Front. Math. China 2012, 7(3): 587–595
DOI 10.1007/s11464-012-0211-8

RESEARCH ARTICLE

Existence of solutions for elliptic equations without superquadraticity condition

Yimin ZHANG[1], **Yaotian SHEN**[2]

1 Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China
2 Department of Mathematics, South China University of Technology, Guangzhou 510641, China

Abstract By weakening or dropping the superquadraticity condition (SQC), the existence of positive solutions for a class of elliptic equations is established. In particular, we deal with the asymptotical linearities as well as the superlinear nonlinearities.

Keywords Mountain pass, superquadraticity condition (SQC), Palais-Smale type condition, weakly superquadraticity condition (WSQC)
MSC 35J65

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary is a smooth manifold, $N \geqslant 2$. As a model problem, we consider the following problem:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f(x,s)$ is a continuous function on $\Omega \times \mathbb{R}$ with subcritical growth, that is,

$$|f(x,s)| \leqslant C_1 + C_2|s|^{p-1}, \quad \forall\, s \in \mathbb{R},\ \text{a.e. } x \in \Omega, \tag{1.2}$$

for some constants $C_1, C_2 > 0$, where

$$2 < p < 2^* = \frac{2N}{N-2}.$$

Received March 30, 2010; accepted March 22, 2012
Corresponding author: Yimin ZHANG, E-mail: ymin.zhang@mail.scut.edu.cn

It is well known that solutions of (1.1) are precisely the critical points of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \tag{1.3}$$

where

$$F(x,s) = \int_0^s f(x,t)\mathrm{d}t.$$

In this paper, about $f(x,s)$ or $F(x,s)$, we assume that

(f_1) $f(x,s) = 0$ for all $s \leqslant 0$, a.e. $x \in \Omega$, and $f(x,s) \geqslant 0$ for all $s > 0$, a.e. $x \in \Omega$;

(f_2) $2F(x,s)/s^2 \to l(x)$ as $s \to \infty$ uniformly for a.e. $x \in \Omega$;

(f_3) $\limsup_{s\to 0} f(x,s)/s = o(1)$ uniformly for a.e. $x \in \Omega$.

If $l(x) = \infty$, it is the superquadratic situation. Almost every author discussing superquadratic problems has made the assumption of superquadraticity condition (SQC) which was originally introduced in [1]. It is usually assumed that

$$0 < \mu F(x,s) \leqslant f(x,s)s, \quad \forall\ |s| \geqslant r,\ \text{uniformly a.e. } x \in \Omega, \tag{SQC}$$

where $\mu > 2$ and $r > 0$. In fact, SQC implies that

$$F(x,s) \geqslant C|s|^\mu, \quad \forall\ |s| \geqslant r,$$

for some $C > 0$ and $r > 0$. Therefore, some authors replaced the assumption of SQC with

$$sf(x,s) - 2F(x,s) \geqslant (\mu - 2)C|s|^\mu, \quad \forall\ |s| \geqslant r,\ \text{uniformly a.e. } x \in \Omega. \tag{1.4}$$

Although (1.4) is a more natural assumption, it is still too restrictive to be desirable. Can SQC be weakened? Recent studies have focused on this problem, for example, see, [3–11] and references therein.

Shen and Guo [10] proved the existence of solution for problem (1.1) with the following condition:

$$sf(x,s) - 2F(x,s) \geqslant C|s|^{\mu_0} - C_1, \quad \forall\ s \in \mathbb{R},\ \text{uniformly a.e. } x \in \Omega, \tag{1.5}$$

for some $\mu_0 > N(m-2)/2$, $C > 0$, and $C_1 \geqslant 0$, where $m \in (2 + \frac{2}{N}, 2^*)$ such that

$$|f(x,s)| \leqslant C + C|s|^{m-1}.$$

By using a weakened version of Palais-Smale type condition introduced by Cerami [2] (we called it the CPS condition), they proved that the Cerami sequence corresponding functional (1.3) is bounded, and then, they obtained the existence of nontrivial solution for problem (1.1). Costa and Magalhães [4] got the same result independently by using the methods similar to [10]. In fact, if $\mu_0 \leqslant \mu$, then (1.4) implies (1.5). Therefore, condition (1.5) allows much more freedom than SQC for function $f(x,s)$.

