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RESEARCH ARTICLE

A simple existence proof of Schubart periodic orbit with arbitrary masses

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Abstract This paper gives an analytic existence proof of the Schubart periodic orbit with arbitrary masses, a periodic orbit with singularities in the collinear three-body problem. A "turning point" technique is introduced to exclude the possibility of extra collisions and the existence of this orbit follows by a continuity argument on differential equations generated by the regularized Hamiltonian.

Keywords Celestial mechanics, Schubart periodic orbit, three-body problem, binary collision, periodic solution with singularity, regularization
MSC 70F10, 70H12, 70H14

1 Introduction

The collinear three-body problem is a system of three points with masses m_1, m_2, and m_3 on a real line attracting each other by the Newtonian gravitational law. Mass m_i locates at position x_i $(i = 1, 2, 3)$ as in Figure 1.

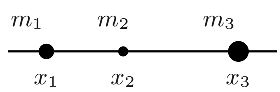

Fig. 1 Mass configuration

In 1976, Schubart [17] numerically discovered a remarkable periodic orbit with singularities in the equal mass collinear three-body problem. In each period of this orbit, the inner mass m_2 alternates between binary collisions (BCs) with the two outer masses: m_1 and m_3. In 1977, Hénon [6] extended Schubart's numerical investigations to the case of unequal masses. Only recently in 2008, did Moeckel [11] and Venturelli [19] analytically prove the

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existence of the Schubart periodic orbit when the two outer masses are equal and the inner mass is arbitrary. Moeckel's proof is topological and uses an idea developed by Conley [5] for the restricted three-body problem. Venturelli's proof is variational wherein he minimizes the Lagrangian action over a well-chosen class of paths. Later, Shibayama [18] gave a variational existence proof for the Schubart periodic orbit with arbitrary masses.

The linear stability of the Schubart periodic orbit was determined numerically by Hietarinta and Mikkola [7]. It revealed that linearly stable Schubart periodic orbit only occurred for certain choices of masses. Recently, Roberts' method [2,3,16] and Maslov index theory [8–10] were introduced to study the linear stability of periodic orbits, such as the Lagrange orbit [9,15] and the figure-eight orbit [4,12] in the three-body problem. However, it is still a big challenge to study the stability of the Schubart periodic orbit analytically due to the existence of collision.

In this paper, we introduce a "turning point" technique [13,14] and give a very simple and direct existence proof of the Schubart periodic orbit with arbitrary masses. The singularity of this orbit can be regularized by a Levi-Civita type transformation and an appropriate scaling of time, as adapted from Aarseth and Zare [1] to this particular problem. Different from the standard variational approach, we first study the connections between the initial condition and the shape of the orbit. Our goal is to show that if the initial value is restricted in a proper interval, then there is no extra collisions, such as total collision and extra binary collisions, between every two connected BCs: BC between m_1 and m_2, and BC between m_2 and m_3. This interval is estimated by a "turning point" technique [14], which is crucial in this work and it can guarantee that the shape of the orbit is exactly the Schubart periodic orbit. Then the existence follows by an intermediate theorem.

The paper is organized as follows. The setting of this problem is introduced in Section 2 and the Hamiltonian is regularized by the Aarseth-Zare method. In Section 3, the "turning point" technique is applied to estimate the value of A so that we can exclude the possibility of extra collisions. Section 4 shows the existence of the Schubart periodic orbit with arbitrary masses in the regularized Hamiltonian system.

2 Preliminaries

As in Fig. 1, we number the three bodies by 1, 2, and 3 from left to right and denote their masses and coordinates on the real line by m_i and x_i with $i = 1$, 2, and 3, respectively. Thus, $x_1 \leqslant x_2 \leqslant x_3$ holds naturally. The Newtonian equations of this system are

$$\ddot{x}_1 = \frac{m_2}{(x_2 - x_1)^2} + \frac{m_3}{(x_3 - x_1)^2}, \tag{1}$$

$$\ddot{x}_2 = -\frac{m_1}{(x_2 - x_1)^2} + \frac{m_3}{(x_3 - x_2)^2}, \tag{2}$$

$$\ddot{x}_3 = -\frac{m_1}{(x_3 - x_1)^2} - \frac{m_2}{(x_3 - x_2)^2}. \tag{3}$$

The Hamiltonian for the system is

$$H = \frac{1}{2}\Big[\frac{w_1^2}{m_1} + \frac{w_2^2}{m_2} + \frac{w_3^2}{m_3}\Big] - \sum_{1 \leqslant i < j \leqslant 3} \frac{m_i m_j}{|x_i - x_j|}, \tag{4}$$

where $w_i = m_i \dot{x}_i$ is the momenta of m_i $(i = 1, 2, 3)$.

