Front. Math. China 2011, 6(4): 581–606
DOI 10.1007/s11464-011-0107-z

RESEARCH ARTICLE

On classification of n-Lie algebras

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Abstract In this paper, we prove the isomorphic criterion theorem for $(n+2)$-dimensional n-Lie algebras, and give a complete classification of $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero.

Keywords n-Lie algebra, classification, multiplication table
MSC 17B05, 17D99

1 Introduction

In 1985, Filippov [9] introduced the concept of n-Lie algebras and classified the $(n+1)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. The structure of n-Lie algebras is very different from that of Lie algebras due to the n-ary multilinear operations involved [16]. The $n = 3$ case, i.e., 3-ary multilinear operation, first appeared in Nambu's work [22] in the description of simultaneous classical dynamics of three particles. In that work, Nambu extended the Poisson bracket and arrived at the generalized Hamiltonian equation involving a 3-ary multilinear bracket $\{\cdot,\cdot,\cdot\}$. Takhtajan [27] investigated the geometrical and algebraic aspects of the generalized Nambu mechanics, and established the connection between the Nambu mechanics and Filippov's theory of n-Lie algebras.

The development of n-Lie algebras has opened a new chapter in the study of Lie theory, attracting much attention in different research areas due to their close connections with dynamics, geometries, as well as string and membrane theories. Bagger and Lambert [1] and Gustavsson [13] proposed a field theory model for multiple M2-branes (BLG model) based on the real metric 3-Lie algebras. Metric n-Lie algebras (for low values of n) had appeared earlier in [8], but this was a different (yet not completely unrelated) context. The metric 3-Lie

Received October 27, 2010; accepted January 12, 2011
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algebras in the BLG model were initially assumed to be positive-definite, but it was soon realized in [10,23], rediscovering independently earlier work [21], that there was a unique nonabelian indecomposable 3-Lie algebra. This prompted the search for metric 3-Lie algebras of indefinite signature. The same examples of Lorentzian (i.e., index 1) metric 3-Lie algebras were found independently in [4,11,14]. These were shown in [18] to be all the indecomposable ones. The classification for index 2 was given in [19] and for general index but subject to the condition of having a maximally isotropic center in [20] and also albeit incompletely in the earlier work [15]. Some of these results were extended from 3-Lie algebras to n-Lie algebras in separate mathematical developments not necessarily related to the BLG proposal [6,7,24].

It is known that, up to isomorphisms, there is a unique finite-dimensional simple n-Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero [17], which is $(n+1)$-dimensional. Cantarini and Kac [5] gave a classification of linearly compact simple n-Lie algebras in zero characteristic. Examples of infinite-dimensional simple n-Lie algebras over fields of characteristic $p \geqslant 0$ are Jacobian algebras and their quotient algebras [25,26]. Bai et al. [2] showed that there exist only $[\frac{n}{2}]+1$ classes of $(n+1)$-dimensional simple n-Lie algebras over a complete field of characteristic 2. They also showed that there are no simple $(n+2)$-dimensional n-Lie algebras.

In [3], 6-dimensional 4-Lie algebras were classified and some basic properties of $(n+2)$-dimensional n-Lie algebras were studied. The purpose of this paper is to classify the $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic zero. Our results are expected to be useful in various applications.

The organization for the rest of this paper is as follows. Section 2 introduces some basic notion. Section 3 is devoted to the properties and classification of the $(n+2)$-dimensional n-Lie algebras.

Throughout this paper, all n-Lie algebras are of finite dimension and over an algebraically closed field F of characteristic zero. Any bracket which is not listed in the multiplication table of an n-Lie algebra is assumed to be zero.

2 Fundamental notion

An *n-Lie algebra* is a vector space A over a field F equipped with an n-multilinear operation $[x_1, \ldots, x_n]$ satisfying

$$[x_1, \ldots, x_n] = \mathrm{sgn}(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], \tag{2.1}$$

and

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n] \tag{2.2}$$

for any $x_1, \ldots, x_n, y_2, \ldots, y_n \in A$ and any permutation $\sigma \in S_n$. Identity (2.2) is usually called the *generalized Jacobi identity*, or simply *the Jacobi identity*.

A *derivation* of an n-Lie algebra A is a linear map D of A into itself satisfying

$$D([x_1,\ldots,x_n]) = \sum_{i=1}^{n}[x_1,\ldots,D(x_i),\ldots,x_n] \tag{2.3}$$

for any $x_1,\ldots,x_n \in A$. Let $\mathrm{Der}(A)$ be the set of all derivations of A. Then $\mathrm{Der}(A)$ is a Lie subalgebra of the general linear Lie algebra $\mathrm{gl}(A)$ and is called *the derivation algebra* of A. The map $\mathrm{ad}(x_1,\ldots,x_{n-1})\colon A \to A$, given by

$$\mathrm{ad}(x_1,\ldots,x_{n-1})(x_n) = [x_1,\ldots,x_n], \quad \forall\ x_n \in A,$$

is called *a left multiplication* defined by elements $x_1, \ldots, x_{n-1} \in A$. It follows from identity (2.2) that $\mathrm{ad}(x_1,\ldots,x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $\mathrm{Der}(A)$, which is denoted by $\mathrm{ad}(A)$. Every derivation in $\mathrm{ad}(A)$ is called *an inner derivation.*

If a subspace B of an n-Lie algebra A satisfying $[x_1,\ldots,x_n] \in B$ for any $x_1,\ldots,x_n \in B$, then B is called *a subalgebra* of A. Let $A_1, A_2, \ldots, A_n$ be subalgebras of an n-Lie algebra A. Denote by $[A_1, A_2, \ldots, A_n]$ the subspace of A generated by all vectors $[x_1,\ldots,x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra $A^1 = [A, A, \ldots, A]$ is called *the derived algebra* of A. If $A^1 = 0$, then A is called *an abelian n-Lie algebra.*

Let H be a proper abelian subalgebra of n-Lie algebra A. Then H is by definition *a toral subalgebra* of A, if A is a complete H-module, that is,

$$A = \oplus_{\alpha\in(H^{n-1})^*} A_\alpha \quad \text{(direct sum as vector spaces)},$$

where

$$A_\alpha = \{x \in A \mid \mathrm{ad}(h_1,\ldots,h_{n-1})(x) = \alpha(h_1,\ldots,h_{n-1})(x), \\ \forall\ (h_1,h_2,\ldots,h_{n-1}) \in H^{n-1}\}.$$

A toral subalgebra H is called *maximal* if there is no toral subalgebra of A properly containing H. An *ideal* I of an n-Lie algebra A is a subspace of A such that $[I, A, \ldots, A] \subseteq I$. If $[I, I, A, \ldots, A] = 0$, then I is called *an abelian ideal.* If $A^1 \neq 0$ and A has no ideal except 0 and itself, then A is called *a simple n-Lie algebra.* An n-Lie algebra A is said to be *decomposable* if there are nonzero ideals I_1, I_2 such that $A = I_1 \oplus I_2$. Otherwise, A is *indecomposable.* Clearly, if A is a simple n-Lie algebra, then A is indecomposable.

The subset

$$Z(A) = \{x \in A \mid [x, y_1,\ldots,y_{n-1}] = 0,\ \forall\ y_1,\ldots,y_{n-1} \in A\}$$

is called the *center* of A. It is clear that $Z(A)$ is an abelian ideal of A.