Jeanjean [6] replaced SQC with the following conditions:

$$\frac{f(x,s)}{s} \to \infty \quad (s \to \infty) \quad \text{uniformly for } x \in \mathbb{R}^N. \tag{1.6}$$

$$\exists\, p \in (2, 2^*),\ \lim_{s\to\infty} f(x,s)s^{1-p} = 0 \quad \text{uniformly for } x \in \mathbb{R}^N, \tag{1.7}$$

and

$$DH(x,s) \geqslant H(x,t), \quad 0 \leqslant t \leqslant s, \tag{1.8}$$

where $D \geqslant 1$ and function $H(x,s)$ is defined by

$$H(x,s) = sf(x,s) - 2F(x,s).$$

Using these conditions, they proved that the Palais-Smale (PS) sequence was bounded for a special mountain pass level and obtained the existence of a nontrivial positive solution for problem (1.1). In this paper, condition (1.6) is called the weakly superquadraticity condition (WSQC). Recently, replaced SQC with WSQC and (1.8), Liu and Li [8] proved the Cerimi sequence was bounded and obtained the existence of infinitely many solutions of problem (1.1) without condition (1.7). Chen et al. [3] obtained a similar results.

Similarly, Li and Zhou [7] replaced SQC with WSQC and

$$\frac{f(x,s)}{s} \text{ is nondecreasing for } s > 0,\ x \in \Omega, \tag{1.9}$$

and then, they obtained the existence of nontrivial solution for problem (1.1). Schechter and Zou [9] explored what happens when SQC was replaced by WSQC and

$$\mu F(x,s) - sf(x,s) \leqslant C(1+s^2), \quad |s| \geqslant r, \tag{1.10}$$

for some $\mu > 2$ and $r \geqslant 0$. Condition (1.10) allows much more freedom for the function $f(x,t)$, however, it still eliminates many superlinear problems. Hence, they weakened (1.10) to

$$H(x,s) \geqslant H(x,t), \quad t \leqslant s, \tag{1.11}$$

and obtained the existence of nontrivial solution of (1.1).

In condition (f_2), taking $l(x) < \infty$, it is easy to see that SQC cannot be supposed. In this case, Jeanjean [6] had considered problem (1.1) replacing SQC by (1.7) and the following condition:

$$H(x,s) \geqslant 0, \quad \forall\ s \geqslant 0,\ \text{a.e. } x \in \mathbb{R}^N,$$

and there is a $\delta > 0$ such that

$$f(x,s)s^{-1} \geqslant K - \delta \Longrightarrow H(x,s) \geqslant \delta. \tag{1.12}$$

Li and Zhou [7] studied the existence of positive solutions of problem (1.1) with condition (f_2) and (1.9). Huang and Zhou [5] dropped condition (1.9), and assumed the following condition:

$$\lim_{s\to 0}\frac{f(x,s)}{s}=a\geqslant 0,\quad \lim_{s\to\infty}\frac{f(x,s)}{s}=b>0, \tag{1.13}$$

where $a<\lambda_1<b$, λ_1 is the first eigenvalue of $-\Delta$. Then they proved that problem (1.1) has a nontrivial positive solution. Recently, Wang and Zhou [11] replaced (1.9) by (1.7) and some other conditions.

In this paper, if $l(x)<\infty$ in (f_2), we drop (1.9) completely. On the other hand, our condition "1 is not the eigenvalue of problem (1.14)" (see Theorem 1 below) is necessary. If $l(x)=\infty$, the SQC replaced only by WSQC. It is a rather surprising result. For all these reasons, we believe that our results both in the cases $l(x)<\infty$ and $l(x)=\infty$ that we treat in a unified way, strongly generalize the previous existence results. That is, we allow much more freedom for the function $f(x,s)$.

The main results of this paper are the following theorems.

