Front. Math. China 2012, 7(5): 919–932
DOI 10.1007/s11464-012-0227-0

RESEARCH ARTICLE

Precise large deviations for widely orthant dependent random variables with dominatedly varying tails

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Abstract For the widely orthant dependent (WOD) structure, this paper mainly investigates the precise large deviations for the partial sums of WOD and non-identically distributed random variables with dominatedly varying tails. The obtained results extend some corresponding results.

Keywords Precise large deviations, widely orthant dependent (WOD), dominatedly varying tails
MSC 60F10, 62E20

1 Introduction

Let X_i $(i \geqslant 1)$ and X be real-valued random variables (r.v.s) with distributions F_i $(i \geqslant 1)$ and F, and finite mean μ_i $(i \geqslant 1)$ and μ, respectively. The partial sums are denoted by

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geqslant 1.$$

This paper will investigate the precise large deviations for these partial sums S_n, $n \geqslant 1$, with heavy-tailed and widely dependent increments. The main results will be given after some heavy-tailed distribution classes and wide dependence structures are introduced.

We first give some notions and notation. For a proper distribution V on $(-\infty, \infty)$, let $\overline{V} = 1 - V$ be its tail. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 1$; write $a(x) \lesssim b(x)$ if

Received July 22, 2010; accepted June 13, 2012
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$\limsup_{x\to\infty} a(x)/b(x) \leqslant 1$; write $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \geqslant 1$; write $a(x) = o(b(x))$ if $\lim_{x\to\infty} a(x)/b(x) = 0$; and write $a(x) = O(b(x))$ if $\limsup_{x\to\infty} a(x)/b(x) < \infty$. Furthermore, for two positive bivariate functions $a(t,x)$ and $b(t,x)$, we write $a(t,x) \sim b(t,x)$ uniformly for all x from some nonempty set Δ as $t \to \infty$, if

$$\lim_{t\to\infty} \sup_{x\in\Delta} \left| \frac{a(t,x)}{b(t,x)} - 1 \right| = 0;$$

write $a(t,x) \lesssim b(t,x)$ uniformly for all $x \in \Delta$ as $t \to \infty$, if

$$\limsup_{t\to\infty} \sup_{x\in\Delta} \frac{a(t,x)}{b(t,x)} \leqslant 1;$$

and write $a(t,x) \gtrsim b(t,x)$ uniformly for all $x \in \Delta$ as $t \to \infty$, if

$$\liminf_{t\to\infty} \inf_{x\in\Delta} \frac{a(t,x)}{b(t,x)} \geqslant 1.$$

The indicator function of a set A is denoted by $\mathbf{1}_A$, and for some real number a, let $a^- = -\min\{a, 0\}$.

1.1 Heavy-tailed distribution classes

In this subsection, we will introduce some subclasses of heavy-tailed distribution classes. An r.v. ξ (or its corresponding distribution V) is called heavy-tailed if for all $\beta > 0$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\beta x} V(\mathrm{d}x) = \infty,$$

otherwise, we say that the r.v. ξ (or V) is light-tailed. One of the heavy-tailed subclasses is the class $\mathscr{D}$ with dominatedly varying tails. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{D}$, if for any $y \in (0,1)$,

$$\overline{V}(xy) = O(\overline{V}(x)), \quad x \to \infty.$$

A smaller class is the class $\mathscr{C}$ with consistently varying tails. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{C}$, if

$$\lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1,$$

or equivalently,

$$\lim_{y\searrow 1} \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

Another important subclass of the heavy-tailed distribution class is the class $\mathscr{L}$ with long tails. Say that a distribution V on $(-\infty, \infty)$ belongs to the class $\mathscr{L}$, if for any $y > 0$,

$$\overline{V}(x+y) \sim \overline{V}(x), \quad x \to \infty.$$

It is well known that these distribution classes have the following relationships:

$$\mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L}$$

(see, e.g., [5,8,9]).

For a distribution V, denote its upper Matuszewska index by

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y},$$

where

$$\overline{V}_*(y) := \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, \quad y > 1,$$

and let

$$L_V = \lim_{y\searrow 1} \overline{V}_*(y).$$

From [1, Chapter 2.1], we know that the following assertions are equivalent:

(i) $V \in \mathscr{D}$;

(ii) $0 < L_V \leqslant 1$;

(iii) $J_V^+ < \infty$.

From the definition of the class $\mathscr{C}$, it holds that $V \in \mathscr{C}$ if and only if $L_V = 1$.

1.2 Wide dependence structures

In this paper, we will mainly discuss the case that the increments $\{X_i,\, i \geqslant 1\}$ of the partial sums S_n, $n \geqslant 1$, have a wide dependence structure and may not be independent. The wide dependence structures are introduced by Wang et al. [23].