3 Classification of $(n+2)$-dimensional n-Lie algebras

We need some symbols for reducing our description. Suppose that $[,\ldots,]_1$ and

$[,\dots,]_2$ are two n-ary multiplications on a vector space A such that $(A,[,\dots,]_1)$ and $(A,[,\dots,]_2)$ are n-Lie algebras. Let $e_1,e_2,\dots,e_{n+2}$ be a basis of A. Set

$$e_{i,j}=[e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+2}]_1=\sum_{k=1}^{n+2}b_{i,j}^k e_k \tag{3.1}$$

for $b_{i,j}^k\in F,\ 1\leqslant i<j\leqslant n+2$. Then

$$(e_{1,2},e_{1,3},\dots,e_{1,n+2},e_{2,3},\dots,e_{2,n+2},\dots,e_{n+1,n+2})=(e_1,e_2,\dots,e_{n+2})B,$$

$$B=\begin{pmatrix} b_{1,2}^1 & b_{1,3}^1 & \cdots & b_{1,n+2}^1 & b_{2,3}^1 & \cdots & b_{n+1,n+2}^1\\ b_{1,2}^2 & b_{1,3}^2 & \cdots & b_{1,n+2}^2 & b_{2,3}^2 & \cdots & b_{n+1,n+2}^2\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ b_{1,2}^{n+2} & b_{1,3}^{n+2} & \cdots & b_{1,n+2}^{n+2} & b_{2,3}^{n+2} & \cdots & b_{n+1,n+2}^{n+2}\end{pmatrix}.$$

The multiplication of $(A,[,\dots,]_1)$ is determined by the $(n+2)\times\frac{(n+1)(n+2)}{2}$ matrix B, and B is called the *structure matrix* of $(A,[,\dots,]_1)$ with respect to the basis $e_1,e_2,\dots,e_{n+2}$.

Similarly, denote $\overline{B}$ as the structure matrix of $(A,[,\dots,]_2)$ with respect to the basis $e_1,e_2,\dots,e_{n+2}$, that is, for $1\leqslant i<j\leqslant n+2$,

$$\overline{e}_{ij}=[e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+2}]_2=\sum_{k=1}^{n+2}\overline{b}_{i,j}^k e_k,\quad \overline{b}_{i,j}^k\in F, \tag{3.2}$$

$$(\overline{e}_{1,2},\overline{e}_{1,3},\dots,\overline{e}_{1,n+2},\overline{e}_{2,3},\dots,\overline{e}_{2,n+2},\dots,\overline{e}_{n+1,n+2})=(e_1,\dots,e_{n+2})\overline{B}.$$

We have the following isomorphic criterion theorem for $(n+2)$-dimensional n-Lie algebras.

Theorem 3.1 *n-Lie algebras $(A,[,\dots,]_1)$ and $(A,[,\dots,]_2)$ with products* (3.1) *and* (3.2) *on an $(n+2)$-dimensional linear space A are isomorphic if and only if there exists a nonsingular $(n+2)\times(n+2)$ matrix $T=(t_{i,j})$ such that*

$$B=T^{-1}\overline{B}T_*, \tag{3.3}$$

where $T_*=(T_{k,l}^{i,j})$ is an $\frac{(n+1)(n+2)}{2}\times\frac{(n+1)(n+2)}{2}$ matrix, and $T_{k,l}^{i,j}\in F$ is the determinant defined by (3.5) *below for $1\leqslant i,j,k,l\leqslant n+2$.*

Proof Suppose that n-Lie algebra $(A,[,\dots,]_1)$ is isomorphic to $(A,[,\dots,]_2)$ under the isomorphism σ. Let $e_1,\dots,e_{n+2}$ be a basis of A, and let (3.1) and (3.2) be their structural matrices with respect to $e_1,\dots,e_{n+2}$, respectively, i.e.,

$$e_{i,j}=[e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+2}]_1=\sum_{k=1}^{n+2}b_{i,j}^k e_k,\quad B=(b_{i,j}^k)_{(n+2)\times\frac{(n+2)\times(n+2)}{2}};$$

and

$$\overline{e}_{i,j} = [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]_2 = \sum_{k=1}^{n+2} \overline{b}_{i,j}^k e_k, \quad \overline{B} = (\overline{b}_{i,j}^k)_{(n+2)\times\frac{(n+2)\times(n+2)}{2}}.$$

Denote $e'_i = \sigma(e_i)$, $1 \leqslant i \leqslant n+2$, and the nonsingular $(n+2)\times(n+2)$ matrix $T = (t_{ij})$ is the transition matrix of σ in the basis $e_1, e_2, \dots, e_{n+2}$, i.e.,

$$(\sigma(e_1), \dots, \sigma(e_{n+2})) = (e'_1, \dots, e'_{n+2}) = (e_1, e_2, \dots, e_{n+2})T. \tag{3.4}$$

Then

$$\begin{aligned} e'_{k,l} &= [e'_1, \dots, \hat{e}'_k, \dots, \hat{e}'_l, \dots, e'_{n+2}]_2 \\ &= \Big[\sum_{m=1}^{n+2} t_{m,1}e_m, \sum_{m=1}^{n+2} t_{m,2}e_m, \dots, \sum_{m=1}^{n+2} t_{m,k-1}e_m, \sum_{m=1}^{n+2} t_{m,k+1}e_m, \\ &\quad \dots, \sum_{m=1}^{n+2} t_{m,l-1}e_m, \sum_{m=1}^{n+2} t_{m,l+1}e_m, \dots, \sum_{m=1}^{n+2} t_{m,n+2}e_m\Big]_2 \\ &= T_{k,l}^{1,2}\overline{e}_{1,2} + T_{k,l}^{1,3}\overline{e}_{1,3} + \dots + T_{k,l}^{1,n+2}\overline{e}_{1,n+2} \\ &\quad + T_{k,l}^{2,3}\overline{e}_{2,3} + \dots + T_{k,l}^{n+1,n+2}\overline{e}_{n+1,n+2}, \end{aligned}$$

where $T_{k,l}^{i,j}$ is the following determinant:

$$\begin{vmatrix} t_{1,1} & \cdots & t_{1,k-1} & t_{1,k+1} & \cdots & t_{1,l-1} & t_{1,l+1} & \cdots & t_{1,n+2} \\ t_{2,1} & \cdots & t_{2,k-1} & t_{2,k+1} & \cdots & t_{2,l-1} & t_{2,l+1} & \cdots & t_{2,n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ t_{i-1,1} & \cdots & t_{i-1,k-1} & t_{i-1,k+1} & \cdots & t_{i-1,l-1} & t_{i-1,l+1} & \cdots & t_{i-1,n+2} \\ t_{i+1,1} & \cdots & t_{i+1,k-1} & t_{i+1,k+1} & \cdots & t_{i+1,l-1} & t_{i+1,l+1} & \cdots & t_{i+1,n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ t_{j-1,1} & \cdots & t_{j-1,k-1} & t_{j-1,k+1} & \cdots & t_{j-1,l-1} & t_{j-1,l+1} & \cdots & t_{j-1,n+2} \\ t_{j+1,1} & \cdots & t_{j+1,k-1} & t_{j+1,k+1} & \cdots & t_{j+1,l-1} & t_{j+1,l+1} & \cdots & t_{j+1,n+2} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ t_{n+1,1} & \cdots & t_{n+1,k-1} & t_{n+1,k+1} & \cdots & t_{n+1,l-1} & t_{n+1,l+1} & \cdots & t_{n+1,n+2} \\ t_{n+2,1} & \cdots & t_{n+2,k-1} & t_{n+2,k+1} & \cdots & t_{n+2,l-1} & t_{n+2,l+1} & \cdots & t_{n+2,n+2} \end{vmatrix}, \tag{3.5}$$