Theorem 1 *Assume that* (1.2) *and* (f_1)–(f_3) *with* $l(x)<\infty$ *hold, and moreover, suppose that* 1 *is not the eigenvalue of problem*

$$\begin{cases}-\Delta u=\lambda l(x)u, & x\in\Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{1.14}$$

Then problem (1.1) *has at least one nontrivial solution.*

Theorem 2 *Assume that* (1.2) *and* (f_1)–(f_3) *with* $l(x)=\infty$ *hold. Then problem* (1.1) *has at least one nontrivial solution.*

2 Preliminaries and some basic properties of $f(x,s)$

In this paper, we denote the norms of u in $H_0^1(\Omega)$ and in $L^p(\Omega)$ by

$$\|u\|=\Big(\int_\Omega|\nabla u|^2\mathrm{d}x\Big)^{1/2},\quad |u|_p=\Big(\int_\Omega|u|^p\mathrm{d}x\Big)^{1/p},$$

respectively. u^+ and u^- denote the positive and negative parts of u, respectively.

Lemma 1 *Assume that* (1.2), (f_1), *and* (f_2) *with* $l(x)<\infty$ *hold. Then the functional* $I(u)$ *satisfies CPS condition* (*see* [2]) *if* 1 *is not the eigenvalue of problem*

$$\begin{cases}-\Delta u=\lambda l(x)u, & x\in\Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Proof By the compactness of Sobolev embedding, it suffices to show that $\{u_n\}\subset H_0^1(\Omega)$ is bounded. By contradiction, we assume $\|u_n\|\to\infty$ and $c\in\mathbb{R}$

such that $I(u_n) \to c$ and $I'(u_n) \to 0$, that is,

$$I(u_n) \to c, \quad \langle I'(u_n), \varphi \rangle \to 0, \quad \langle I'(u_n), u_n \rangle \to 0, \tag{2.1}$$

for all $\varphi \in H_0^1(\Omega)$. Set $v_n = u_n/\|u_n\|$ and let v be such that (up to a subsequence) $\{v_n\}$ converges weakly to v in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ almost everywhere.

Dividing (2.1) by $\|u_n\|$, we have

$$\int_\Omega \nabla v_n \nabla \varphi \mathrm{d}x - \int_\Omega \frac{f(x, u_n^+)}{u_n^+} v_n^+ \varphi \mathrm{d}x \to 0. \tag{2.2}$$

If $v(x) > 0$, it is easy to know $u_n(x) \to \infty$. Hence, from (f_2), using the Vitali theorem and Lebesgue dominated convergence theorem, as [10], it is easy to obtain that

$$\frac{f(x, u_n^+)}{u_n^+} v_n^+ \varphi \to l(x) v^+ \varphi \quad \text{in } L^1(\Omega) \tag{2.3}$$

for all $\varphi \in H_0^1(\Omega)$, and

$$\frac{F(x, u_n^+)}{(u_n^+)^2} (v_n^+)^2 \to \frac{1}{2} l(x)(v^+)^2 \quad \text{in } L^1(\Omega). \tag{2.4}$$

(2.4) together with (2.1) gives

$$\frac{1}{2} \int_\Omega |\nabla v_n|^2 \mathrm{d}x - \frac{1}{2} \int_\Omega l(x)(v^+)^2 \mathrm{d}x \to c. \tag{2.5}$$

From (2.2), (2.3), and (2.5), it is easy to obtain that $v_n \to v$ in $H_0^1(\Omega)$. Then we have

$$\int_\Omega \nabla v \nabla \varphi \mathrm{d}x - \int_\Omega l(x) v^+ \varphi \mathrm{d}x = 0.$$

Taking $\varphi = v^-$ in the above formulate, we have

$$\int_\Omega |\nabla v^-|^2 \mathrm{d}x = 0,$$

which implies $v^- \equiv 0$. Therefore,

$$v = v^+ \geqslant 0 \quad (\neq 0)$$

since

$$\|v\| = \lim_{n\to\infty} \|v_n\| = 1.$$

It shows that 1 is an eigenvalue of (1.14), which is a contradiction. □

Remark 1 If 1 is the ith eigenvalue of (1.14), taking $u_n = ne_i(l(x))$, where $e_i(l(x)) > 0$ is the eigenfunction corresponding eigenvalue 1, then we can deduce that

$$\|I'(u)\| \, \|u\| = 0.$$

This indicates that it does not satisfy CPS condition. Hence, that the condition 1 is not the eigenvalue of (1.14) is necessary.