Definition 1 For the r.v.s $\{\xi_n,\, n \geqslant 1\}$, if there exists a finite real sequence $\{g_U(n),\, n \geqslant 1\}$ satisfying for each integer $n \geqslant 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P\biggl(\bigcap_{i=1}^{n}\{\xi_i > x_i\}\biggr) \leqslant g_U(n) \prod_{i=1}^{n} P(\xi_i > x_i), \tag{1}$$

then we say that the r.v.s $\{\xi_n,\, n \geqslant 1\}$ are widely upper orthant dependent (WUOD) with dominating coefficients $g_U(n)$, $n \geqslant 1$; if there exists a finite real sequence $\{g_L(n),\, n \geqslant 1\}$ satisfying for each integer $n \geqslant 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leqslant i \leqslant n$,

$$P\biggl(\bigcap_{i=1}^{n}\{\xi_i \leqslant x_i\}\biggr) \leqslant g_L(n) \prod_{i=1}^{n} P(\xi_i \leqslant x_i), \tag{2}$$

then we say that the r.v.s $\{\xi_n,\, n \geqslant 1\}$ are widely lower orthant dependent (WLOD) with dominating coefficients $g_L(n)$, $n \geqslant 1$; if they are both WUOD and WLOD, then we say that the r.v.s $\{\xi_n,\, n \geqslant 1\}$ are widely orthant dependent (WOD). The WUOD, WLOD, and WOD r.v.s are called by a joint name wide dependent (WD) r.v.s.

Definition 1 shows that the wide dependence structures contain some common negative dependence structures. Indeed, if

$$g_L(n) = g_U(n) \equiv 1$$

for any integer $n \geqslant 1$ in (1) and (2), then the r.v.s $\{\xi_n,\, n \geqslant 1\}$ are called negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD), respectively. The r.v.s $\{\xi_n,\, n \geqslant 1\}$ are called negatively orthant dependent (NOD) if $\{\xi_n,\, n \geqslant 1\}$ are both NUOD and NLOD (see, e.g. [2,7]). If there exists a positive constant M such that for all integer $n \geqslant 1$, both (1) and (2) hold with

$$g_L(n) = g_U(n) \equiv M,$$

then the r.v.s $\{\xi_n,\, n \geqslant 1\}$ are called extendedly negatively orthant dependent (ENOD) (see, e.g. [3,4,14]). Wang et al. [23] also gave some examples to show that the WUOD and WLOD structures can contain some positively dependent r.v.s.

1.3 Motivation and main results

For the precise large deviations of the partial sums S_n, $n \geqslant 1$, when $\{X_i,\, i \geqslant 1\}$ are independent and identically distributed (i.i.d.) r.v.s, some earlier work can be found in [6,10–12,16–19], among others. The recent result is [20], which investigated the case that $\{X_i,\, i \geqslant 1\}$ are i.i.d. nonnegative r.v.s and obtained the following result.

Theorem A *Let $\{X_i,\, i \geqslant 1\}$ be a sequence of i.i.d. nonnegative r.v.s with common distribution $F_1 \in \mathscr{C}$ and finite mean μ_1. Then for any $\gamma > 0$,*

$$\lim_{n\to\infty} \sup_{x\geqslant \gamma n} \left| \frac{P(S_n - n\mu_1 > x)}{n\overline{F_1}(x)} - 1 \right| = 0. \tag{3}$$

Now, many studies of precise large deviations are devoted to the dependent r.v.s. Wang et al. [24] and Liu [15] considered the nonnegative negatively associated (NA) (for the definition, see [13]) r.v.s. Tang [21] considered a weaker dependence structure than the NA dependence structure and investigated the precise large deviations for real-valued r.v.s.

Theorem B *Let $\{X_i,\, i \geqslant 1\}$ be a sequence of NOD identically distributed real-valued r.v.s with common distribution $F_1 \in \mathscr{C}$ and finite mean $\mu_1 = 0$ satisfying*

$$xF_1(-x) = o(\overline{F_1}(x)), \quad x \to \infty.$$

If there exists some $r > 1$ such that $E(X_1^-)^r < \infty$, then for any $\gamma > 0$, relation (3) holds.

On the basis of [21], Liu [14] discussed a more general case that $\{X_i,\, i \geqslant 1\}$ are real-valued ENOD and nonidentically distributed r.v.s with consistently varying tails. Yang and Wang [25] extended Liu's results to the dominatedly-varying-tailed distribution class by adding the following two assumptions.