Following the work of Hietarinta and Mikkola [7], we introduce a canonical transformation to eliminate the center of mass and total linear momentum. Specially, set

$$\begin{aligned} q_1 &= x_2 - x_1, \quad p_1 = -w_1 + \frac{m_1(w_1 + w_2 + w_3)}{m_1 + m_2 + m_3}, \\ q_2 &= x_3 - x_2, \quad p_2 = w_3 - \frac{m_2(w_1 + w_2 + w_3)}{m_1 + m_2 + m_3}, \\ q_3 &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}, \quad p_3 = w_1 + w_2 + w_3. \end{aligned}$$

By setting $q_3 = 0$ and $p_3 = 0$, the transformation becomes

$$q_1 = x_2 - x_1, \quad q_2 = x_3 - x_2, \quad p_1 = -w_1, \quad p_2 = w_3,$$

and the new Hamiltonian becomes

$$\begin{aligned} H = {} & \frac{1}{2}\Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)p_1^2 + \frac{1}{2}\Big(\frac{1}{m_2} + \frac{1}{m_3}\Big)p_2^2 - \frac{p_1 p_2}{m_2} \\ & - \frac{m_1 m_2}{q_1} - \frac{m_2 m_3}{q_2} - \frac{m_1 m_3}{q_1 + q_2}. \end{aligned} \tag{5}$$

The Hamiltonian (5) can be regularized by the Aarseth-Zare method [1]. We define $q_i = Q_i^2$, and the new canonical momenta are $P_i = 2Q_i p_i$ $(i = 1, 2)$. Let the new time variable s satisfy $\mathrm{d}t/\mathrm{d}s = q_1 q_2$. Then the regularized Hamiltonian $\Gamma = q_1 q_2 (H - E)$ becomes

$$\begin{aligned} \Gamma = {} & \frac{1}{8}\Big[\Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)P_1^2 Q_2^2 + \Big(\frac{1}{m_2} + \frac{1}{m_3}\Big)P_2^2 Q_1^2 - \frac{2}{m_2} P_1 P_2 Q_1 Q_2\Big] \\ & - m_2 m_3 Q_1^2 - m_1 m_2 Q_2^2 - m_1 m_3 \frac{Q_1^2 Q_2^2}{Q_1^2 + Q_2^2} - Q_1^2 Q_2^2 E, \end{aligned} \tag{6}$$

where E is the total energy.

Without loss of generality, we can assume that $H = -1$, $m_2 = 1$, and $m_1 \geqslant m_3$ in this paper.

We start at BC between m_1 and m_2, and set the initial conditions to be

$$x_1(0) = x_2(0), \quad x_3(0) = A > 0, \quad \dot{x}_1(0) = -\infty, \quad \dot{x}_2(0) = +\infty, \quad \dot{x}_3(0) = 0.$$

Note that the center of mass at time $t = 0$ is

$$0 = m_1 x_1(0) + m_2 x_2(0) + m_3 x_3(0) = x_2(0)(m_1 + 1) + m_3 A.$$

Then

$$x_1(0) = x_2(0) = -\frac{m_3 A}{1 + m_1}.$$

The corresponding coordinates in (p_i, q_i) $(i = 1, 2)$ are

$$q_1(0) = 0, \quad q_2(0) = \frac{1 + m_1 + m_3}{1 + m_1} A, \quad p_1(0) = +\infty, \quad p_2(0) = 0,$$

which is also a singular point. To analyze the motion nearby, it is necessary to deal with the singularity in the regularized Hamiltonian system. The corresponding initial conditions at $s = 0$ in the new coordinate system are

$$Q_1(0) = 0, \quad Q_2(0) = R, \quad P_1(0) = \frac{2\sqrt{2}\, m_1 m_2}{\sqrt{m_1 + m_2}} = \frac{2\sqrt{2}\, m_1}{\sqrt{m_1 + 1}}, \quad P_2(0) = 0, \tag{7}$$

where

$$R = \sqrt{\frac{1 + m_1 + m_3}{1 + m_1} A}.$$

Note that this initial point is a regular point in the regularized Hamiltonian system. In the next section, we will show that there exists an interval such that for any A in this interval, there is no collision before the first BC between bodies 2 and 3 happens.

3 Estimation of A

Intuitively, if the relative distance between bodies 2 and 3 is sufficiently large, there will be multiple BCs between bodies 1 and 2 before the first collision of bodies 2 and 3 happens. In order to find the Schubart periodic orbit, we will have to give an estimation of A such that there is no extra binary collision between bodies 1 and 2 before bodies 2 and 3 collides for the first time. In this section, we apply a "turning point" technique [14] to estimate the value of A.

Definition 1 Let $t = t^*$ be the time when the velocity of a body is 0, i.e., $v(t^*) = 0$. If there exists a time interval $[t_m, t_n]$, such that $t_m < t^* < t_n$, and v is positive for $t \in [t_m, t^*)$ and is negative for $t \in (t^*, t_n]$, or v is negative for $t \in [t_m, t^*)$ and is positive for $t \in (t^*, t_n]$, then we call t^* a *turning time* and the position of the body at t^* is called a *turning point* (Figure 2).

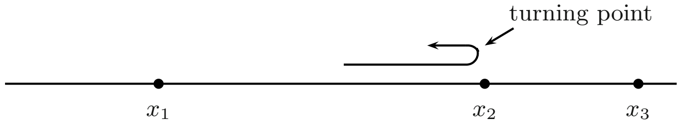

Fig. 2 Turning point of body 2

We first show that if body 2 has a turning point before it collides with body 3 for the first time, then A must have a positive lower bound.

