}^{t} a_{ji} b_{bs} - \sum_{1\leqslant l,s\leqslant n} C_{ls}^{j} b_{bl} b_{st}\Big) f_t = 0.$$

Case 4 $x^* = f_l^*$ and $y^* = f_t^*$.

By (5), we have

$$\Big(\sum_{1\leqslant k,s\leqslant n} C_{ks}^{d} b_{kl} b_{st} - \sum_{j=1}^{m}\sum_{s=1}^{n} C_{js}^{t} a_{dj} b_{ls} + \sum_{j=1}^{m}\sum_{s=1}^{n} C_{js}^{l} a_{dj} b_{st}\Big) e_d = 0.$$

Therefore, it is easy to see that r is a solution of super JYBE in J if and only if r satisfies (5). $\square$

Now, we introduce the notion of super $\mathscr{O}$-operator of a Jordan superalgebra.

Definition 2 Let J be a Jordan superalgebra, and let (ρ, V) be a J-representation. A linear map $T\colon V \to J$ with $|T| = 0$ is called a *super $\mathscr{O}$-operator* associated to ρ if T satisfies

$$T(u)T(v) = T(\rho(T(u))v + (-1)^{|u|\,|v|}\rho(T(v))u), \quad \forall\, u, v \in V. \tag{6}$$

In the case $(\rho, V) = (L, J)$, the super $\mathscr{O}$-operator is called a *super Rota-Baxter operator*.

Let J be a Jordan superalgebra, and let (ρ, V) be a J-representation. A linear map $T\colon V \to J$ with $|T| = 0$ can be identified as an element in

$$J \otimes V^* \subset (J \ltimes_{\rho^*} V^*) \otimes (J \ltimes_{\rho^*} V^*)$$

as follows. Let $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ be a basis of J. Let $\{u_1, \dots, u_p, v_1, \dots, v_q\}$ be a basis of V, and let $\{u_1^*, \dots, u_p^*, v_1^*, \dots, v_q^*\}$ be its dual basis. Since $|T| = 0$, we set

$$T(u_i) = \sum_{a=1}^{m} a_{ia} e_a \ (1 \leqslant i \leqslant p), \quad T(v_s) = \sum_{b=1}^{n} b_{sb} f_b \ (1 \leqslant s \leqslant q).$$

Since as vector space,

$$\mathrm{Hom}(V, J) \cong J \otimes V^*,$$

we have

$$\begin{aligned} T &= \sum_{i=1}^{p} T(u_i) \otimes u_i^* + \sum_{s=1}^{q} T(v_s) \otimes v_s^* \\ &= \sum_{i=1}^{p} \sum_{a=1}^{m} a_{ia} e_a \otimes u_i^* + \sum_{s=1}^{q} \sum_{b=1}^{n} b_{sb} f_b \otimes v_s^* \subset (J \ltimes_{\rho^*} V^*) \otimes (J \ltimes_{\rho^*} V^*). \end{aligned} \tag{7}$$

Theorem 2 *We have $r = T - \sigma(T)$ is a skew-supersymmetric solution of super JYBE in the Jordan superalgebra $J \ltimes_{\rho^*} V^*$ if and only if T is a super $\mathscr{O}$-operator associated to ρ.*

Proof From (7), we have

$$r = T - \sigma(T) = \sum_{i=1}^{p} T(u_i) \otimes u_i^* + \sum_{s=1}^{q} T(v_s) \otimes v_s^* - \sum_{i=1}^{p} u_i^* \otimes T(u_i) + \sum_{s=1}^{q} v_s^* \otimes T(v_s).$$

Thus,

$$\begin{aligned} r_{12} r_{13} = \ & \sum_{1 \leqslant i,j \leqslant p} (T(u_i)T(u_j) \otimes u_i^* \otimes u_j^* - \rho^*(T(u_i)) u_j^* \otimes u_i^* \otimes T(u_j) \\ & - \rho^*(T(u_j)) u_i^* \otimes T(u_i) \otimes u_j^*) + \sum_{i=1}^{p} \sum_{t=1}^{q} (T(u_i)T(v_t) \otimes u_i^* \otimes v_t^* \\ & + \rho^*(T(u_i)) v_t^* \otimes u_i^* \otimes T(v_t) - \rho^*(T(v_t)) u_i^* \otimes T(u_i) \otimes v_t^*) \\ & + \sum_{s=1}^{q} \sum_{j=1}^{p} (T(v_s)T(u_j) \otimes v_s^* \otimes u_j^* - \rho^*(T(v_s)) u_j^* \otimes v_s^* \otimes T(u_j) \\ & + \rho^*(T(u_j)) v_s^* \otimes T(v_s) \otimes u_j^*) + \sum_{1 \leqslant s,t \leqslant q} (-T(v_s)T(v_t) \otimes v_s^* \otimes v_t^* \\ & - \rho^*(T(v_s)) v_t^* \otimes v_s^* \otimes T(v_t) + \rho^*(T(v_t)) v_s^* \otimes T(v_s)) \otimes v_t^*). \end{aligned}$$