Assumption 1 For some $T > 0$,

$$0 < \liminf_{n\to\infty} \inf_{x\geqslant T} \frac{\sum_{i=1}^{n} \overline{F_i}(x)}{n\overline{F}(x)} \leqslant \limsup_{n\to\infty} \sup_{x\geqslant T} \frac{\sum_{i=1}^{n} \overline{F_i}(x)}{n\overline{F}(x)} < \infty,$$

$$0 < \liminf_{n\to\infty} \inf_{x\geqslant T} \frac{\sum_{i=1}^{n} F_i(-x)}{nF(-x)} \leqslant \limsup_{n\to\infty} \sup_{x\geqslant T} \frac{\sum_{i=1}^{n} F_i(-x)}{nF(-x)} < \infty.$$

Assumption 2 For all $i \geqslant 1$, $F_i \in \mathscr{D}$. Furthermore, assume that for any $\varepsilon > 0$, there exist some $w_1 = w_1(\varepsilon) > 1$ and $x_1 = x_1(\varepsilon) > 0$, irrespective of i, such that for all $i \geqslant 1$, $1 \leqslant w \leqslant w_1$, and $x \geqslant x_1$,

$$\frac{\overline{F_i}(wx)}{\overline{F_i}(x)} \geqslant L_{F_i} - \varepsilon,$$

or, equivalently, for any $\varepsilon > 0$, there exist some $0 < w_2 = w_2(\varepsilon) < 1$ and $x_2 = x_2(\varepsilon) > 0$, irrespective of i, such that for all $i \geqslant 1$, $w_2 \leqslant w \leqslant 1$, and $x \geqslant x_2$,

$$\frac{\overline{F_i}(wx)}{\overline{F_i}(x)} \leqslant L_{F_i}^{-1} + \varepsilon.$$

Theorem C *Let $\{X_i,\, i \geqslant 1\}$ be a sequence of ENOD real-valued r.v.s with finite mean $\mu_i = 0,\ i \geqslant 1$. If Assumptions 1 and 2 hold, then for any $\gamma > 0$,*

$$\limsup_{x\to\infty} \sup_{x\geqslant \gamma n} \frac{P(S_n > x)}{\sum_{i=1}^{n} L_{F_i}^{-1}\overline{F_i}(x)} \leqslant 1. \tag{4}$$

Furthermore, if

$$F(-x) = o(\overline{F}(x)), \quad x \to \infty, \tag{5}$$

and there exists some $r > 1$ such that $E(X_i^-)^r < \infty,\ i \geqslant 1$, and $E(X^-)^r < \infty$, then for any $\gamma > 0$,

$$\liminf_{x\to\infty} \inf_{x\geqslant \gamma n} \frac{P(S_n > x)}{\sum_{i=1}^{n} L_{F_i}\overline{F_i}(x)} \geqslant 1. \tag{6}$$

In this paper, we still investigate the precise large deviations of the partial sums S_n, $n \geqslant 1$, and consider the case that $\{X_i,\, i \geqslant 1\}$ are real-valued and nonidentically distributed r.v.s with dominatedly varying tails, but have a wider dependence structure (i.e., wide dependence structure) than the above results. Under Assumptions 1 and 2, the upper bound of the precise large deviations of the partial sums S_n, $n \geqslant 1$, can be obtained.

Theorem 1 *Let* $\{X_i,\ i\geqslant 1\}$ *be a sequence of real-valued r.v.s with* $\mu_i=0$ $(i\geqslant 1)$ *and satisfying Assumptions* 1 *and* 2. *If* $\{X_i,\ i\geqslant 1\}$ *are WUOD r.v.s with dominating coefficients* $g_U(n)$ $(n\geqslant 1)$ *satisfying for any fixed* $0<\alpha<1$,

$$\lim_{n\to\infty} g_U(n)(n\overline{F}(n))^{\alpha}=0, \tag{7}$$

then relation (4) *holds.*

In order to obtain the lower bound of the precise large deviations of the partial sums S_n, $n\geqslant 1$, we will use a stronger assumption than Assumption 2.

Assumption 3 For all $i\geqslant 1$, $F_i\in\mathscr{D}$. Furthermore, assume that for any $\delta\in(0,1)$, there exist some $v_1=v_1(\delta)>1$ and $x_1=x_1(\delta)>0$, irrespective of i, such that for all $i\geqslant 1$, $1\leqslant v\leqslant v_1$, and $x\geqslant x_1$,

$$\frac{\overline{F_i}(vx)}{\overline{F_i}(x)}\geqslant \delta L_{F_i},$$

or, equivalently, for any $\delta>1$, there exist some $0<v_2=v_2(\delta)<1$ and $x_2=x_2(\delta)>0$, irrespective of i, such that for all $i\geqslant 1$, $v_2\leqslant v\leqslant 1$, and $x\geqslant x_2$,

$$\frac{\overline{F_i}(vx)}{\overline{F_i}(x)}\leqslant \delta L_{F_i}^{-1}.$$

Remark 1 (i) It is noted that, generally, Assumption 3 is stronger than Assumption 2. In fact, for any $\varepsilon\in(0,1)$, taking some fixed $\delta\in(1-\varepsilon,1)$, since $0<L_{F_i}\leqslant 1$, by Assumption 3, there exist some $v_1=v_1(\delta)>1$ and $x_1=x_1(\delta)>0$, irrespective of i, such that for all $i\geqslant 1$, $1\leqslant v\leqslant v_1$, and $x\geqslant x_1$,

$$\frac{\overline{F_i}(vx)}{\overline{F_i}(x)}\geqslant \delta L_{F_i}>(1-\varepsilon)L_{F_i}\geqslant L_{F_i}-\varepsilon.$$

This shows that Assumption 2 holds.