Lemma 2 *Let t_1 be the first time of collision between bodies 2 and 3. If body 2 has a turning point for $t \in (0, t_1)$, then*

$$A \geqslant \min\Big\{\frac{m_1^2 m_3}{m_1+m_3}, \frac{m_3(m_1+m_3)}{m_1}\Big\}.$$

Proof Assume that body 2 has a turning point before the first collision between bodies 2 and 3 happens. Let $t = t^* < t_1$ be the first turning time. Then $\dot{x}_2(t^*) = 0$, $\ddot{x}_2(t^*) \leqslant 0$, and $\dot{x}_2(t) > 0$ for any $t \in (0, t^*)$. At $t = t^*$, we assume that the position of body 3 is $x_3(t^*) = A^*$ and the position of body 2 is $x_2(t^*) = aA^*$.

Since the center of mass $m_1x_1 + x_2 + m_3x_3 = 0$, the coordinates of three bodies at $t = 0$ and $t = t^*$ are

$$x_1(0) = x_2(0) = -\frac{m_3}{1+m_1}A, \quad x_3(0) = A,$$
$$x_1(t^*) = -\frac{a+m_3}{m_1}A^*, \quad x_2(t^*) = aA^*, \quad x_3(t^*) = A^*.$$

Note that

$$x_3(t^*) > x_2(t^*) > x_1(t^*).$$

Then

$$-\frac{m_3}{1+m_1} < a < 1. \tag{8}$$

Since $\ddot{x}_2(t^*) \leqslant 0$, by the Newtonian equation (2) of x_2, we have

$$\begin{aligned} 0 &\geqslant \ddot{x}_2(t^*) \\ &= \Big[-\frac{m_1}{(a+\frac{a+m_3}{m_1})^2} + \frac{m_3}{(1-a)^2}\Big]\frac{1}{x_3^2(t^*)} \\ &= \frac{[m_3(m_1a+a+m_3)^2 - m_1^3(1-a)^2]}{(m_1a+a+m_3)^2(1-a)^2}\cdot\frac{1}{x_3^2(t^*)}, \end{aligned} \tag{9}$$

i.e.,

$$m_3^{1/2}(m_1a+a+m_3) \leqslant m_1^{3/2}(1-a). \tag{10}$$

Consider the Newtonian equation (3) of x_3 for $t \in (0, t^*]$:

$$\ddot{x}_3 = -\Big[\frac{m_1}{(x_3-x_1)^2} + \frac{1}{(x_3-x_2)^2}\Big].$$

Since $x_1 = -(x_2 + m_3x_3)/m_1$, it can be rewritten as

$$\ddot{x}_3 = -\Big[\frac{m_1^3}{[(m_1+m_3)x_3+x_2]^2} + \frac{1}{(x_3-x_2)^2}\Big].$$

Let

$$f(y)=\frac{m_1^3}{[(m_1+m_3)x_3+y]^2}+\frac{1}{(x_3-y)^2}.$$

Then

$$\begin{aligned}f'(y)&=\frac{-2m_1^3}{[(m_1+m_3)x_3+y]^3}+\frac{2}{(x_3-y)^3}\\&=\frac{2[(m_1+m_3)x_3+y]^3-2m_1^3(x_3-y)^3}{[(m_1+m_3)x_3+y]^3(x_3-y)^3}.\end{aligned}$$

Hence, $f'(y)\geqslant 0$ if and only if

$$(m_1+m_3)x_3+y\geqslant m_1(x_3-y),$$

i.e.,

$$y\geqslant -\frac{m_3x_3}{1+m_1}.$$

Therefore, $f(y)$ is an increasing function when $y\geqslant -m_3x_3/(1+m_1)$.

In our case, by the center of mass equal to 0 and $x_1\leqslant x_2$, we have

$$(m_1+1)x_2+m_3x_3\geqslant m_1x_1+x_2+m_3x_3=0,$$

i.e.,

$$x_2\geqslant -\frac{m_3x_3}{1+m_1}.$$

Since $-m_3x_3/(1+m_1)\leqslant x_2\leqslant aA^*$ for $t\in[0,t^*]$ and $f(y)$ is increasing whenever $y\geqslant -m_3x_3/(1+m_1)$, we have $f(x_2)\leqslant f(aA^*)$. Therefore,

$$\ddot{x}_3\geqslant -\Big[\frac{m_1^3}{[(m_1+m_3)x_3+aA^*]^2}+\frac{1}{(x_3-aA^*)^2}\Big]. \tag{11}$$

Note that $\dot{x}_3<0$ for $t\in(0,t^*]$. Multiplying both sides of inequality (11) by $\dot{x}_3$, we have

$$\dot{x}_3\ddot{x}_3\leqslant -\dot{x}_3\Big[\frac{m_1^3}{[(m_1+m_3)x_3+aA^*]^2}+\frac{1}{(x_3-aA^*)^2}\Big]. \tag{12}$$

Integrating (11) from $t=0$ to $t=t^*$, we get

$$\int_0^{t^*}\dot{x}_3\ddot{x}_3\mathrm{d}t\leqslant -\int_0^{t^*}\dot{x}_3\Big[\frac{m_1^3}{[(m_1+m_3)x_3+aA^*]^2}+\frac{1}{(x_3-aA^*)^2}\Big]\mathrm{d}t,$$

$$\frac{(\dot{x}_3(t^*))^2-(\dot{x}_3(0))^2}{2}\leqslant\frac{m_1^3}{m_1+m_3}\cdot\frac{1}{(m_1+m_3)x_3(t)+aA^*}\Big|_0^{t^*}+\frac{1}{x_3(t)-aA^*}\Big|_0^{t^*}.$$