By the definition of dual representation, we have

$$\rho^*(T(u_i))u_j^* = \sum_{k=1}^{p} u_j^*(\rho(T(u_i))u_k)u_k^*.$$

Therefore,

$$\begin{aligned}\sum_{1\leqslant i,j\leqslant p} \rho^*(T(u_i))u_j^* \otimes u_i^* \otimes T(u_j) &= \sum_{1\leqslant i,j,k\leqslant p} u_j^*(\rho(T(u_i))u_k)u_k^* \otimes u_i^* \otimes T(u_j)\\ &= \sum_{1\leqslant i,k\leqslant p} u_k^* \otimes u_i^* \otimes T\left(\sum_{j=1}^{p} u_j^*(\rho(T(u_i))u_k)u_j\right)\\ &= \sum_{1\leqslant i,k\leqslant p} u_k^* \otimes u_i^* \otimes T(\rho(T(u_i))u_k).\end{aligned}$$

Then we get

$$\begin{aligned}r_{12}r_{13} = &\sum_{1\leqslant i,j\leqslant p}(T(u_i)T(u_j)\otimes u_i^*\otimes u_j^* - u_i^*\otimes u_j^*\otimes T(\rho(T(u_j))u_i)\\ &- u_i^*\otimes T(\rho(T(u_j))u_i)\otimes u_j^*) + \sum_{i=1}^{p}\sum_{t=1}^{q}((T(u_i)T(v_t)\otimes u_i^*\otimes v_t^*\\ &+ u_i^*\otimes v_t^*\otimes T(\rho(T(v_t))u_i) - u_i^*\otimes T(\rho(T(v_t))u_i)\otimes v_t^*)\\ &+ \sum_{s=1}^{q}\sum_{j=1}^{p}(T(v_s)T(u_j)\otimes v_s^*\otimes u_j^* + v_s^*\otimes u_j^*\otimes T(\rho(T(u_j))v_s)\\ &+ v_s^*\otimes T(\rho(T(u_j))v_s)\otimes u_j^*) + \sum_{1\leqslant s,t\leqslant q}(-T(v_s)T(v_t)\otimes v_s^*\otimes v_t^*\\ &- v_s^*\otimes v_t^*\otimes T(\rho(T(v_t))v_s)) - v_s^*\otimes T(\rho(T(v_t))v_s)\otimes v_t^*),\end{aligned}$$

$$\begin{aligned}r_{13}r_{23} = &\sum_{1\leqslant i,j\leqslant p}(-T(\rho(T(u_i))u_j)\otimes u_i^*\otimes u_j^* + u_i^*\otimes u_j^*\otimes T(u_i)T(u_j)\\ &- u_i^*\otimes T(\rho(T(u_i))u_j)\otimes u_j^*) + \sum_{i=1}^{p}\sum_{t=1}^{q}(-T(\rho(T(u_i))v_t)\otimes u_i^*\otimes v_t^*\\ &- u_i^*\otimes v_t^*\otimes T(u_i)T(v_t) - u_i^*\otimes T(\rho(T(u_i))v_t)\otimes v_t^*)\\ &+ \sum_{s=1}^{q}\sum_{j=1}^{p}(-T(\rho(T(v_s))u_j)\otimes v_s^*\otimes u_j^* - v_s^*\otimes u_j^*\otimes T(v_s)T(u_j)\\ &+ v_s^*\otimes T(\rho(T(v_s))u_j)\otimes u_j^*) + \sum_{1\leqslant s,t\leqslant q}(T(\rho(T(v_s))v_t)\otimes v_s^*\otimes v_t^*\\ &- v_s^*\otimes v_t^*\otimes T(v_s)T(v_t) + v_s^*\otimes T(\rho(T(v_s))v_t)\otimes v_t^*),\end{aligned}$$