However, in some particular case, for example, if there exists a positive constant a, such that for any $i\geqslant 1$, $L_{F_i}\geqslant a$, then Assumption 2 can imply Assumption 3. Indeed, for any $\delta\in(0,1)$, taking some fixed $\varepsilon\in(0,(1-\delta)a)$, by Assumption 2, there exist some $w_1=w_1(\varepsilon)>1$ and $x_1=x_1(\varepsilon)>0$, irrespective of i, such that for all $i\geqslant 1$, $1\leqslant w\leqslant w_1$, and $x\geqslant x_1$,

$$\frac{\overline{F_i}(wx)}{\overline{F_i}(x)}\geqslant L_{F_i}-\varepsilon>L_{F_i}-(1-\delta)a\geqslant \delta L_{F_i},$$

which is Assumption 3.

(ii) Assumptions 2 and 3 actually require the distributions of X_i, $i\geqslant 1$, do not differ too much from each other. Especially, if there exists a positive integer i_0 such that for all $i\geqslant i_0$, $F_i=F_{i_0}$, then by $F_i\in\mathscr{D}$, we know that Assumptions 2 and 3 are satisfied.

Under Assumptions 1 and 3, the lower bound of the precise large deviations of the partial sums S_n, $n \geqslant 1$, can be given.

Theorem 2 *Let* $\{X_i,\, i \geqslant 1\}$ *be a sequence of real-valued r.v.s with* $\mu_i = 0$ $(i \geqslant 1)$ *and satisfying Assumptions* 1 *and* 3*. Suppose that* $E(X_i^-)^r < \infty$, $i \geqslant 1$, $E(X^-)^r < \infty$ *for some* $r > 1$*, and relation* (5) *holds. If* $\{X_i,\, i \geqslant 1\}$ *are WOD r.v.s with dominating coefficients* $g_U(n)$ $(n \geqslant 1)$ *and* $g_L(n)$ $(n \geqslant 1)$ *satisfying* (7) *and for any* $\alpha \in (0,1)$,

$$\lim_{n\to\infty} g_L(n) n^{-\alpha} = 0. \tag{8}$$

Then relation (6) *holds.*

If $\{X_i,\, i \geqslant 1\}$ and X are identically distributed r.v.s with common distribution $F \in \mathscr{D}$, then Assumptions 1–3 are satisfied. Hence, from Theorems 1 and 2, the following two corollaries can be obtained.

Corollary 1 *Let* $\{X_i,\, i \geqslant 1,\, X\}$ *be identically distributed real-valued r.v.s with common distribution* $F \in \mathscr{D}$ *and finite mean* $\mu_1 = 0$*. If* $\{X_i,\, i \geqslant 1\}$ *are WUOD r.v.s with dominating coefficients* $g_U(n)$ $(n \geqslant 1)$ *satisfying* (7)*, then for any* $\gamma > 0$,

$$\limsup_{x\to\infty} \sup_{x\geqslant\gamma n} \frac{P(S_n > x)}{n\overline{F}(x)} \leqslant L_F^{-1}.$$

Corollary 2 *Let* $\{X_i,\, i \geqslant 1,\, X\}$ *be identically distributed real-valued r.v.s with common distribution* $F \in \mathscr{D}$ *and finite mean* $\mu_1 = 0$*. Suppose that* $E(X_1^-)^r < \infty$ *for some* $r > 1$ *and relation* (5) *holds. If* $\{X_i,\, i \geqslant 1\}$ *are WOD r.v.s with dominating coefficients* $g_U(n)$ $(n \geqslant 1)$ *and* $g_L(n)$ $(n \geqslant 1)$ *satisfying* (7) *and* (8)*, then for any* $\gamma > 0$,

$$\liminf_{x\to\infty} \inf_{x\geqslant\gamma n} \frac{P(S_n > x)}{n\overline{F}(x)} \geqslant L_F.$$

Remark 2 (i) When $\{X_i,\, i \geqslant 1\}$ are i.i.d. nonnegative r.v.s with common distribution $F_1 \in \mathscr{C}$, let $Y_i = X_i - \mu_1,\ i \geqslant 1$. Then $EY_i = 0,\ i \geqslant 1$, and

$$P(S_n - n\mu_1 > x) = P\biggl(\sum_{i=1}^n Y_i > x\biggr).$$

Let G be the distribution of $Y_i,\ i \geqslant 1$. Since $F_1 \in \mathscr{C} \subset \mathscr{L}$, it knows that $\overline{G}(x) \sim \overline{F_1}(x)$ as $x \to \infty$. From this, we get $G \in \mathscr{C} \subset \mathscr{D}$ and $L_G = 1$. Hence, $\{Y_i,\, i \geqslant 1\}$ satisfy the conditions of Corollaries 1 and 2, and Theorem A can be obtained from Corollaries 1 and 2.