$1 \leqslant i < j \leqslant n+2$, $1 \leqslant k \neq l \leqslant n+2$. Denote

$$T_* = \begin{pmatrix} T_{1,2}^{1,2} & T_{1,3}^{1,2} & \cdots & T_{1,n+2}^{1,2} & T_{2,3}^{1,2} & \cdots & T_{n+1,n+2}^{1,2} \\ T_{1,2}^{1,3} & T_{1,3}^{1,3} & \cdots & T_{1,n+2}^{1,3} & T_{2,3}^{1,3} & \cdots & T_{n+1,n+2}^{1,3} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ T_{1,2}^{n,n+2} & T_{1,3}^{n,n+2} & \cdots & T_{1,n+2}^{n,n+2} & T_{2,3}^{n,n+2} & \cdots & T_{n+1,n+2}^{n,n+2} \\ T_{1,2}^{n+1,n+2} & T_{1,3}^{n+1,n+2} & \cdots & T_{1,n+2}^{n+1,n+2} & T_{2,3}^{n+1,n+2} & \cdots & T_{n+1,n+2}^{n+1,n+2} \end{pmatrix}. \tag{3.6}$$

Then T_* is an $\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2}$ matrix, and

$$(e'_{1,2}, \ldots, e'_{1,n+2}, e'_{2,3}, \ldots, e'_{n+1,n+2}) = (\overline{e}_{1,2}, \ldots, \overline{e}_{1,n+2}, \overline{e}_{2,3}, \ldots, \overline{e}_{n+1,n+2})T_*, \tag{3.7}$$

$$(e'_{1,2}, e'_{1,3}, \ldots, e'_{1,n+2}, e'_{2,3}, \ldots, e'_{n+1,n+2}) = (e_1, e_2, \ldots, e_{n+2})\overline{B}T_*. \tag{3.8}$$

$$\begin{aligned} e'_{k,l} &= [e'_1, \ldots, \hat{e}'_k, \ldots, \hat{e}'_l, \ldots, e'_{n+2}]_2 \\ &= \sigma(e_{kl}) \\ &= \sigma([e_1, \ldots, \hat{e}_k, \ldots, \hat{e}_l, \ldots, e_{n+2}]_1) \\ &= \sum_{i=1}^{n+2} b^i_{kl}\sigma(e_i) \\ &= \sum_{s=1}^{n+2} \Big(\sum_{i=1}^{n+2} b^i_{kl}\Big) t_{si} e_s, \end{aligned}$$

we have

$$(e'_{1,2}, e'_{1,3}, \ldots, e'_{1,n+2}, e'_{2,3}, \ldots, e'_{n+1,n+2}) = (e_1, e_2, \ldots, e_{n+2})TB. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$TB = \overline{B}T_*, \quad \text{i.e.,} \quad B = T^{-1}\overline{B}T_*.$$

On the other hand, let σ be a linear transformation of A, and let

$$\sigma(e_1, \ldots, e_{n+2}) = (e_1, \ldots, e_{n+2})T.$$

By the completely similar discussion to the above, σ is an n-Lie isomorphism from $(A, [, \ldots,]_1)$ to $(A, [, \ldots,]_2)$. $\square$

It is complex when we use Theorem 3.1 to judge the isomorphism of two $(n+2)$-dimensional n-Lie algebras due to the massive computations involved. However, from (3.5) and (3.3), the computation is orderly, and thus, it is easy to use the computer.

Before giving the classification theorem of $(n+2)$-dimensional n-Lie algebras, we need to refine the classification of $(n+1)$-dimensional n-Lie algebras given by Filippov [9].

Lemma 3.1 [9] *Let A be an $(n+1)$-dimensional n-Lie algebra over F and e_1, e_2, …, e_{n+1} be a basis of A ($n \geqslant 3$). Then one and only one of the following possibilities holds up to isomorphisms*:

(a) *If $\dim A^1 = 0$, then A is an abelian n-Lie algebra.*

(b) *If $\dim A^1 = 1$, let $A^1 = Fe_1$. Then in the case that $A^1 \subseteq Z(A)$,*

(b^1)

$$[e_2, \ldots, e_{n+1}] = e_1; \tag{3.10}$$

if A^1 is not contained in $Z(A)$, then

(b^2)

$$[e_1, \dots, e_n] = e_1. \tag{3.11}$$

(c) *If* $\dim A^1 = 2$, *let* $A^1 = Fe_1 + Fe_2$. *Then*

(c^1)

$$\begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases} \tag{3.12a}$$

(c^2)

$$\begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases} \tag{3.12b}$$

(c^3)

$$\begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, e_{n+1}] = e_2, \end{cases} \tag{3.12c}$$

where $\alpha \in F$ *and* $\alpha \neq 0$.

(d) *If* $\dim A^1 = r,\ 3 \leqslant r \leqslant n+1$, *let* $A^1 = Fe_1 + Fe_2 + \cdots + Fe_r$. *Then*

(d^r)

$$[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = e_i, \quad 1 \leqslant i \leqslant r, \tag{3.13}$$

where symbol $\hat{e}_i$ *means that* e_i *is omitted.*

Proof By [9], we only need to study the case $\dim A^1 = 2$. Let

$$A^1 = Fe_1 + Fe_2, \quad e^i = (-1)^{n+1+i}[e_1, \dots, \hat{e}_i, \dots, e_{n+1}], \quad 3 \leqslant i \leqslant n+1.$$

Then we have

$$(1) \quad \begin{cases} e^1 = (-1)^{n+1+1}[e_2, \dots, e_{n+1}] = ae_1 + ce_2, & a, b, c, d \in F, \\ e^2 = (-1)^{n+1+2}[e_1, e_3, \dots, e_{n+1}] = be_1 + de_2, & ad - bc \neq 0. \end{cases}$$

If $a \neq 0$, let

$$P = \begin{pmatrix} 1/\sqrt{a} & 0 \\ -c/\sqrt{(ad-bc)a} & \sqrt{a}/\sqrt{ad-bc} \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$(\det P^{-1})PAP' = \begin{pmatrix} \sqrt{ad-bc} & b-c \\ 0 & \sqrt{ad-bc} \end{pmatrix}.$$

By [9, Theorem 2], (1) is isomorphic to

$$(1') \quad \begin{cases} e^1 = (-1)^n[e_2, \dots, e_{n+1}] = \sqrt{ad-bc}\, e_1, \\ e^2 = (-1)^{n+1}[e_1, e_3, \dots, e_{n+1}] = (b-c)e_1 + \sqrt{ad-bc}\, e_2. \end{cases}$$