$$
\begin{aligned}
-r_{12}r_{23} = \sum_{1\leqslant i,j\leqslant p} & (-T(\rho(T(u_j))u_i)\otimes u_i^*\otimes u_j^* - u_i^*\otimes u_j^*\otimes T(\rho(T(u_i))u_j) \\
& + u_i^*\otimes T(u_i)T(u_j)\otimes u_j^*) + \sum_{i=1}^{p}\sum_{t=1}^{q}(-T(\rho(T(v_t))u_i)\otimes u_i^*\otimes v_t^* \\
& + u_i^*\otimes v_t^*\otimes T(\rho(T(u_i))v_t) + u_i^*\otimes T(u_i)T(v_t)\otimes v_t^*) \\
& + \sum_{s=1}^{q}\sum_{j=1}^{p}(-T(\rho(T(u_j))v_s)\otimes v_s^*\otimes u_j^* + v_s^*\otimes u_j^*\otimes T(\rho(T(v_s))u_j \\
& - v_s^*\otimes T(v_s)T(u_j)\otimes u_j^*) + \sum_{1\leqslant s,t\leqslant q}(-T(\rho(T(v_t))v_s)\otimes v_s^*\otimes v_t^* \\
& + v_s^*\otimes v_t^*\otimes T(\rho(T(v_s))v_t) - v_s^*\otimes T(v_s)T(v_t)\otimes v_t^*).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& r_{12}r_{13} + r_{13}r_{23} - r_{12}r_{23} \\
& \quad = \sum_{1\leqslant i,j\leqslant p}\{[T(u_i)T(u_j) - T(\rho(T(u_i))u_j) - T(\rho(T(u_j))u_i)]\otimes u_i^*\otimes u_j^* \\
& \quad + u_i^*\otimes u_j^*\otimes[T(u_i)T(u_j) - T(\rho(T(u_i))u_j) - T(\rho(T(u_j))u_i)] \\
& \quad + u_i^*\otimes[T(u_i)T(u_j) - T(\rho(T(u_i))u_j) - T(\rho(T(u_j))u_i)]\otimes u_j^*\} \\
& \quad + \sum_{i=1}^{p}\sum_{t=1}^{q}\{[T(u_i)T(v_t) - T(\rho(T(u_i))v_t) - T(\rho(T(v_t))u_i)]\otimes u_i^*\otimes v_t^* \\
& \quad - u_i^*\otimes v_t^*\otimes[T(u_i)T(v_t) - T(\rho(T(u_i))v_t) - T(\rho(T(v_t))u_i)] \\
& \quad + u_i^*\otimes[T(u_i)T(v_t)) - T(\rho(T(u_i))v_t) - T(\rho(T(v_t))u_i)]\otimes v_t^*\} \\
& \quad + \sum_{s=1}^{q}\sum_{j=1}^{p}\{[T(v_s)T(u_j) - T(\rho(T(v_s))u_j) - T(\rho(T(u_j))v_s)]\otimes v_s^*\otimes u_j^* \\
& \quad - v_s^*\otimes u_j^*\otimes[T(v_s)T(u_j) - T(\rho(T(v_s))u_j) - T(\rho(T(u_j))v_s)] \\
& \quad - v_s^*\otimes[T(v_s)T(u_j) - T(\rho(T(v_s))u_j) - T(\rho(T(u_j))v_s)]\otimes u_j^*\} \\
& \quad + \sum_{1\leqslant s,t\leqslant q}\{-[T(v_s)T(v_t) - T(\rho(T(v_s))v_t) + T(\rho(T(v_t))v_s)]\otimes v_s^*\otimes v_t^* \\
& \quad - v_s^*\otimes v_t^*\otimes[T(v_s)T(v_t) - T(\rho(T(v_s))v_t) + T(\rho(T(v_t))v_s)] \\
& \quad - v_s^*\otimes[T(v_s)T(v_t) - T(\rho(T(v_s))v_t) + T(\rho(T(v_t))v_s)]\otimes v_t^*\}.
\end{aligned}
$$

Obviously, r is a solution of super JYBE in $J\ltimes_{\rho^*}V^*$ if and only if T is a super $\mathscr{O}$-operator associated to ρ. □

In fact, Theorem 2 gives a relation between super $\mathscr{O}$-operator and super JYBE. Then, we get a direct conclusion from Theorems 1 and 2.