Lemma 2 *Assume that* (1.2) *and* (f_1)–(f_3) *with* $l(x) = \infty$ *hold. Then the functional* $I(u)$ *satisfies PS condition.*

Proof Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it suffices to show that $\{u_n\} \subset H_0^1(\Omega)$ is bounded. Assume the contrary, $\|u_n\| \to \infty$ and $\{u_n\}$ satisfies

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \int_\Omega F(x,u_n)\mathrm{d}x \to c, \tag{2.6}$$

$$\int_\Omega \nabla u_n \nabla \varphi \mathrm{d}x - \int_\Omega f(x,u_n)\varphi \mathrm{d}x = o(1)\|\varphi\|, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{2.7}$$

Set

$$v_n = \frac{u_n}{\|u_n\|},$$

and let v be such that (up to a subsequence) $\{v_n\}$ converges weakly to v in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ almost everywhere.

Dividing (2.6) by $\|u_n\|^2$, we have

$$\int_\Omega \frac{F(x,u_n)}{u_n^2}\, v_n^2 \mathrm{d}x \to \frac{1}{2}. \tag{2.8}$$

Set

$$\Omega_1 = \{x \in \Omega\colon v(x) \neq 0\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

From (f_2), we have

$$\frac{F(x,u_n)}{u_n^2}\, v_n^2 \to \infty, \quad x \in \Omega_1.$$

If Ω_1 has positive measure, then

$$\int_\Omega \frac{F(x,u_n)}{u_n^2}\, v_n^2 \mathrm{d}x = \int_{\Omega_1} \frac{F(x,u_n)}{u_n^2}\, v_n^2 \mathrm{d}x + \int_{\Omega_2} \frac{F(x,u_n)}{u_n^2}\, v_n^2 \mathrm{d}x \to \infty,$$

which contradicts with (2.8).

Dividing (2.7) by $\|u_n\|$ and taking $n \to \infty$, we can obtain

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|}\, \varphi \mathrm{d}x \to 0, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{2.9}$$

Set

$$T_n(\varphi) = \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\, \varphi \mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega).$$

Then it is easy to see that $\{T_n\}$ is a family of bounded linear functionals defined on $H_0^1(\Omega)$. From (2.9) and the Resonance Theorem, we know that $\{|T_n|\}$ is bounded, where $|T_n|$ is the norm of $T_n(\varphi)$ defined on $H_0^1(\Omega)$. It means that

$$|T_n| \leqslant C. \tag{2.10}$$

Since $H_0^1(\Omega) \subset L^p(\Omega)$, using the Hahn-Banach Theorem, there exists a continuous linear functional $\hat{T}_n$ defined on $L^p(\Omega)$ such that $\hat{T}_n$ is an extension of T_n, and

$$\hat{T}_n(\varphi) = T_n(\varphi), \quad \forall\, \varphi \in H_0^1(\Omega), \tag{2.11}$$

$$\|\hat{T}_n\|_{p'} = |T_n|, \tag{2.12}$$

where $\|\hat{T}_n\|_{p'}$ is the norm of $\hat{T}_n(\varphi)$ in $L^{p'}(\Omega)$ which is defined on $L^p(\Omega)$.

From the definition of the functional on $L^p(\Omega)$, we know that there is a function $h_n(x) \in L^{p'}(\Omega)$ such that

$$\hat{T}_n(\varphi) = \int_\Omega h_n(x)\varphi(x)\mathrm{d}x, \quad \forall\, \varphi \in L^p(\Omega). \tag{2.13}$$

From (2.11) and (2.13), we have

$$\int_\Omega h_n(x)\varphi(x)\mathrm{d}x = \int_\Omega \frac{f(x,u_n)}{\|u_n\|}\varphi\mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega),$$

which means that

$$\int_\Omega \Big(h_n(x) - \frac{f(x,u_n)}{\|u_n\|}\Big)\varphi(x)\mathrm{d}x = 0, \quad \forall\, \varphi \in H_0^1(\Omega).$$

Using the basic lemma of variational, we can obtain

$$h_n(x) = \frac{f(x,u_n)}{\|u_n\|} \quad \text{a.e. } x \in \Omega.$$

Using (2.10) and (2.12), we have

$$\|\hat{T}_n\|_{p'} = \|h_n(x)\|_{p'} = |T_n| < C. \tag{2.14}$$

From (2.7), (2.9), and taking $\varphi = v_n - v$, we have

$$\int_\Omega |\nabla v_n - \nabla v|^2\mathrm{d}x - \int_\Omega \frac{f(x,u_n)}{\|u_n\|} v_n\mathrm{d}x = 0. \tag{2.15}$$