Note that $\dot{x}_3(0)=0,\ x_3(0)=A$, and $x_3(t^*)=A^*$. Then

$$\frac{1}{2}\,(\dot{x}_3(t^*))^2\leqslant\frac{1}{A^*}\Big[\frac{m_1^3}{(m_1+m_3)(m_1+m_3+a)}+\frac{1}{1-a}\Big]$$

$$-\frac{m_1^3}{(m_1+m_3)[(m_1+m_3)A+aA^*]}-\frac{1}{A-aA^*}. \tag{13}$$

At time $t=t^*$, the coordinates of three bodies are

$$x_1(t^*)=-\frac{a+m_3}{m_1}A^*,\quad x_2(t^*)=aA^*,\quad x_3(t^*)=A^*,$$

and the velocities satisfy

$$\dot{x}_2(t^*)=0,\quad \dot{x}_1(t^*)=-\frac{m_3}{m_1}\dot{x}_3(t^*).$$

Since the total energy E is -1, the Hamiltonian H at $t=t^*$ satisfies

$$\begin{aligned}-1&=\frac{1}{2}m_1(\dot{x}_1(t^*))^2+\frac{1}{2}m_3(\dot{x}_3(t^*))^2-\frac{1}{A^*}\Big[\frac{m_1}{a+\frac{a+m_3}{m_1}}+\frac{m_3}{1-a}+\frac{m_1m_3}{1+\frac{a+m_3}{m_1}}\Big]\\&=\frac{m_3(m_1+m_3)}{2m_1}(\dot{x}_3(t^*))^2-\frac{1}{A^*}\Big[\frac{m_1^2}{m_1a+a+m_3}+\frac{m_3}{1-a}+\frac{m_1^2m_3}{m_1+a+m_3}\Big].\end{aligned}$$

Applying inequality (13) to the above equality, we have

$$\begin{aligned}-1\leqslant&\frac{m_3(m_1+m_3)}{m_1A^*}\Big[\frac{m_1^3}{(m_1+m_3)(m_1+m_3+a)}+\frac{1}{1-a}\Big]-\frac{m_1^2m_3}{(m_1+m_3)A+aA^*}\\&-\frac{m_3(m_1+m_3)}{m_1(A-aA^*)}-\frac{1}{A^*}\Big[\frac{m_1^2}{m_1a+a+m_3}+\frac{m_3}{1-a}+\frac{m_1^2m_3}{m_1+a+m_3}\Big]\\=&\frac{m_3^2(m_1a+a+m_3)-m_1^3(1-a)}{m_1(1-a)(m_1a+a+m_3)A^*}-\frac{m_1^2m_3}{(m_1+m_3)A+aA^*}-\frac{m_3(m_1+m_3)}{m_1(A-aA^*)}.\end{aligned}$$

Note that a satisfies inequality (10) and $m_1\geqslant m_3$ by our assumption. Then

$$\begin{aligned}m_3^2(m_1a+a+m_3)-m_1^3(1-a)&\leqslant m_3^2(m_1a+a+m_3)-m_3^{3/2}m_1^{3/2}(1-a)\\&=m_3^{3/2}[m_3^{1/2}(m_1a+a+m_3)-m_1^{3/2}(1-a)]\\&\leqslant 0.\end{aligned}$$

Hence,

$$-1\leqslant-\frac{m_1^2m_3}{(m_1+m_3)A+aA^*}-\frac{m_3(m_1+m_3)}{m_1(A-aA^*)},$$

i.e.,

$$A\geqslant\frac{m_1^2m_3A}{(m_1+m_3)A+aA^*}+\frac{m_3(m_1+m_3)A}{m_1(A-aA^*)}. \tag{14}$$

It is clear that $A>0$ and $A^*>0$. If $a\geqslant 0$, (14) implies

$$A\geqslant\frac{m_3(m_1+m_3)A}{m_1(A-aA^*)}\geqslant\frac{m_3(m_1+m_3)}{m_1}.$$

If $a < 0$, by (14), we obtain

$$A \geqslant \frac{m_1^2 m_3 A}{(m_1+m_3)A + aA^*} \geqslant \frac{m_1^2 m_3}{m_1+m_3}.$$

Therefore,

$$A \geqslant \min\Big\{\frac{m_1^2 m_3}{m_1+m_3}, \frac{m_3(m_1+m_3)}{m_1}\Big\}. \qquad \square$$

Define a set $\wp = \{A \mid \text{body 2 has at least one turning point for } t \in (0, t_1)\}$. We first show that the set $\wp$ is not empty.

If not, then for all positive values of A, body 2 has no turning point when $t \in (0, t_1)$, that is, $\dot{x}_2(t) > 0$ for any $t \in (0, t_1)$. By the regularization theory of binary collision, $\lim_{A\to+\infty} \dot{x}_2(t) \geqslant 0$ for any $t \in (0, t_1)$. However, when $A = +\infty$, the Newtonian equations (1)–(3) become

$$\ddot{x}_1 = \frac{m_2}{(x_2-x_1)^2}, \quad \ddot{x}_2 = -\frac{m_1}{(x_2-x_1)^2}, \quad \ddot{x}_3 = 0.$$