Corollary 1 *Let J be a Jordan superalgebra, and let (ρ, V) be a*

J-representation. Set

$$\hat{J} = J \ltimes_{\rho^*} V^*.$$

Let $T\colon V \to J$ be a linear map with $|T| = 0$. Then the following three conditions are equivalent:

(i) *T is a super $\mathscr{O}$-operator of J associated to ρ;*

(ii) *$T - \sigma(T)$ is a skew-supersymmetric solution of the super JYBE in $\hat{J}$;*

(iii) *$T - \sigma(T)$ is super $\mathscr{O}$-operator of the Jordan superalgebra $\hat{J}$ associated to L^*.*

4 Super $\mathscr{O}$-operators of Jordan superalgebras and pre-Jordan superalgebras

In this section, we introduce the notion of pre-Jordan superalgebras. Then we study the relations among Jordan superalgebras, pre-Jordan superalgebras, and dendriform superalgebras.

Definition 3 A *pre-Jordan superalgebra* A is a super vector space

$$A = A_{\overline{0}} \oplus A_{\overline{1}}$$

equipped with a bilinear product $(x, y) \to x \cdot y$ satisfying

$$A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}, \quad \forall\ \alpha, \beta \in \mathbb{Z}_2,$$

and the following equations ($\forall\ x, y, z, u \in A$):

$$\begin{aligned}
&x \cdot [y \cdot (z \cdot u)] + (-1)^{|z|(|x|+|y|)+|x|\,|y|} z \cdot [y \cdot (x \cdot u)] + (-1)^{|z|\,|y|}[(x \circ z) \circ y] \cdot u \\
&\quad = x \cdot [(y \circ z) \cdot u] + (-1)^{|x|(|y|+|z|)} y \cdot [(z \circ x) \cdot u] + (-1)^{|z|(|x|+|y|)} z \cdot [(x \circ y) \cdot u], \quad (8)\\
&(x \circ y) \cdot (z \cdot u) + (-1)^{|x|(|y|+|z|)}(y \circ z) \cdot (x \cdot u) + (-1)^{|z|(|x|+|y|)}(z \circ x) \cdot (y \cdot u) \\
&\quad = x \cdot [(y \circ z) \cdot u] + (-1)^{|x|(|y|+|z|)} y \cdot [(z \circ x) \cdot u] + (-1)^{|z|(|x|+|y|)} z \cdot [(x \circ y) \cdot u], \quad (9)
\end{aligned}$$

where

$$x \circ y = x \cdot y + (-1)^{|x|\,|y|} y \cdot x.$$

In fact, (8) and (9) are equivalent to the following equations:

$$\begin{aligned}
&(x, y, z \cdot u) - (-1)^{|y|\,|z|}(x \cdot z, y, u) \\
&+ (-1)^{|x|(|y|+|z|)}(y, z, x, u)_2 + (-1)^{|x|\,|y|}(y, x, z, u)_2 \\
&+ (-1)^{|x|\,|y|+|x|\,|z|+|y|\,|z|}(z, y, x \cdot u) - (-1)^{|x|\,|z|+|y|\,|z|}(z \cdot x, y, u) = 0, \qquad (10)\\
&(x, y, z, u)_1 + (-1)^{|x|(|y|+|z|)}(y, z, x, u)_1 \\
&+ (-1)^{|z|(|x|+|y|)}(z, x, y, u)_1 + (-1)^{|x|\,|y|}(y, x, z, u)_1 \\
&+ (-1)^{|y|\,|z|}(x, z, y, u)_1 + (-1)^{|x|\,|y|+|x|\,|z|)+|y|\,|z|}(z, y, x, u)_1 = 0, \qquad (11)
\end{aligned}$$

where

$$(x,y,z)=(x\cdot y)\cdot z-x\cdot(y\cdot z),$$
$$(x,y,z,u)_1=(x\cdot y)\cdot(z\cdot u)-x\cdot[(y\cdot z)\cdot u],$$
$$(x,y,z,u)_2=(x\cdot y)\cdot(z\cdot u)-[x\cdot[(y\cdot z)]\cdot u.$$

Definition 4 A *dendriform superalgebra* D is a super vector space

$$D=D_{\overline{0}}\oplus D_{\overline{1}}$$

equipped with two bilinear products denoted by

$$\succ,\prec\colon D\otimes D\to D$$

satisfying

$$D_\alpha D_\beta\subseteq D_{\alpha+\beta},\quad \forall\ \alpha,\beta\in\mathbb{Z}_2,$$

and the following equations ($\forall\ x,y,z\in D$):

$$x\succ(y\succ z)=(x*y)\succ z,$$
$$(x\succ y)\prec z=x\succ(y\prec z),$$
$$(x\prec y)\prec z=x\prec(y*z),$$

where

$$x*y=x\succ y+x\prec y.$$

Analogous to the connection between associative superalgebras and Jordan superalgebras, dendriform superalgebras are closely related to pre-Jordan superalgebrs.