By the properly linear transformations on $e_1, \dots, e_{n+1}$, in the case of $b-c=0$, (1) is isomorphic to (c^1); in the case of $b-c \neq 0$, (1) is isomorphic to (c^2), and

$$\alpha = -\frac{ad-bc}{(b-c)^2} \neq 0.$$

If $a=0$, then in the cases of $d \neq 0$ or $b+c \neq 0$, (1) is isomorphic to the case (c^1) or (c^2); in the case of $d=b+c=0$, (1) is isomorphic to (c^3). □

Lemma 3.2 [3] *Let A be a nonabelian $(n+2)$-dimensional n-Lie algebra over F. If $\dim A^1 \neq 3$, then there exists a nonabelian subalgebra of codimension 1 containing A^1.*

Lemma 3.3 [3] *Let A be an $(n+2)$-dimensional n-Lie algebra over F. Then we have $\dim A^1 \leqslant n+1$.*

Theorem 3.2 *Let A be an $(n+2)$-dimensional n-Lie algebra over F with a basis $e_1, \dots, e_{n+2}$. Then one and only one of the following possibilities holds up to isomorphisms*:

(a) *If $\dim A^1 = 0$, then A is an abelian n-Lie algebra.*

(b) *If $\dim A^1 = 1$, let $A^1 = Fe_1$. Then we have*

(b^1) *in the case that $A^1 \subseteq Z(A)$,* $[e_2, \dots, e_{n+1}] = e_1$;

(b^2) *in the case that A^1 is not contained in $Z(A)$,* $[e_1, \dots, e_n] = e_1$.

(c) *If $\dim A^1 = 2$, let $A^1 = Fe_1 + Fe_2$. Then we have*

$$(c^1) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_2; \end{cases}$$

$$(c^2) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{cases}$$

$$(c^3) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases}$$

$$(c^4) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{cases}$$

$$(c^5) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases}$$

$$(c^6) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{cases}$$

$$(\mathrm{c}^7)\quad \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \end{cases}$$

where $\alpha \in F$, *and* $\alpha \neq 0$.

(d) *If* $\dim A^1 = 3$, *let*

$$A^1 = Fe_1 + Fe_2 + Fe_3.$$

Then we have

$$(\mathrm{d}^1)\quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = -e_2, \\ [e_3, \dots, e_{n+2}] = e_3; \end{cases}$$

$$(\mathrm{d}^2)\quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_3 + \alpha e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{cases}$$

$$(\mathrm{d}^3)\quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, e_4, \dots, e_{n+2}] = e_3, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = 2e_1; \end{cases}$$

$$(\mathrm{d}^4)\quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3; \end{cases}$$

$$(\mathrm{d}^5)\quad \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = \beta e_2 + (1+\beta) e_3, \quad \beta \in F,\ \beta \neq 0, 1; \end{cases}$$

$$(\mathrm{d}^6)\quad \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_3, e_4, \dots, e_{n+2}] = e_3; \end{cases}$$

$$(\mathrm{d}^7)\quad \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = s e_1 + t e_2 + u e_3, \quad s, t, u \in F,\ s \neq 0. \end{cases}$$

And n-Lie algebras corresponding to case (d^7) *with coefficients* s, t, u *and* s', t', u' *are isomorphic if and only if there exists a nonzero element* $r \in F$ *such that*

$$s = r^3 s', \quad t = r^2 t', \quad u = r u', \quad s, s', t, t', u, u' \in F.$$

(r) *If* $\dim A^1 = r,\ 4 \leqslant r \leqslant n+1,$ *let*

$$A^1 = Fe_1 + \dots + Fe_r.$$

Then we have

$$(\mathrm{r}^1)\quad \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_3,\ldots,e_{n+2}]=e_2,\\ \ldots,\\ [e_2,\ldots,\hat{e_i},\ldots,e_r,\ldots,e_{n+2}]=e_i,\\ \ldots,\\ [e_2,\ldots,e_{r-1},e_{r+1},\ldots,e_{n+2}]=e_r; \end{cases}$$

$$(\mathrm{r}^2)\quad \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ \ldots,\\ [e_1,\ldots,\hat{e_i},\ldots,e_r,\ldots,e_{n+1}]=e_i,\\ \ldots,\\ [e_1,\ldots,e_{r-1},e_{r+1},\ldots,e_{n+1}]=e_r. \end{cases}$$

Proof 1. Case (a) is trivial.

2. Case (b). Suppose $A^1=Fe_1$. Then from Lemmas 3.1–3.3, the multiplication table of A has the following possibilities:

$$(1)\quad \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_1,\ldots,\hat{e_i},\ldots,\hat{e_j},\ldots,e_{n+1},e_{n+2}]=b_{ij}e_1; \end{cases}$$

$$(2)\quad \begin{cases} [e_1,\ldots,e_n]=e_1,\\ [e_1,\ldots,\hat{e_i},\ldots,\hat{e_j},\ldots,e_{n+1},e_{n+2}]=b_{ij}e_1, \end{cases}$$

where $b_{ij}\in F,\ 1\leqslant i<j\leqslant n+1$.

First, substituting the first identity of (1) into the other equations and using the Jacobi identities, we get

$$\begin{aligned} b_{ij}e_1 &= [e_1,e_2,\ldots,\hat{e_i},\ldots,\hat{e_j},\ldots,e_{n+1},e_{n+2}]\\ &= [[e_2,\ldots,e_{n+1}],e_2,\ldots,\hat{e_i},\ldots,\hat{e_j},\ldots,e_{n+1},e_{n+2}]\\ &= b_{1j}[e_1,e_2,\ldots,\hat{e_i},\ldots,e_{n+1}]+b_{1i}[e_1,e_2,\ldots,\hat{e_j},\ldots,\ldots e_{n+1}]\\ &= 0,\quad 2\leqslant i<j\leqslant n+1. \end{aligned}$$

Then (1) is in the form of

$$(1')\quad \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_2,\ldots,\hat{e_j},\ldots,e_{n+1},e_{n+2}]=b_{1j}e_1,\quad 2\leqslant j\leqslant n+1. \end{cases}$$

Replacing e_{n+2} by

$$e_{n+2}-\sum_{j=2}^{n+1}(-1)^{n+1-j}b_{1j}e_j$$

in $(1')$, we get that (1) is isomorphic to (b^1).

By the similar discussion to above, (2) is isomorphic to (b^2). And (b^1) is not isomorphic to (b^2) since (b^1) has a nonzero center.

3. If $\dim A^1=2$, suppose $A^1=Fe_1+Fe_2$. Then the multiplication table of A in the basis $e_1,\ldots,e_{n+2}$ has the following possibilities:

$$(1)\quad \begin{cases} [e_2,\dots,e_{n+1}]=e_1, \\ [e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2; \end{cases}$$

$$(2)\quad \begin{cases} [e_1,\dots,e_n]=e_1, \\ [e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2; \end{cases}$$

$$(3)\quad \begin{cases} [e_2,\dots,e_{n+1}]=e_1, \\ [e_1,e_3,\dots,e_{n+1}]=e_2, \\ [e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2; \end{cases}$$

$$(4)\quad \begin{cases} [e_2,\dots,e_{n+1}]=\alpha e_1+e_2, \\ [e_1,e_3,\dots,e_{n+1}]=e_2, \\ [e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2, \end{cases}$$

$$(5)\quad \begin{cases} [e_1,e_3,\dots,e_{n+1}]=e_1, \\ [e_2,e_3,\dots,e_{n+1}]=e_2, \\ [e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2, \end{cases}$$

where $b_{ij}\in F,\ 1\leqslant i<j\leqslant n+1$.