In this case, bodies 1 and 2 have the same equations of motion as the one-dimensional Kepler two-body problem. Note that the total energy is -1, the theory of Kepler two-body problem indicates that body 2 has at least one turning point. It contradicts with the inequality $\lim_{A\to+\infty} \dot{x}_2(t) \geqslant 0$. Therefore,

$$\wp \neq \emptyset.$$

Let $A_0 = \inf \wp$. By Lemma 2,

$$A_0 \geqslant \min\Big\{\frac{m_1^2 m_3}{m_1+m_3}, \frac{m_3(m_1+m_3)}{m_1}\Big\} > 0.$$

Also, by the definition of A_0, body 2 has no turning point for $t \in (0, t_1)$ whenever $A < A_0$. For each $A \in \wp$, there exists a smallest $t^* > 0$ such that

$$\dot{x}_2(t^*) = 0, \quad \ddot{x}_2(t^*) \leqslant 0.$$

Next, we show that when $A = A_0$, there exists some $t^* < t_1$ such that $\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0$. Since $\dot{x}_2(t)$ is continuous for $t \in (0, t_1)$, and $\lim_{t\to 0^+} \dot{x}_2(t) = +\infty$, the proof can be ended by the following result.

Theorem 3 *For $A = A_0$, there exists a unique $t^* \in (0, t_1)$ such that $\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0$. Furthermore, $\dddot{x}_2(t^*) \geqslant 0$ and the equality holds if and only if $m_1 = m_3$.*

Proof We first show that in the case $A = A_0$, $\dot{x}_2(t) \geqslant 0$ for any $t \in (0, t_1)$.

For a given $A > 0$ such that $x_3(0) = A$ and $x_2(0) = x_1(0)$, the solution $x_2(t)$ has a turning point in the interval $(0, t_1)$, by continuity of solution with respect to initial condition, changing a bit A, the solution $x_2(t)$ still has a turning point in $(0, t_1)$ (here t_1 is a function of A). By the definition of A_0, body 2 is free of turning point in the interval $(0, t_1)$. Therefore, $t \mapsto x_2(t)$ is an increasing function in the interval $(0, t_1)$. Hence, $\dot{x}_2(t) \geqslant 0$ for $t \in (0, t_1)$.

Consider the function

$$g(y) = m_3(m_1 y + y + m_3)^2 - m_1^3(1 - y)^2.$$

It is easy to see that $g(y)$ is strictly increasing for $y > -m_3/(1 + m_1)$. By (9), $\ddot{x}_2(t^*)$ has the same sign as $g(a)$. Let $1 > a_0 > -m_3/(1 + m_1)$ be the unique value such that $g(a_0) = 0$. When $g(a_0) = 0$, we solve for a_0:

$$m_3^{1/2}(m_1 a_0 + a_0 + m_3) = m_1^{3/2}(1 - a_0),$$

$$a_0[(1 + m_1)m_3^{1/2} + m_1^{3/2}] = m_1^{3/2} - m_3^{3/2}.$$

By our assumption, $m_1 \geqslant m_3$. Then $a_0 \geqslant 0$ and $a_0 = 0$ if and only if $m_1 = m_3$.

Next, we show that the function $t \mapsto [x_2(t) - a_0 x_3(t)]$ has exactly one zero t^*. Since $t \mapsto x_2(t)$ is an increasing function, $t \mapsto x_3(t)$ is a strictly decreasing function and $a_0 \geqslant 0$, $t \mapsto [x_2(t) - a_0 x_3(t)]$ is an increasing function in $(0, t_1)$ and $\frac{\mathrm{d}}{\mathrm{d}t}[x_2(t) - a_0 x_3(t)] = 0$ if and only if $\dot{x}_2(t) = 0,\ a_0 = 0$. Therefore, there exists only one value $t^* \in (0, t_1)$ such that

$$x_2(t^*) - a_0 x_3(t^*) = 0.$$

And

$$\frac{-m_3}{1 + m_1} < \frac{x_2(t)}{x_3(t)} < a_0, \quad t \in (0, t^*),$$
$$\frac{x_2(t)}{x_3(t)} > a_0, \quad t \in (t^*, t_1).$$

Thus,

$$\ddot{x}_2(t) \begin{cases} < 0, & t \in (0, t^*), \\ = 0, & t = t^*, \\ > 0, & t \in (t^*, t_1). \end{cases}$$

Note that at $A = A_0,\ \dot{x}_2(t) \geqslant 0$ for $t \in (0, t_1)$. Assume now for the sake of contradiction that $\dot{x}_2(t^*) > 0$. Then by the sign of $\ddot{x}_2(t)$, the relative minimum of $\dot{x}_2(t)$ is achieved at $t = t^*$, and hence,

$$\dot{x}_2(t) \geqslant \dot{x}_2(t^*) > 0, \quad \forall\, t \in (0, t_1).$$

By the continuity with respect to the initial conditions, this fact would persist changing a bit A, and therefore, for A closed to A_0, the corresponding solution would be free of turning point. This contradicts the definition of A_0. Hence, at $A = A_0$, we have

$$\dot{x}_2(t^*) = \ddot{x}_2(t^*) = 0.$$

Furthermore, we consider the sign of $\dddot{x}_2(t^*)$ in the case when $A = A_0$. Derivative equation (2) with respect to t evaluated at t^*:

$$\begin{aligned}\dddot{x}_2(t^*) &= \frac{2m_1(\dot{x}_2(t^*) - \dot{x}_1(t^*))}{(x_2(t^*) - x_1(t^*))^3} - \frac{2m_3(\dot{x}_3(t^*) - \dot{x}_2(t^*))}{(x_3(t^*) - x_2(t^*))^3} \\ &= \frac{-2m_1\dot{x}_1(t^*)}{(x_2(t^*) - x_1(t^*))^3} - \frac{2m_3\dot{x}_3(t^*)}{(x_3(t^*) - x_2(t^*))^3} \\ &= \frac{2m_1^3 m_3\dot{x}_3(t^*)}{(m_1a + a + m_3)^3(A^*)^3} - \frac{2m_3\dot{x}_3(t^*)}{(1-a)^3(A^*)^3} \\ &= \frac{2m_3\dot{x}_3(t^*)[m_1^3(1-a)^3 - (m_1a + a + m_3)^3]}{(A^*)^3(m_1a + a + m_3)^3(1-a)^3}.\end{aligned}$$

Note that $\ddot{x}_2(t^*) = 0$. By equation (9), we obtain

$$m_3(m_1a + a + m_3)^2 - m_1^3(1-a)^2 = 0.$$

Then

$$m_1^3(1-a)^3 - (m_1a + a + m_3)^3 = \frac{m_1^3}{m_3^{3/2}}(1-a)^3(m_3^{3/2} - m_1^{3/2}).$$

Since $\dot{x}_3(t^*) < 0$ and $m_1 \geqslant m_3$, we have

$$\dddot{x}_2(t^*) = \frac{2m_3\dot{x}_3(t^*)}{(A^*)^3(m_1a + a + m_3)^3(1-a)^3}\frac{m_1^3}{m_3^{3/2}}(1-a)^3(m_3^{3/2} - m_1^{3/2}) \geqslant 0.$$

When $\dddot{x}_2(t^*) = 0$, we have $m_1 = m_3$. □

Remark When $A = A_0$, $m_1 = m_3$,

$$x_2^{(n)}(t^*) = 0, \quad \forall\ n \geqslant 1,$$

where $x_2^{(n)}$ means the n-th derivative of x_2 with respect to t. That is, body 2 will stay at $x_2(t^*)$ forever, bodies 1 and 3 move towards to it, and they will end up with a total collision.

Corollary 4 *For $A \in (0, A_0)$, bodies 1 and 2 have no other collision before bodies 2 and 3 collide at the first time.*

Proof If bodies 1 and 2 have another binary collision at time $t = t_0$, then

$$\lim_{t\to t_0} \dot{x}_1 = +\infty, \quad \lim_{t\to t_0} \dot{x}_2 = -\infty.$$

($\lim_{t\to t_0} \dot{x}_2 \neq +\infty$ because the total linear momentum is 0). By continuity, there exists some t^*, such that $\dot{x}_2(t^*) = 0$ and $\ddot{x}_2(t^*) \leqslant 0$. It contradicts the definition of A_0. Therefore, when $A \in (0, A_0)$, bodies 1 and 2 have no other collision before bodies 2 and 3 collide for the first time. □

4 Existence of Schubart periodic orbit

Recall the regularized Hamiltonian Γ in (6), where $E = -1$ is the total energy. By the assumption, $m_2 = 1$. The initial conditions at $s = 0$ are as in (7).

By Theorem 3, when

$$0 < R = \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\, A} < \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\, A_0},$$

$Q_2^2 = x_3 - x_2$ decreases from $s = 0$ to $s = s_1$, where s_1 is the time when the first collision between bodies 2 and 3 happens.

The equations of motion from the regularized Hamiltonian Γ are

$$Q_1' = \frac{m_1 + 1}{4m_1}\, P_1 Q_2^2 - \frac{1}{4}\, P_2 Q_1 Q_2, \tag{15}$$

$$Q_2' = \frac{m_3 + 1}{4m_3}\, P_2 Q_1^2 - \frac{1}{4}\, P_1 Q_1 Q_2, \tag{16}$$

$$P_1' = -\frac{m_3 + 1}{4m_3}\, P_2^2 Q_1 + \frac{1}{4}\, P_1 P_2 Q_2 + 2m_3 Q_1 + \frac{2m_1 m_3 Q_1 Q_2^4}{(Q_1^2 + Q_2^2)^2} - 2Q_1 Q_2^2, \tag{17}$$

$$P_2' = -\frac{m_1 + 1}{4m_1}\, P_1^2 Q_2 + \frac{1}{4}\, P_1 P_2 Q_1 + 2m_1 Q_2 + \frac{2m_1 m_3 Q_2 Q_1^4}{(Q_1^2 + Q_2^2)^2} - 2Q_2 Q_1^2, \tag{18}$$

where $(')$ is the derivative with respect to s.

At the time $s = s_1$, the coordinates of P_i and Q_i $(i = 1, 2)$ are

$$Q_2(s_1) = 0, \quad Q_1(s_1) = R_1 > 0, \quad P_2(s_1) = -\frac{2\sqrt{2}\, m_3}{\sqrt{1 + m_3}}.$$

To prove the existence of the Schubart periodic orbit, we need to find a suitable value of R, such that

$$P_1(s_1) = P_1(s_1, R) = 0.$$

Corollary 5 *For the differential equations* (15)–(18), *let the initial conditions be*

$$Q_1(0) = 0, \quad Q_2(0) = R = \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\, A},$$
$$P_1(0) = \frac{2\sqrt{2}\, m_1}{\sqrt{m_1 + 1}}, \quad P_2(0) = 0,$$

where

$$R \in \left(0, \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\, A_0}\right).$$

Let s_1 be the time when bodies 2 and 3 have the first collision. Then $P_1(s_1, R)$ is a continuous function of R.