Proposition 1 *Let $(D,\succ,\prec)$ be a dendriform superalgebra. Then the product*

$$x\cdot y=x\succ y+(-1)^{|x|\,|y|}y\prec x,\quad \forall\ x,y\in D,$$

defines a pre-Jordan superalgebra structure on D.

Proof It can be straightforward proved by (10) and (11). □

Let $(A,\cdot)$ be a pre-Jordan superalgebra. Then it is easy to see that the product

$$x\circ y=x\cdot y+(-1)^{|x|\,|y|}y\cdot x,\quad \forall\ x,y\in A, \tag{12}$$

defines a Jordan superalgebra $(J(A),\circ)$, which is called the associated Jordan superalgebra of $(A,\cdot)$ and $(A,\cdot)$ is called a compatible pre-Jordan superalgebra structure on the Jordan superalgebra $(J(A),\circ)$. Next, we give a sufficient and necessary condition for a Jordan superalgebra with a compatible pre-Jordan superalgebra structure.

Proposition 2 *A superalgebra $(J,\cdot)$ is a pre-Jordan superalgebra if and only if $(J,\circ)$ is a Jordan superalgebra and (L,J) is a representation of $(J,\circ)$, where*

$$x\circ y=x\cdot y+(-1)^{|x|\,|y|}y\cdot x,\quad \forall\ x,y\in J.$$

Summarizing the above study, we have the following commutative diagram of categories:

$$\begin{array}{ccc} \text{dendriform superalgebra} & \xrightarrow{+} & \text{associative superalgebra} \\ \downarrow + & & \downarrow + \\ \text{pre-Jordan superalgebra} & \xrightarrow{+} & \text{Jordan superalgebra} \end{array}$$

Furthermore, we can construct pre-Jordan superalgebras from super $\mathscr{O}$-operators of Jordan superalgebras.

Theorem 3 *Let J be a Jordan superalgebra and (ρ, V) be its representation. Let $T\colon V \to J$ be a super $\mathscr{O}$-operator of J associated to ρ. Then there exists a pre-Jordan superalgebra structure on V given by*

$$u * v = \rho(T(u))v, \quad \forall\ u, v \in V. \tag{13}$$

Proof Let $u, v, w, a \in V$. Set

$$x = T(u), \quad y = T(v), \quad z = T(w),$$

and

$$u \circ v = u * v + (-1)^{|u|\,|v|} v * u.$$

Hence, we have

$$u * [(v * w) * a] = \rho(T(u))\rho(T(v))\rho(T(w))a = \rho(x)\rho(y)\rho(z)a,$$

$$\begin{aligned} [(u \circ w) \circ v] * a &= \rho\{T[\rho(T(u \circ w))v + (-1)^{(|u|+|w|)|v|}\rho(T(v))(u \circ w)]\}a \\ &= \rho\{T(u \circ w) \circ T(v)\}a \\ &= \rho\{T[\rho(T(u))w + (-1)^{|u|\,|w|}\rho(T(w)u] \circ T(v)\}a \\ &= \rho\{(T(u) \circ T(w)) \circ T(v)\}a \\ &= \rho((x \circ z) \circ y)a, \end{aligned}$$

$$\begin{aligned} u * [(v \circ w) * a] &= \rho(T(u))\rho\{T[\rho(Tv))w + (-1)^{|w|\,|v|}\rho(T(w))v]\}a \\ &= \rho(T(u))\rho[T(v) \circ T(w)]a \\ &= \rho(x)\rho(y \circ z)a, \end{aligned}$$

$$\begin{aligned} (u \circ v) * (w * a) &= [\rho(T(u))v + (-1)^{|u|\,|v|}\rho(T(v))u] * (\rho(T(w))a) \\ &= \rho\{T[\rho(T(u))v + (-1)^{|u|\,|v|}\rho(T(v))u]\}\rho(T(w))a \\ &= \rho[T(u) \circ T(v)]\rho(T(w))a \\ &= \rho(x \circ y)\rho(z)a. \end{aligned}$$