First, imposing the Jacobi identities on (1), we get

$$\begin{aligned} b_{ij}^1e_1+b_{ij}^2e_2 &= [e_1,e_2,e_3,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}] \\ &= [[e_2,\dots,e_{n+1}],e_2,e_3,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}] \\ &= 0, \quad 3\leqslant i<j\leqslant n+1. \end{aligned}$$

If $i=2$ and $3\leqslant j\leqslant n+1$, then

$$\begin{aligned} b_{2j}^1e_1+b_{2j}^2e_2 &= [e_1,e_3,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}] \\ &= [[e_2,\dots,e_{n+1}],e_3,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}] \\ &= b_{1j}^2e_1. \end{aligned}$$

Replacing e_{n+2} by

$$e_{n+2}-\sum_{j=2}^{n+1}(-1)^{n+1-j}b_{1j}^1e_j,$$

we get

$$(1')\quad \begin{cases} [e_2,\dots,e_{n+1}]=e_1, \\ [e_3,\dots,e_{n+2}]=b_{12}^2e_2, \\ [e_2,e_3,\dots,\hat{e}_j\dots,e_{n+1},e_{n+2}]=b_{1j}^2e_2, \quad 3\leqslant j\leqslant n+1, \\ [e_1,e_3,\dots,\hat{e}_j\dots,e_{n+1},e_{n+2}]=b_{1j}^2e_1, \quad 3\leqslant j\leqslant n+1. \end{cases}$$

If $b_{12}^2\neq 0,\ b_{1j}^2=0,\ 3\leqslant j\leqslant n+1$, substituting e_{n+2}/b_{12}^2 for e_{n+2} in $(1')$, (1) is isomorphic to (c^1). If there exists j such that $b_{1j}^2\neq 0,\ 3\leqslant j\leqslant n+1$, then we might as well suppose $b^2_{13} \neq 0$. Substituting

$$e_3 + \sum_{j=4}^{n+1} (-1)^{j-3} \frac{b^2_{1j}}{b^2_{13}}\, e_j - \frac{b^2_{12}}{b^2_{13}}\, e_2$$

for e_3 and e_{n+2}/b^2_{13} for e_{n+2} in $(1')$, we get (c^2).

Second, substituting

$$e_{n+2} - \sum_{i=1}^{n} (-1)^{n-i} b^1_{in+1} e_i$$

for e_{n+2} in (2), we get

$$\begin{cases} [e_1, \ldots, e_n] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}] = b^1_{ij} e_1 + b^2_{ij} e_2, \quad 1 \leqslant i < j \leqslant n, \\ [e_1, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}] = b^2_{in+1} e_2, \quad 1 \leqslant i \leqslant n. \end{cases}$$

By the Jacobi identities, we have $b^2_{2n+1} = b^2_{1n+1} = 0$ and $b^2_{in+1} = 0,\ 3 \leqslant i \leqslant n$. Then (2) is isomorphic to

$$\begin{cases} [e_1, \ldots, e_n] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{ij} e_1 + b^2_{ij} e_2, \quad 1 \leqslant i < j \leqslant n. \end{cases}$$

If $i = 1,\ 2 \leqslant j \leqslant n$, since

$$[[e_1, e_3, \ldots, e_n, e_{n+2}], e_2, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}] = (-1)^{2n-3} b^2_{1j} e_1 = 0,$$

we have $b^2_{1j} = 0,\ 2 \leqslant j \leqslant n$. If $i = 2,\ 3 \leqslant j \leqslant n$, by

$$\begin{aligned} b^1_{2j} e_1 + b^2_{2j} e_2 &= [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}] \\ &= [[e_1, \ldots, e_n], e_3, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}] \\ &= b^1_{2j} e_1 + b^2_{1j} e_1 \\ &= b^1_{2j} e_1, \end{aligned}$$

we obtain $b^2_{2j} = 0,\ 3 \leqslant j \leqslant n$. If $3 \leqslant i < j \leqslant n$, by

$$\begin{aligned} b^1_{ij} e_1 + b^2_{ij} e_2 &= [e_1, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] \\ &= [[e_1, \ldots, e_n], e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] \\ &= b^1_{ij} e_1, \end{aligned}$$

we get $b^2_{ij} = 0,\ 3 \leqslant i < j \leqslant n$. Then (2) is isomorphic to

$$\begin{cases} [e_1, \ldots, e_n] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{ij} e_1, \quad 1 \leqslant i < j \leqslant n. \end{cases}$$

This contradicts $\dim A^1 = 2$. Therefore, table (2) is not realized.

Third, we study case (3). For $i = 1,\ 3 \leqslant j \leqslant n+1$, since

$$\begin{aligned} b_{1j}^1 e_1 + b_{1j}^2 e_2 &= [e_2, \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= [[e_1, e_3, \ldots, e_{n+1}], \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= b_{2j}^1 e_2 + b_{2j}^2 e_1, \end{aligned}$$

we have

$$b_{2j}^1 = b_{1j}^2, \quad b_{2j}^2 = b_{1j}^1, \quad 3 \leqslant j \leqslant n+1.$$

For $3 \leqslant i < j \leqslant n+1$, from

$$\begin{aligned} b_{ij}^1 e_1 + b_{ij}^2 e_2 &= [e_1, e_2, e_3, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] \\ &= [[e_2, \ldots, e_{n+1}], e_2, e_3, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] \\ &= 0, \end{aligned}$$

we have

$$b_{ij}^1 = b_{ij}^2 = 0, \quad 3 \leqslant i < j \leqslant n+1.$$

Again, substituting

$$e_{n+2} + \sum_{j=2}^{n+1} (-1)^{n+2-j} b_{1j}^1 e_j + (-1)^n b_{12}^2 e_1$$

for e_{n+2}, we obtain

$$(3') \quad \begin{cases} [e_2, \ldots, e_{n+1}] = e_1, \\ [e_1, e_3, \ldots, e_{n+1}] = e_2, \\ [e_2, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_2, \quad 3 \leqslant j \leqslant n+1, \\ [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_1, \quad 3 \leqslant j \leqslant n+1. \end{cases}$$

If $b_{1j}^2 = 0,\ 3 \leqslant j \leqslant n+1$, then (3) is isomorphic to (c^3).

If there exists $b_{1j}^2 \neq 0$ for $3 \leqslant j \leqslant n+1$, then we might as well suppose $b_{13}^2 \neq 0$. Replacing e_3 and e_{n+2} by

$$e_3 + \sum_{j=4}^{n+1} (-1)^{j-3} \frac{b_{1j}^2}{b_{13}^2} e_j$$

and e_{n+2}/b_{13}^2 in ($3'$), respectively, we get (c^4).