(ii) For Theorem B, since $F_1 \in \mathscr{C} \subset \mathscr{D}$, the NOD structure is stronger than the WOD structure and satisfies (7) and (8), the conditions of Corollaries 1 and 2 are satisfied. Therefore, Corollaries 1 and 2 extend Theorem B.

(iii) Since the ENOD structure is stronger than the WUOD structure, Theorem 1 extends result (4) in Theorem C.

2 Proofs of main results

Before proving Theorems 1 and 2, we first give some lemmas. The following lemma is similar to [14, Lemma 3.3]. We omit the proof here.

Lemma 1 *Assume that* $E(X_i^-)^q < \infty$ $(i \geqslant 1)$ *and* $E(X^-)^q < \infty$ *for some* $q \geqslant 1$. *If Assumption 1 holds, then there exists some finite constant* $\hat{\mu}_q^-$ *such that for all integer* $n \geqslant 1$,

$$\sum_{i=1}^{n} E(X_i^-)^q \leqslant n\hat{\mu}_q^-.$$

The following lemma can be obtained by [22, Lemma 3.5].

Lemma 2 *If* $V \in \mathscr{D}$, *then it holds for any* $p > J_V^+$ *that* $\lim_{x\to\infty} x^{-p}/\overline{V}(x) = 0$.

Wang et al. [23] obtained the following properties for the WUOD and WLOD r.v.s, which also can be proved by the argument of the proof of [3, Lemma 2.2].

Lemma 3 (i) *Let* $\{\xi_n, n \geqslant 1\}$ *be WLOD* (*resp. WUOD*) *with dominating coefficients* $g_L(n)$ $(n \geqslant 1)$ (*resp.* $g_U(n)$ $(n \geqslant 1)$). *If* $\{f_n(\cdot), n \geqslant 1\}$ *are non-decreasing, then* $\{f_n(\xi_n), n \geqslant 1\}$ *are still WLOD* (*resp. WUOD*) *with dominating coefficients* $g_L(n)$ $(n \geqslant 1)$ (*resp.* $g_U(n)$ $(n \geqslant 1)$); *If* $\{f_n(\cdot), n \geqslant 1\}$ *are nonincreasing, then* $\{f_n(\xi_n), n \geqslant 1\}$ *are WUOD* (*resp. WLOD*) *with dominating coefficients* $g_L(n)$ $(n \geqslant 1)$ (*resp.* $g_U(n)$ $(n \geqslant 1)$).

(ii) *If* $\{\xi_n, n \geqslant 1\}$ *are nonnegative and WUOD with dominating coefficients* $g_U(n)$ $(n \geqslant 1)$, *then for each* $n \geqslant 1$,

$$E\prod_{i=1}^{n}\xi_i \leqslant g_U(n)\prod_{i=1}^{n}E\xi_i.$$

In particular, if $\{\xi_n, n \geqslant 1\}$ *are WUOD with dominating coefficients* $g_U(n)$ $(n \geqslant 1)$, *then for each* $n \geqslant 1$ *and any* $s > 0$,

$$E\exp\left\{s\sum_{i=1}^{n}\xi_i\right\} \leqslant g_U(n)\prod_{i=1}^{n}E\exp\{s\xi_i\}.$$

Proof of Theorem 1 In the sequel, C always represents some finite and positive constant whose value may vary in different places.

Since $F_i \in \mathscr{D}$, $i \geqslant 1$, by Assumption 1, we have $F \in \mathscr{D}$. Relation (4) will be shown by using the line of [14], whose idea is from [21]. For any fixed positive integer m and any fixed $v \in (0, m/(m+1))$, we set

$$\widetilde{X}_i = \min\{X_i, vx\}, \quad i \geqslant 1.$$

By Lemma 3, $\{\widetilde{X}_i,\, i \geqslant 1\}$ are still WUOD. Let

$$\widetilde{S}_n = \sum_{i=1}^{n} \widetilde{X}_i, \quad n \geqslant 1.$$

A standard argument shows that

$$P(S_n > x) \leqslant \sum_{i=1}^{n} \overline{F_i}(vx) + P(\widetilde{S}_n > x). \tag{9}$$