Proof Since the Hamiltonian Γ is regularized, the solutions $P_i = P_i(s, R)$ and $Q_i = Q_i(s, R)$ are continuous functions with respect to s and R. We are going to show that $s_1 = s_1(R)$ is a continuous function of R. By Theorem 3 and the transformation, we have $Q_1^2(s_1) > 0$, $Q_2(s_1) = 0$, and also $Q_2^2(s) > 0$ for $0 < s < s_1$. In order to apply the implicit function theorem for $Q_2 = Q_2(s_1, R) = 0$, we need to show that

$$\frac{\partial Q_2}{\partial s}\,(s_1, R) \neq 0.$$

By the regularized Hamiltonian Γ, we have

$$\frac{\partial Q_2}{\partial s}\Big|_{(s_1,R)} = \frac{\partial \Gamma}{\partial P_2}\Big|_{(s_2,R)} = \Big[\frac{m_3+1}{4m_3}\,P_2Q_1^2 - \frac{1}{4}\,P_1Q_1Q_2\Big]\Big|_{(s_1,R)}.$$

Note that for any fixed

$$R \in \Big(0, \sqrt{\frac{1+m_1+m_3}{1+m_1}\,A_0}\Big),$$

$\Gamma = 0$ at any time s. At $s = s_1$, we have

$$Q_1 = Q_1(s_1, R) \neq 0, \quad Q_2 = Q_2(s_1, R) = 0,$$

and then

$$P_2^2 = P_2^2(s_1, R) = \frac{8m_3^2}{1+m_3}.$$

Therefore,

$$\frac{\partial Q_2}{\partial s}\Big|_{(s_1,R)} = \Big[\frac{m_3+1}{4m_3}\,P_2Q_1^2 - \frac{1}{4}\,P_1Q_1Q_2\Big]\Big|_{(s_1,R)} = \frac{m_3+1}{4m_3}\,P_2Q_1^2\Big|_{(s_1,R)} \neq 0.$$

By the implicit function theorem, s_1 is a continuous function of R. Hence, $P_1(s_1, R)$ is also a continuous function of R. □

Theorem 6 *There exists some*

$$R \in \Big(0, \sqrt{\frac{1+m_1+m_3}{1+m_1}\,A_0}\Big)$$

such that

$$P_1(s_1) = P_1(s_1, R) = 0.$$

Proof The existence is proved by the intermediate value theorem. First, we show that there exists an $R > 0$ such that $P_1(s_1) > 0$.

By equations (15)–(18), we have

$$\begin{aligned}(P_1Q_1 + P_2Q_2)' &= 2m_1Q_2^2 + 2m_3Q_1^2 + \frac{2m_1m_3Q_1^2Q_2^2}{Q_1^2+Q_2^2} - 4Q_1^2Q_2^2 \\ &= 2m_1Q_2^2 + 2Q_1^2\Big[m_3 + \frac{2m_1m_3Q_2^2}{Q_1^2+Q_2^2} - 2Q_2^2\Big].\end{aligned}$$

Note that for $s \in [0, s_1]$, Q_2 is decreasing and $0 \leqslant Q_2 \leqslant R$.
Choosing

$$R = \min\left\{\sqrt{\frac{m_3}{4}}, \sqrt{\frac{(1+m_1+m_3)}{2(1+m_1)}\frac{m_1^2 m_3}{m_1+m_3}}, \sqrt{\frac{(1+m_1+m_3)}{2(1+m_1)}\frac{m_3(m_1+m_3)}{m_1}}\right\},$$

by Lemma 2, we have

$$R^2 \leqslant \frac{1+m_1+m_3}{2(1+m_1)}A_0 < \frac{1+m_1+m_3}{1+m_1}A_0, \quad Q_2^2 \leqslant R^2 \leqslant \frac{m_3}{4} < \frac{m_3}{2}.$$

Thus,

$$(P_1Q_1 + P_2Q_2)' = 2m_1Q_2^2 + 2Q_1^2\Big[\frac{2m_1m_3Q_2^2}{Q_1^2+Q_2^2} + m_3 - 2Q_2^2\Big] > 0.$$

From the initial conditions,

$$(P_1Q_1 + P_2Q_2)\mid_{s=0} = 0,$$

hence,

$$P_1(s_1)Q_1(s_1) = \int_0^{s_1}(P_1Q_1 + P_2Q_2)'\mathrm{d}s > 0.$$

By Corollary 4, $Q_1(s_1) \geqslant 0$. For the above choice of R, there is no total collision for $s \in [0, s_1]$, and then, $Q_1(s_1) > 0$. Therefore, under the above definition of R, we have $P_1(s_1) > 0$.

Next, we show that $P_1(s_1) < 0$ for some proper choice of R. We prove it in two cases: $m_1 \neq m_3$ and $m_1 = m_3$.