Fourth, we study case (4). For $i = 1,\ 3 \leqslant j \leqslant n+1$, by

$$\begin{aligned} b_{1j}^1 e_1 + b_{1j}^2 e_2 &= [e_2, \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= [[e_1, e_3, \ldots, e_{n+1}], \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= b_{2j}^1 e_2 + b_{2j}^2 \alpha e_1 + b_{2j}^2 e_2, \end{aligned}$$

we have

$$b^1_{1j} = b^2_{2j}\alpha, \quad b^2_{1j} = b^1_{2j} + b^2_{2j}, \quad 3 \leqslant j \leqslant n+1.$$

For $3 \leqslant i < j \leqslant n+1$, we have

$$b^1_{ij} = b^2_{ij} = 0, \quad 3 \leqslant i < j \leqslant n+1,$$

since

$$\begin{aligned} b^1_{ij}e_1 + b^2_{ij}e_2 &= [e_1, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= \frac{1}{\alpha}\,[[e_2, \ldots, e_{n+1}] - e_2, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= \frac{1}{\alpha}\,[e_2, \ldots, [e_i, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+2}], \ldots, e_{n+1}] \\ &= 0. \end{aligned}$$

If substituting

$$\sum_{j=3}^{n+1}(-1)^{n-j}b^2_{2j}e_j + (-1)^n b^2_{12}e_1 + e_{n+2}$$

for e_{n+2} in (4), we get

$$(4')\quad \begin{cases} [e_2, \ldots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \ldots, e_{n+1}] = e_2, \\ [e_3, \ldots, e_{n+2}] = b^1_{12}e_1, \\ [e_2, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{2j}e_2, \quad 3 \leqslant j \leqslant n+1, \\ [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{2j}e_1, \quad 3 \leqslant j \leqslant n+1. \end{cases}$$

If $b^1_{2j} = 0$ for $3 \leqslant j \leqslant n+1$, (4) is isomorphic to ($c^5$). If there exists $b^1_{2j} \neq 0$ for some $3 \leqslant j \leqslant n+1$, we might as well suppose $b^1_{23} \neq 0$. Substituting

$$e_3 + \sum_{j=4}^{n+1}(-1)^{j-3}\frac{b^1_{2j}}{b^1_{23}}\,e_j - \frac{b^1_{12}}{b^1_{23}}\,e_1$$

and e_{n+2}/b^1_{23} for e_3 and e_{n+2} in (4′), respectively, we get (c^6).

Lastly, we study case (5). For $3 \leqslant i < j \leqslant n+1$, we have $b^1_{ij} = b^2_{ij} = 0$ since

$$[e_1, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+2}] = [[e_1, e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+2}], e_3, \ldots, e_{n+1}].$$

Then if substituting

$$e_{n+2} + (-1)^n b^1_{12}e_1 - \sum_{i=1}^{n+1}(-1)^{n+1-i}b^2_{1i}e_i$$

for e_{n+2} in (5), we get

$$(5')\quad \begin{cases} [e_1, e_3, \ldots, e_{n+1}] = e_1, \\ [e_2, \ldots, e_{n+1}] = e_2, \\ [e_2, \ldots, \hat{e}_j, \ldots, e_{n+2}] = b^1_{1j} e_1, \quad 3 \leqslant j \leqslant n+1, \\ [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}] = b^1_{2j} e_1 + b^2_{2j} e_2, \quad 3 \leqslant j \leqslant n+1. \end{cases}$$

We discuss (5′) in two steps.

Step 1 If $b^1_{1j} = 0$ for $3 \leqslant j \leqslant n+1$, then (5) is isomorphic to

$$(5'')\quad \begin{cases} [e_1, e_3, \ldots, e_{n+1}] = e_1, \\ [e_2, e_3, \ldots, e_{n+1}] = e_2, \\ [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{2j} e_1 + b^2_{2j} e_2, \quad 3 \leqslant j \leqslant n+1. \end{cases}$$

For every j, $4 \leqslant j \leqslant n+1$, since

$$\begin{aligned} b^1_{23}[e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}] &= [[e_1, e_4, \ldots, e_{n+2}], e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}] \\ &= b^1_{2j}[e_1, e_4, \ldots, e_{n+2}], \end{aligned}$$

we have $b^1_{23} b^2_{2j} = b^2_{23} b^1_{2j}$. Then $[e_1, e_4, \ldots, e_{n+2}]$ and $[e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}]$, $4 \leqslant j \leqslant n+1$, are linearly dependent. If

$$[e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}] = 0, \quad 3 \leqslant j \leqslant n+1,$$

then (5) is isomorphic to (c^7). If there exists

$$[e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+2}] \neq 0, \quad 3 \leqslant j \leqslant n+1,$$

then A is isomorphic to (c^4) when $b^1_{23} \neq 0$, and A is isomorphic to (c^2) when $b^2_{23} \neq 0$.

Step 2 If there exists $b^1_{1j} \neq 0$ for some $3 \leqslant j \leqslant n+1$. The we might as well suppose $b^1_{13} \neq 0$. Substituting

$$e_3 - \sum_{j=4}^{n+1} (-1)^{j-4} \frac{b^1_{1j}}{b^1_{13}} e_j$$

and e_{n+2}/b^1_{13} for e_3 and e_{n+2} in (5′), respectively, (5) is isomorphic to

$$\begin{cases} [e_1, e_3, \ldots, e_{n+1}] = e_1, \\ [e_2, e_3, \ldots, e_{n+1}] = e_2, \\ [e_2, e_4, \ldots, e_{n+1}, e_{n+2}] = e_1, \\ [e_1, e_3, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = b^1_{2j} e_1 + b^2_{2j} e_2, \quad 3 \leqslant j \leqslant n+1. \end{cases}$$

The discussion is completely similar to Step 1. (5) is isomorphic to (c^2), (c^4), or (c^6) for the cases that b^1_{2j}, b^2_{2j} being to zero simultaneously, or not.

Now, we prove that (c^i) is not isomorphic to (c^j) when $i \neq j$ for $1 \leqslant i, j \leqslant 7$. Case (c^1) is not isomorphic to (c^3), (c^5), and (c^7) since it is indecomposable. By

Lemma 3.1, (c^i) is not isomorphic to (c^j) when $i \neq j$ for $i,j=3,5,7$. And (c^j) for $j=1,3,5,7$ are not isomorphic to (c^2), (c^4), (c^6) since they have nonzero center.

For cases (c^i), $i=2,4,6$, we have Lie algebras $A_i=A$ (as vector spaces) for $i=2,4,6$, respectively, with product $[\cdot,\cdot]_1$ as follows:

$$(\mathrm{c}^2)_1\begin{cases}[e_2,e_3]_1=e_1,\\ [e_2,e'_{n+2}]_1=e_2,\\ [e_1,e'_{n+2}]_1=e_1;\end{cases}$$

$$(\mathrm{c}^4)_1\begin{cases}[e_2,e_3]_1=e_1,\\ [e_1,e_3]_1=e_2,\\ [e_2,e'_{n+2}]_1=e_2,\\ [e_1,e_{n+2}]_1=e_1;\end{cases}$$

$$(\mathrm{c}^6)_1\begin{cases}[e_2,e_3]_1=\alpha e_1+e_2,\\ [e_1,e_3]_1=e_2,\\ [e_2,e'_{n+2}]_1=e_2,\\ [e_1,e'_{n+2}]_1=e_1,\end{cases}$$