By Assumption 1, there exists a constant $C > 0$ such that for all large n and all $x \geqslant \gamma n$,

$$\frac{P(\widetilde{S}_n > x)}{\sum_{i=1}^{n} \overline{F_i}(vx)} \leqslant \frac{P(\widetilde{S}_n > x)}{Cn\overline{F}(vx)}.$$

Write

$$a = \max\{-m^{-1} \log(n\overline{F}(vx)), 1\}.$$

Then

$$\liminf_{n\to\infty} \inf_{x\geqslant\gamma n} a = \infty.$$

Thus, for any fixed $h > 0$, by Markov's inequality and Lemma 3, when n is sufficiently large, we have

$$\begin{aligned}
\frac{P(\widetilde{S}_n > x)}{n\overline{F}(vx)} &\leqslant g_U(n)\mathrm{e}^{-hx+ma} \prod_{i=1}^{n} E\mathrm{e}^{h\widetilde{X}_i} \\
&= g_U(n)\mathrm{e}^{-hx+ma} \prod_{i=1}^{n} \left\{ \int_{-\infty}^{vx} (\mathrm{e}^{hy} - 1)F_i(\mathrm{d}y) + (\mathrm{e}^{hvx} - 1)\overline{F_i}(vx) + 1 \right\} \\
&\leqslant g_U(n) \exp\Bigg\{ - hx + ma \\
&\quad + \sum_{i=1}^{n} \int_{-\infty}^{vx} (\mathrm{e}^{hy} - 1)F_i(\mathrm{d}y) + (\mathrm{e}^{hvx} - 1) \sum_{i=1}^{n} \overline{F_i}(vx) \Bigg\}.
\end{aligned} \tag{10}$$

For any fixed $k > 1$ and some $\rho > J_F^+$, let

$$h = \frac{ma - k\rho \log a}{vx}.$$

Then

$$\lim_{n\to\infty} \sup_{x\geqslant\gamma n} h = 0.$$

By the proof of [14, Lemma 3.6], for all large n and all $x \geqslant \gamma n$, it holds that

$$\begin{aligned}
&\exp\left\{ - hx + ma + \sum_{i=1}^{n} \int_{-\infty}^{vx} (\mathrm{e}^{hy} - 1)F_i(\mathrm{d}y) \right\} \\
&\quad \leqslant \exp\{-hx + ma + (\mathrm{e}^{hvx/a^k} - 1)\widehat{\mu}_1^- nh + o(nh) + C\} \\
&\quad = \exp\{-hx + ma + o(nh) + C\} \\
&\quad = \exp\{m(1 - v^{-1})a + o(a)\}.
\end{aligned} \tag{11}$$

By Assumption 1, for all large n and all $x \geqslant \gamma n$, we have

$$
\begin{aligned}
(\mathrm{e}^{hvx}-1)\sum_{i=1}^{n}\overline{F_i}(vx) &\leqslant C(\mathrm{e}^{hvx}-1)n\overline{F}(vx)\\
&= C(a^{-k\rho}-\mathrm{e}^{-ma})\\
&= o(1). \qquad (12)
\end{aligned}
$$

Thus, by (10)–(12), for all large n and all $x \geqslant \gamma n$, we have

$$
\begin{aligned}
\frac{P(\widetilde{S}_n > x)}{n\overline{F}(vx)} &\leqslant g_U(n)\mathrm{e}^{-\alpha}\exp\{m(1-v^{-1})a+o(a)+o(1)+a\}\\
&= g_U(n)(n\overline{F}(vx))^{1/m}\exp\{m(1-v^{-1})a+o(a)+o(1)+a\}.
\end{aligned}
$$

By (7) and $F \in \mathscr{D}$, we get

$$
\begin{aligned}
\limsup_{n\to\infty}\sup_{x\geqslant\gamma n} g_U(n)(n\overline{F}(vx))^{1/m} &\leqslant \limsup_{n\to\infty} g_U(n)(n\overline{F}(n))^{1/m}\Big(\frac{\overline{F}(v\gamma n)}{\overline{F}(n)}\Big)^{1/m}\\
&= 0.
\end{aligned}
$$

Thus,

$$
\limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{P(\widetilde{S}_n > x)}{n\overline{F}(vx)} = 0,
$$

which, together with (9), yields that

$$
\limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{P(S_n > x)}{\sum_{i=1}^{n}\overline{F_i}(vx)} \leqslant 1. \qquad (13)
$$