From the Hölder inequality and (2.14), we can deduce that

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|} v_n\mathrm{d}x \to 0.$$

Then from (2.15), we can deduce that

$$v_n \to v \quad \text{in } H_0^1(\Omega).$$

This is a contradiction since $\|v_n\| = 1$ and $v \equiv 0$. Thus, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. □

Example 1 Let

$$f(x,s)=\begin{cases} 2q(x)s\log(1+|s|)+\dfrac{q(x)s^2}{1+|s|}, & s\geqslant 0,\\ 0, & s<0. \end{cases}$$

It is obvious that $f(x,s)$ does not satisfy SQC, but it satisfies (1.5) for $\mu_0=2$, (1.7), and (1.11).

Example 2 Let

$$f(x,s)=\begin{cases} 2q(x)s\log(1+|s|)(2+\sin s)+\dfrac{q(x)s^2(2+\sin s)}{1+|s|} & \\ \quad +q(x)s^2\log(1+|s|)\cos s, & s\geqslant 0,\\ 0, & s<0. \end{cases}$$

Then $f(x,s)$ does not satisfy (1.5) or (1.7) or (1.11), but it satisfies WSQC.

3 Proofs of theorems

Proof of Theorem 1 A standard method gives that $I(u)$ satisfies the mountain pass geometry. From Lemma 1 and the Mountain Pass Theorem, (1.1) has a nontrivial solution. □

Proof of Theorem 2 A standard method gives that $I(u)\geqslant a$ for $\rho=\|u\|>0$ small enough. Choosing $\phi\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that supp$\phi=\overline{B}_1$, where B_1 denote the unit ball centered at the origin, from (f_2), we have

$$\begin{aligned} I(t\phi)&=\frac{1}{2}t^2\int_\Omega|\nabla\phi|^2\mathrm{d}x-\int_\Omega F(x,t\phi)\mathrm{d}x\\ &=\frac{1}{2}t^2\left[\|\phi\|^2-\int_\Omega\frac{2F(x,t\phi)}{t^2\phi^2}\phi^2\mathrm{d}x\right]\\ &\to\infty \quad (t\to\infty). \end{aligned}$$

From Lemma 2 and the Mountain Pass Theorem, (1.1) has a nontrivial solution.

Acknowledgements This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11101418, 10771074) and the Fundamental Research Funds for the Central Universities, South China University of Technology (No. 2009zm0014).

References

1. Ambrosetti A, Rabinewitz P H. Dual variational method in critical point theory and applications. J Funct Anal, 1973, 14: 349–381
2. Cerami G. Un criterio de esistenza per i punti critici su varietà illimitate. Rc Ist Lomb Sci Lett, 1978, 112: 332–336

3. Chen Z H, Shen Y T, Yao Y X. Some existence results of solutions for p-Laplacian. Acta Math Sci, 2003, 23B: 487–496
4. Costa D G, Magalhães C A. Variational elliptic problems which are nonquadratic at infinity. Nonlinear Anal, 1994, 23: 1401–1412
5. Huang Y S, Zhou H S. Positive solution for $-\Delta_p u = f(x,u)$ with $f(x,u)$ growing as u^{p-1} at infinity. Appl Math Lett, 2004, 17: 881–887
6. Jeanjean L. On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer-type problem set on $\mathbb{R}^N$. Proc Roy Soc Edinburgh, 1999, 129A: 787–809
7. Li G B, Zhou H S. Asymptotically "linear" Dirichlet problem for the p-Laplacian. Nonlinear Anal, 2001, 43: 1043–1055
8. Liu S B, Li S J. Infinitely many solutions for a superlinear elliptic equation. Acta Math Sinica, 2003, 46(4): 625–630
9. Schechter M, Zou W M. Superlinear problems. Pacific J Math, 2004, 214: 145–160
10. Shen Y T, Guo X K. Discussion of nontrivial critical points of the functional $\int_\Omega F(x,u,\mathrm{D}u)\mathrm{d}x$. Acta Math Sci, 1987, 10: 249–258
11. Wang Z P, Zhou H S. Positive solutions for a nonhomogeneous elliptic equation on $\mathbb{R}^N$ without (AR) condition. J Math Anal Appl, 2009, 353: 470–479