where

$$[x,y]_1=[x,y,e_4,\ldots,e_{n+1}],\quad x,y\in A,$$

$$e'_{n+2}=(-1)^n e_{n+2}.$$

And A_i has decomposition $A_i=Z(A_i)\dot{+}B_i$ (the direct sum as ideals), where

$$B_i=Fe_1+Fe_2+Fe_3+Fe_{n+2},\quad i=2,4,6,$$

are 4-dimensional solvable Lie algebras with multiplication table $(\mathrm{c}^i)_1$, respectively. The subalgebra

$$H=Fe_3+\cdots+Fe_{n+2}$$

is a Cartan subalgebra of (c^2), (c^4), and (c^6), and the vectors $e_4,\ldots,e_{n+1}$ have the symmetric status in the multiplication. Then (c^i) is isomorphic to (c^j) if and only if the Lie algebra $(\mathrm{c}^i)_1$ is isomorphic to $(\mathrm{c}^j)_1$. By the classification [12] of 4-dimensional solvable Lie algebras, $(\mathrm{c}^i)_1$ is not isomorphic to $(\mathrm{c}^j)_1$ for $i\neq j$. Then we get (c^i) is not isomorphic to (c^j) when $i\neq j$. And the n-Lie algebra of case (c^6) with coefficient α is isomorphic to that with coefficient α' if and only if $\alpha=\alpha'$.

Summarizing, we get that (c^i) is not isomorphic to (c^j) if $1\leqslant i\neq j\leqslant 7$.

4. Let $\dim A^1=3$ and

$$A^1=Fe_1+Fe_2+Fe_3.$$

Then the multiplication table of A has only the following possibilities:

$$(1)\ \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(2)\ \begin{cases} [e_1,\ldots,e_n]=e_1,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(3)\ \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_1,e_3,\ldots,e_{n+1}]=e_2,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(4)\ \begin{cases} [e_2,\ldots,e_{n+1}]=\alpha e_1+e_2,\\ [e_1,e_3,\ldots,e_{n+1}]=e_2,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(5)\ \begin{cases} [e_1,e_3,\ldots,e_{n+1}]=e_1,\\ [e_2,e_3,\ldots,e_{n+1}]=e_2,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(6)\ \begin{cases} [e_2,\ldots,e_{n+1}]=e_1,\\ [e_1,e_3,\ldots,e_{n+1}]=e_2,\\ [e_1,e_2,e_4,\ldots,e_{n+1}]=e_3,\\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}]=b_{ij}^1e_1+b_{ij}^2e_2+b_{ij}^3e_3; \end{cases}$$

$$(7)\ \begin{cases} [e_3,e_4,\ldots,e_{n+2}]=b_{12}^1e_1+b_{12}^2e_2+b_{12}^3e_3,\\ [e_2,e_4,\ldots,e_{n+2}]=b_{13}^1e_1+b_{13}^2e_2+b_{13}^3e_3,\\ [e_1,e_4,\ldots,e_{n+2}]=b_{23}^1e_1+b_{23}^2e_2+b_{23}^3e_3, \end{cases}$$

where $b_{ij}\in F,\ 1\leqslant i<j\leqslant n+1$.

First, we study case (1). Substituting the first identity into the other equations, we get

$$\sum_{k=1}^{3} b_{ij}^k e_k=0,\quad 4\leqslant i<j\leqslant n+1;$$
$$\sum_{k=1}^{3} b_{2j}^k e_k=b_{1j}^2e_1,\quad 4\leqslant j\leqslant n+1;$$
$$\sum_{k=1}^{3} b_{3j}^k e_k=-b_{1j}^3e_1,\quad 4\leqslant j\leqslant n+1;$$
$$\sum_{k=1}^{3} b_{23}^k e_k=b_{13}^2e_1+b_{12}^3e_1.$$

Replace

$$e_{n+2}-\sum_{j=2}^{n+1}(-1)^{n+1-j}b_{1j}^1e_j$$

for e_{n+2}. Then (1) is isomorphic to

$$(1')\quad \begin{cases} [e_2,\dots,e_{n+1}]=e_1,\\ [e_3,\dots,e_{n+2}]=b_{12}^2e_2+b_{12}^3e_3,\\ [e_2,e_4,\dots,e_{n+2}]=b_{13}^2e_2+b_{13}^3e_3,\\ [e_2,e_3,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{1j}^2e_2+b_{1j}^3e_3,\quad 4\leqslant j\leqslant n+1,\\ [e_1,e_4,e_5,\dots,e_{n+2}]=b_{23}^1e_1=(b_{13}^2+b_{12}^3)e_1,\\ [e_1,e_3,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{2j}^1e_1=b_{1j}^2e_1,\quad 4\leqslant j\leqslant n+1,\\ [e_1,e_2,\dots,\hat{e}_j,\dots,e_{n+1},e_{n+2}]=b_{3j}^1e_1=-b_{1j}^3e_1,\quad 4\leqslant j\leqslant n+1. \end{cases}$$

Fixing e_{n+2} in the n-ary multiplication of A, we get an $(n+2)$-dimensional $(n-1)$-Lie algebra $A_0=A$ (as vector space) with the product $[,\dots,]_0$, and the multiplication table of A_0 in the basis $e_1,\dots,e_{n+2}$ is as follows:

$$\begin{cases} [e_3,e_4,\dots,e_{n+1}]_0=b_{12}^2e_2+b_{12}^3e_3,\\ [e_2,e_4,\dots,e_{n+1}]_0=b_{13}^2e_2+b_{13}^3e_3,\\ [e_2,e_3,\dots,\hat{e}_j,\dots,e_{n+1}]_0=b_{1j}^2e_2+b_{1j}^3e_3,\quad 4\leqslant j\leqslant n+1,\\ [e_1,e_4,e_5,\dots,e_{n+1}]_0=b_{23}^1e_1=(b_{13}^2+b_{12}^3)e_1,\\ [e_1,e_3,\dots,\hat{e}_j,\dots,e_{n+1}]_0=b_{2j}^1e_1=b_{1j}^2e_1,\quad 4\leqslant j\leqslant n+1,\\ [e_1,e_2,\dots,\hat{e}_j,\dots,e_{n+1}]_0=b_{3j}^1e_1=-b_{1j}^3e_1,\quad 4\leqslant j\leqslant n+1. \end{cases}$$

Set $B=Fe_2+\cdots+Fe_{n+1}$. Then B is a subalgebra of A_0, $\dim B^1=2$ since $\dim A_0^1=\dim A^1=3$. The multiplication table of B in the basis $e_2,\dots,e_{n+1}$ is as follows:

$$\begin{cases} [e_3,e_4,\dots,e_{n+1}]_0=b_{12}^2e_2+b_{12}^3e_3,\\ [e_2,e_4,\dots,e_{n+1}]_0=b_{13}^2e_2+b_{13}^3e_3,\\ [e_2,e_3,e_4,\dots,\hat{e}_j,\dots,e_{n+1}]_0=b_{1j}^2e_2+b_{1j}^3e_3,\quad 4\leqslant j\leqslant n+1. \end{cases}$$

By discussion completely similar to [9],

$$\Delta=\begin{vmatrix} b_{12}^2 & b_{12}^3\\ b_{13}^2 & b_{13}^3 \end{vmatrix}\neq 0,\quad b_{1j}^2=b_{1j}^3=0,\quad 4\leqslant j\leqslant n.$$