Then

$$\begin{aligned} &u * [v * (w * a)] + (-1)^{|u|(|w|+|v|)+|w|\,|v|} w * [v * (u * a)] \\ &\quad + (-1)^{|w|\,|v|}[(u \circ w) \circ v] * a \end{aligned}$$

$$
\begin{aligned}
&= \rho(x)\rho(y)\rho(z)a + (-1)^{|x|(|y|+|z|)+|y|\,|z|}\rho(z)\rho(y)\rho(x)a + (-1)^{|z|\,|y|}\rho((x\circ z)\circ y)a \\
&= (-1)^{|z|(|x|+|y|)}\rho(z)\rho(x\circ y)a + \rho(x)\rho(y\circ z)a + (1)^{|x|(|y|+|z|)}\rho(y)\rho(z\circ x)a \\
&= (-1)^{|w|(|u|+|v|)} w * [(u\circ v) * a] + u * [(v\circ w) * a] \\
&\quad + (-1)^{|u|(|v|+|w|)} v * [(w\circ u) * a],
\end{aligned}
$$

$$
\begin{aligned}
&(u\circ v) * (w * a) + (-1)^{(|v|+|w|)|u|}(v\circ w) * (u * a) \\
&\quad + (-1)^{|w|(|u|+|v|)}(w\circ u) * (v * a) \\
&= (-1)^{|z|(|x|+|y|)}\rho(x\circ y)\rho(z)a + (-1)^{|x|(|y|+|z|)}\rho(y\circ z)\rho(x)a \\
&\quad + (-1)^{|z|(|x|+|y|)}\rho(z\circ x)\rho(y)a \\
&= (-1)^{|z|(|x|+|y|)}\rho(z)\rho(x\circ y)a + \rho(x)\rho(y\circ z)a + (-1)^{|x|(|z|+|y|)}\rho(y)\rho(z\circ x)a \\
&= (-1)^{|w|(|u|+|v|)} w * (u\circ v) * a + u * [(v\circ u) * a] \\
&\quad + (-1)^{|u|(|v|+|w|)} v * [(w\circ u) * a].
\end{aligned}
$$

Thus, $(V, *)$ is a pre-Jordan superalgebra. □

Therefore, there exists a Jordan superaglebra structure on V given by (12) and T is a homomorphism of Jordan superalgebras. Furthermore, there is an induced pre-Jordan superalgebra structure on $T(V)$ given by

$$T(u)\cdot T(v) = T(u * v), \quad \forall\ u, v \in V. \tag{14}$$

Moreover, the corresponding associated Jordan superalgebra structure on $T(V)$ given by (12) is just a Jordan supersubalgebra structure of J and T becomes a homomorphism of pre-Jordan superalgebra.

Corollary 2 *Let $(J, \circ)$ be a Jordan superalgebra, and let R be a super Rota-Baxter operator. Then there is a pre-Jordan superalgebra given by*

$$x\cdot y = R(x)\circ y, \quad \forall\ x, y \in J.$$

Corollary 3 *Let $(J, \circ)$ be a Jordan superalgebra. Then there is a compatible pre-Jordan superalgebra structure on J if and only if there is an invertible super $\mathscr{O}$-operator of J.*

Proof Let T be an invertible super $\mathscr{O}$-operator of J associated to a representation (ρ, V). From (13) and (14), letting

$$x = T(u), \quad y = T(v),$$

we can get a pre-Jordan superalgebra structure on J defined by

$$x\cdot y = T(\rho(x)T^{-1}(y)), \quad \forall\ x, y \in J.$$

Conversely, let $(J, \cdot)$ be a pre-Jordan superalgebra, and let $(J, \circ)$ be the associated Jordan superalgebra. Then the identity map $id\colon J \to J$ is a super $\mathscr{O}$-operator of $(J, \circ)$ associated to the representation (L, J). □

Corollary 4 *Let $(A, \cdot)$ be a pre-Jordan superalgebra. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)$$

is a skew-supersymmetric solution of super Jordan Yang-Baxter equation in the Jordan superalgebra $J(A) \ltimes_{L^*} J(A^*)$, where $\{e_1, \ldots, e_n\}$ is a basis of A and $\{e_1^*, \ldots, e_n^*\}$ is the dual basis.

Proof Since id is an $\mathscr{O}$-operator of the associated Jordan superalgebra $(J(A), \circ)$ associated to the representation (L, A), the conclusion follows from Theorem 2. □

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