From Assumption 2, for any $\varepsilon > 0$, take positive integer m such that $m/(m+1) > w_2$. Then for all $v \in (w_2, m/(m+1))$, $x \geqslant \gamma n$, and $n \geqslant \gamma^{-1}x_2$, we have

$$
\sum_{i=1}^{n}\overline{F_i}(vx) \leqslant \sum_{i=1}^{n}(L_{F_i}^{-1}+\varepsilon)\overline{F_i}(x) \leqslant (1+\varepsilon)\sum_{i=1}^{n}L_{F_i}^{-1}\overline{F_i}(x),
$$

which, combining with (13) and letting $\varepsilon \downarrow 0$, yields that (4) holds. □

Proof of Theorem 2 We will prove (6) by the line of the proof of [14, Lemma 3.7]. For every $1 \leqslant i \leqslant n$ and any fixed $w > 1$, let

$$
A_i = \{X_i > wx, \max_{1\leqslant j\neq i\leqslant n} X_j \leqslant wx\},
$$

which are pairwise disjoint sets. Thus,

$$
P(S_n > x) \geqslant P\Big(S_n > x, \bigcup_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n} P(S_n \leqslant x, A_i) =: I_1 - I_2. \qquad (14)
$$

Since $\{X_i,\, i \geqslant 1\}$ are WUOD, we have

$$I_1 \geqslant \sum_{i=1}^{n} \overline{F_i}(wx) - g_U(n)\biggl(\sum_{i=1}^{n} \overline{F_i}(wx)\biggr)^2.$$

It follows from (7), $F \in \mathscr{D}$, and Assumption 1, that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{x\geqslant\gamma n} g_U(n) \sum_{i=1}^{n} \overline{F_i}(wx) \\
&\quad\leqslant \limsup_{n\to\infty} g_U(n) n\overline{F}(n) \limsup_{n\to\infty} \frac{\overline{F}(w\gamma n)}{\overline{F}(n)} \limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{\sum_{i=1}^{n} \overline{F_i}(wx)}{n\overline{F}(wx)} \\
&\quad= 0.
\end{aligned}$$

Thus,

$$\liminf_{n\to\infty} \inf_{x\geqslant\gamma n} \frac{I_1}{\sum_{i=1}^{n} \overline{F_i}(wx)} \geqslant 1. \tag{15}$$

Now, we estimate I_2. For any fixed $u \in (0,1)$, we write

$$Y_i = -X_i, \quad \widetilde{Y}_i = \min\{Y_i, ux\}, \quad i \geqslant 1.$$

Then

$$\begin{aligned}
I_2 &\leqslant \sum_{i=1}^{n} P\biggl(\sum_{j\neq i} Y_j \geqslant (w-1)x, A_i\biggr) \\
&\leqslant \sum_{i=1}^{n} P\biggl(A_i, \bigcup_{j\neq i}\{Y_j > ux\}\biggr) + \sum_{i=1}^{n} P\biggl(\sum_{j\neq i} \widetilde{Y}_j \geqslant (w-1)x\biggr) \\
&\leqslant \sum_{j=1}^{n} P(Y_j > ux) + \sum_{i=1}^{n} P\biggl(\sum_{j\neq i} \widetilde{Y}_j \geqslant (w-1)x\biggr) \\
&=: I_{21} + I_{22}.
\end{aligned} \tag{16}$$

By Assumption 1, for all large n and all $x \geqslant \gamma n$, it holds that

$$I_{21} \leqslant \sum_{j=1}^{n} F_j(-ux) \leqslant CnF(-ux).$$

Therefore, by (5) and $F \in \mathscr{D}$, we can get

$$\limsup_{n\to\infty} \sup_{x\geqslant\gamma n} \frac{I_{21}}{n\overline{F}(x)} = 0. \tag{17}$$

For I_{22}, since $\{X_i,\ i\geqslant 1\}$ are WLOD, by Lemma 3, $\{\widetilde{Y}_i,\ i\leqslant 1\}$ are WUOD. For any $1\leqslant i\leqslant n$ and any fixed $h>0$, we obtain from Markov's inequality and Lemma 3 that

$$\begin{aligned}
&P\Big(\sum_{j\neq i}\widetilde{Y}_j\geqslant (w-1)x\Big)\\
&\quad\leqslant g_L(n)\mathrm{e}^{-h(w-1)x}\prod_{j\neq i}\Big\{\int_{-\infty}^{ux}(\mathrm{e}^{hy}-1)F_{Y_j}(\mathrm{d}y)+(\mathrm{e}^{hux}-1)\overline{F_{Y_j}}(ux)+1\Big\}\\
&\quad\leqslant g_L(n)\exp\Big\{-h(w-1)x\\
&\qquad+\sum_{j\neq i}\Big(\int_{-\infty}^{ux}(\mathrm{e}^{hy}-1)F_{Y_j}(\mathrm{d}y)+(\mathrm{e}^{hux}-1)F_j(-ux)\Big)\Big\}. \qquad (18)
\end{aligned}$$