Case 1 $m_1 \neq m_3$. Choose $R > 0$, such that

$$R^2 = \frac{1+m_1+m_3}{1+m_1}A_0.$$

At $A = A_0$, by Theorem 3, there exists some t^* such that $\dot{x}_2(t^*) = 0$. Then

$$\dot{x}_1(t^*) = -\frac{m_3}{m_1}\dot{x}_3(t^*) > 0.$$

By equation (1), we have $\ddot{x}_1 \geqslant 0$ for any $t \in (0, t_1)$, and then $\dot{x}_1(t_1) > 0$. By Corollary 4 and the continuity of Q_1, we have

$$Q_1(s_1) = Q_1(s_1, R) > 0.$$

Hence,

$$P_1(s_1) = 2Q_1(s_1)p_1(t_1) = -2Q_1(s_1)m_1\dot{x}_1(t_1) < 0.$$

Case 2 $m_1 = m_3$. As we know, when $A = A_0$, the three bodies end up at a total collision. Therefore,

$$\dot{x}_1(t) = \dot{x}_1(t, A_0) > 0, \quad \forall\, t \in (t^*, t_1).$$

By the continuity with respect to A, there exists some small $\varepsilon > 0$ such that when $A = A_0 - \varepsilon$,

$$\dot{x}_1(t_0) = \dot{x}_1(t_0, A) > 0$$

for some $t_0 < t_1$. Since $\ddot{x}_1 \geqslant 0$ for any $t \in (0, t_1)$, it follows that $\dot{x}_1(t_1) > 0$. Then when

$$R = \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\,(A_0 - \varepsilon)},$$

we have

$$Q_1(s_1) = Q_1(s_1, R) > 0, \quad P_1(s_1) = 2Q_1(s_1)p_1(t_1) = -2Q_1(s_1)m_1\dot{x}_1(t_1) < 0.$$

Therefore, by the intermediate theorem, there exists an

$$R \in \left(0, \sqrt{\frac{1 + m_1 + m_3}{1 + m_1}\,A_0}\right)$$

such that

$$P_1(s_1) = P_1(s_1, R) = 0. \qquad \square$$

Theorem 7 *If R satisfies*

$$P_1(s_1) = P_1(s_1, R) = 0,$$

then the solution of the differential system is exactly the Schubart-like periodic orbit.

Proof The proof follows by the uniqueness of solution of ordinary differential equations.

At time $s = 0$, a BC happens between bodies 1 and 2. At time $s = s_1$, another BC occurs between bodies 2 and 3. Since the system is regularized, the solutions $\{P_i, Q_i\}$ $(i = 1, 2)$ are continuous.

At time $s = 0$,

$$Q_1(0) = 0, \quad Q_2(0) = R, \quad P_1(0) = \frac{2\sqrt{2}\,m_1}{\sqrt{m_1 + 1}}, \quad P_2(0) = 0.$$

At time $s = s_1$,

$$Q_1(s_1) = R_1, \quad Q_2(s_1) = 0, \quad P_1(s_1) = 0, \quad P_2(s_1) = -\frac{2\sqrt{2}\,m_3}{\sqrt{m_3 + 1}},$$

where R_1 is a positive number.

From the Hamiltonian Γ, we can see that

$$Q_1'(s_1) = 0, \quad Q_1''(s_1) < 0, \quad Q_2'(s_1) < 0.$$

This means that $Q_1(s_1)$ is a relative maximum of Q_1.

Compare the motion for $s \in [0, s_1]$ with the motion for $s \in [s_1, 2s_1]$. By the uniqueness of the regularized Hamiltonian system, the orbit for $s \in [s_1, 2s_1]$ can be generated from the orbit of $s \in [0, s_1]$ by the following symmetry:

$$\begin{aligned} Q_1(2s_1 - s) &= Q_1(s), \quad P_1(2s_1 - s) = -P_1(s), \\ Q_2(2s_1 - s) &= -Q_2(s), \quad P_2(2s_1 - s) = P_2(s), \end{aligned}$$

where $s \in [0, s_1]$. Then at the time $s = 2s_1$ when the second BC occurs,

$$Q_1(2s_1) = 0, \quad Q_2(2s_1) = -R, \quad P_1(2s_1) = -\frac{2\sqrt{2}\,m_1}{\sqrt{m_1+1}}, \quad P_2(2s_1) = 0.$$

By the symmetry and the uniqueness again, at time $s = 3s_1$,

$$Q_1(3s_1) = -R_1, \quad Q_2(3s_1) = 0, \quad P_1(3s_1) = 0, \quad P_2(3s_1) = \frac{2\sqrt{2}\,m_3}{\sqrt{m_3+1}}.$$

At time $s = 4s_1$,

$$Q_1(4s_1) = 0, \quad Q_2(4s_1) = R, \quad P_1(4s_1) = \frac{2\sqrt{2}\,m_1}{\sqrt{m_1+1}}, \quad P_2(4s_1) = 0,$$

which is exactly the same as the initial condition at $s = 0$. Then the orbit from $s = 0$ to $s = 4s_1$ generates one period.

By the symmetry of differential equations (15)–(18), we have

$$Q_i(s + 2s_1) = -Q_i(s), \quad P_i(s + 2s_1) = -P_i(s), \quad i = 1, 2.$$

However, since $q_i = Q_i^2$ and $P_i = 2Q_i p_i$, in the variables (q_i, p_i), $i = 1, 2$, and in physical time t, the solution is periodic with period $2t_1$ (and not $4t_1$). □

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