Take

$$h=\frac{1}{ux}\log\Big(\frac{u^{q-1}x^q}{n\hat{\mu}_q^-}+1\Big).$$

Then

$$\lim_{n\to\infty}\sup_{x\geqslant\gamma n}h=0.$$

By the proof of [14, Lemma 3.7], for any fixed $q\in(1,\min\{r,2\})$, when n is sufficiently large, for all $x\geqslant\gamma n$, it holds that

$$\begin{aligned}
&\exp\Big\{-h(w-1)x+\sum_{j\neq i}\int_{-\infty}^{ux}(\mathrm{e}^{hy}-1)F_{Y_j}(\mathrm{d}y)\Big\}\\
&\quad\leqslant \mathrm{e}^{u^{-1}}\Big(\frac{u^{q-1}x^q}{n\hat{\mu}_q^-}\Big)^{-(w-1)/(2u)}\\
&\quad\leqslant \mathrm{e}^{u^{-1}}\Big(\frac{\gamma u^{q-1}}{\hat{\mu}_q^-}\Big)^{-(w-1)/(2u)}x^{-(q-1)(w-1)/(2u)}\\
&\quad= C_1x^{-(q-1)(w-1)/(2u)}, \qquad (19)
\end{aligned}$$

where

$$C_1=\mathrm{e}^{u^{-1}}\Big(\frac{\gamma u^{q-1}}{\hat{\mu}_q^-}\Big)^{-(w-1)/(2u)}.$$

It follows from Assumption 1 and $E(X^-)^q<\infty$ that there exists a constant $C_2>0$ such that when n is sufficiently large, for all $x\geqslant\gamma n$, it holds that

$$\begin{aligned}
\exp\Big\{\sum_{j\neq i}(\mathrm{e}^{hux}-1)F_j(-ux)\Big\}&\leqslant \exp\{C(\mathrm{e}^{hux}-1)nF(-ux)\}\\
&=\exp\Big\{C\,\frac{u^{q-1}x^q}{\hat{\mu}_q^-}\,F(-ux)\Big\}\\
&\leqslant C_2,
\end{aligned}$$

which, together with (18) and (19), yields that for any fixed $\alpha > 0$, when n is sufficiently large, for all $x \geqslant \gamma n$, it holds that

$$P\left(\sum_{j\neq i}\widetilde{Y}_j \geqslant (w-1)x\right) \leqslant g_L(n)n^{-\alpha}C_1C_2\gamma^{-\alpha}x^{-\frac{(q-1)(w-1)}{2u}+\alpha}. \tag{20}$$

For any fixed $w > 1$, we take sufficiently small $u > 0$ such that

$$\frac{(q-1)(w-1)}{2u} - \alpha > J_F^+.$$

Thus, from (16), (17), (20), (8), and Lemma 2, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{I_2}{n\overline{F}(x)} &\leqslant C_1C_2\gamma^{-\alpha}\limsup_{n\to\infty}g_L(n)n^{-\alpha}\limsup_{x\to\infty}\frac{x^{-\frac{(q-1)(w-1)}{2u}+\alpha}}{\overline{F}(x)}\\ &= 0.\end{aligned}$$

Since $0 < L_{F_i} \leqslant 1,\ i \geqslant 1$, by Assumption 1, it holds that

$$\begin{aligned}\limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{I_2}{\sum_{i=1}^n L_{F_i}^{-1}\overline{F_i}(x)} &\leqslant \limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{I_2}{n\overline{F}(x)}\limsup_{n\to\infty}\sup_{x\geqslant\gamma n}\frac{n\overline{F}(x)}{\sum_{i=1}^n\overline{F_i}(x)}\\ &= 0,\end{aligned}$$

that is, for any $\varepsilon > 0$, when n is sufficiently large, for all $x \geqslant \gamma n$,

$$I_2 \leqslant \varepsilon\sum_{i=1}^n L_{F_i}^{-1}\overline{F_i}(x). \tag{21}$$

Again by Assumption 3, for any $\delta \in (0,1)$, there exist constants $v_1 > 1$ and $x_1 > 0$ such that for any $w \in (1, v_1)$, all $x \geqslant \gamma n$, and $n \geqslant \gamma^{-1}x_1$,

$$\sum_{i=1}^n\overline{F_i}(wx) \geqslant \delta\sum_{i=1}^n L_{F_i}\overline{F_i}(x),$$

which, combining with (14), (15), (21), and the arbitrariness of δ and ε, yields that (6) holds. □

Acknowledgements The authors wish to thank the referees and Yuebao Wang for their very valuable comments by which the presentation of the paper has been improved. This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11071182, 11001052, 11171065), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 10KJB110010), the Natural Science Foundation of Jiangsu Province (Nos. BK2012165, BK2011058), the Postdoctoral Research Program of Jiangsu Province (No. 0901029C), and the Research Foundation of SUST.

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