Therefore, $(1')$ has the form

$$(1'')\quad \begin{cases} [e_2,\dots,e_{n+1}]=e_1,\\ [e_3,\dots,e_{n+2}]=b_{12}^2e_2+b_{12}^3e_3,\\ [e_2,e_4,\dots,e_{n+2}]=b_{13}^2e_2+b_{13}^3e_3,\\ [e_1,e_4,e_5,\dots,e_{n+2}]=(b_{13}^2+b_{12}^3)e_1. \end{cases}$$

If $b_{13}^2+b_{12}^3=0$ and $b_{12}^2\neq 0$, taking a linear transformation for basis $e_1,\dots,e_{n+2}$ by replacing

$$\frac{2\sqrt{\Delta}}{b_{12}^2}e_1,\quad e_2+\frac{b_{12}^3-\sqrt{\Delta}}{b_{12}^2}e_3,\quad e_2+\frac{b_{12}^3+\sqrt{\Delta}}{b_{12}^2}e_3,\quad \dots,\quad \frac{1}{\sqrt{\Delta}}e_{n+2}$$

for $e_1,e_2,e_3,\dots,e_{n+2}$ in $(1'')$, respectively, (1) is isomorphic to (d^1). If $b_{12}^2=0$, (1) is also isomorphic to (d^1).

If $b_{13}^2 + b_{12}^3 \neq 0$, and $b_{12}^2 \neq 0$, taking a linear transformation for basis $e_1, \ldots, e_{n+2}$ by replacing

$$\frac{\Delta}{b_{12}^2(b_{13}^2 + b_{12}^3)}\, e_1, \quad e_2 + \frac{b_{12}^3}{b_{12}^2}\, e_3,$$
$$e_2 + \frac{1}{b_{12}^2}\Big(b_{12}^3 + \frac{\Delta}{b_{13}^2 + b_{12}^3}\Big)\, e_3, \quad \ldots, \quad \frac{1}{b_{13}^2 + b_{12}^3}\, e_{n+2}$$

for $e_1, e_2, e_3, \ldots, e_{n+2}$ in ($1''$), (1) is isomorphic to (d^2), and

$$\alpha = \frac{\Delta}{(b_{13}^2 + b_{12}^3)^2} \neq 0.$$

If $b_{12}^2 = 0$, (1) is isomorphic to (d^2) in the case of $b_{13}^3 \neq 0$, or $b_{13}^3 = 0$ and $b_{13}^2 \neq b_{12}^3$. In the case of $b_{13}^3 = 0$ and $b_{13}^2 = b_{12}^3$, it is evident that (1) is isomorphic to (d^3).

Second, substituting

$$e_{n+2} - \sum_{i=1}^{n} (-1)^{n-i} b_{in+1}^1 e_i$$

for e_{n+2} in (2), and using the Jacobi identities for

$$\{[e_1, \ldots, e_n], e_3, \ldots, e_n, e_{n+2}\}, \quad \{[e_1, \ldots, e_n], e_2, e_4, \ldots, e_n, e_{n+2}\},$$
$$\{[e_1, \ldots, e_n], e_2, e_3, e_4, \ldots, \hat{e}_i, \ldots, e_n, e_{n+2}\}, \quad 4 \leqslant i \leqslant n,$$
$$\{[e_1, e_2, e_4, \ldots, e_n, e_{n+2}], e_2, \ldots, \hat{e}_j, \ldots, e_{n+1}\}, \quad 2 \leqslant j \leqslant n,$$
$$\{[e_2, \ldots, e_{n+1}], e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}], \quad 2 \leqslant i \leqslant j \leqslant n,$$

we get $b_{ij}^3 = 0$ for $1 \leqslant i < j \leqslant n$. Hence, (2) is of the form

$$\begin{cases} [e_1, \ldots, e_n] = e_1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_n, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2, \quad 1 \leqslant i < j \leqslant n. \end{cases}$$

This contradicts $\dim A^1 = 3$. Therefore, case (2) is not realized.

Third, cases (3)–(5) are not realized by discussion similar to case (2).

Fourth, for $4 \leqslant i < j \leqslant n+1$, from table (6),

$$\sum_{k=1}^{3} b_{ij}^k e_k = [e_1, e_2, e_3, e_4, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = 0.$$

Then (6) has the form

$$(6') \quad \begin{cases} [e_2, \ldots, e_{n+1}] = e_1, \\ [e_1, e_3, \ldots, e_n] = e_2, \\ [e_1, e_2, e_4, \ldots, e_{n+1}] = e_3, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{3} b_{ij}^k e_k, \\ \qquad\qquad 1 \leqslant i \leqslant 3,\ i < j \leqslant n+1. \end{cases}$$

For $4 \leqslant j \leqslant n+1$, imposing the Jacobi identities for

$$\{[e_1, e_3, \ldots, e_{n+1}], e_3, e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{e_2, [e_1, e_2, e_4, \ldots, e_{n+1}], e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{[e_2, \ldots, e_{n+1}], e_3, e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{e_1, [e_1, e_2, e_4, \ldots, e_{n+1}], e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{[e_2, \ldots, e_{n+1}], e_2, e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{e_1, [e_1, e_3, \ldots, e_{n+1}], e_4, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\},$$
$$\{[e_1, e_2, e_4, \ldots, e_{n+1}], e_4, \ldots, e_{n+2}\},$$
$$\{[e_1, e_3, \ldots, e_{n+1}], e_4, \ldots, e_{n+2}\},$$
$$\{[e_2, \ldots, e_{n+1}], e_4, \ldots, e_{n+2}\},$$

we get

$$b_{1j}^3 = 0, \quad b_{1j}^1 = b_{2j}^2, \quad b_{1j}^2 = b_{2j}^1;$$
$$b_{1j}^2 = 0, \quad b_{1j}^3 = -b_{3j}^1, \quad b_{1j}^1 = b_{3j}^3;$$
$$b_{2j}^3 = 0, \quad b_{2j}^1 = b_{1j}^2, \quad b_{2j}^2 = b_{1j}^1;$$
$$b_{2j}^1 = 0, \quad b_{2j}^2 = b_{3j}^3, \quad b_{2j}^3 = b_{3j}^2;$$
$$b_{3j}^2 = 0, \quad b_{3j}^1 = -b_{1j}^3, \quad b_{3j}^3 = b_{1j}^1;$$
$$b_{3j}^1 = 0, \quad b_{3j}^2 = b_{2j}^3, \quad b_{3j}^3 = b_{2j}^2;$$
$$b_{12}^1 = -b_{23}^3, \quad b_{12}^2 = b_{13}^3, \quad b_{12}^3 = b_{13}^2 + b_{23}^1;$$
$$b_{13}^1 = b_{23}^2, \quad b_{13}^2 = b_{12}^3 + b_{23}^1, \quad b_{13}^3 = b_{12}^2;$$
$$b_{23}^1 = b_{12}^3 + b_{13}^2, \quad b_{23}^2 = b_{13}^1, \quad b_{23}^3 = -b_{12}^1,$$

respectively. Substituting

$$e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j - (-1)^{n-1} b_{12}^2 e_1$$

for e_{n+2}, we get (d^4).

Lastly, we discuss case (7). It follows the direct computation that there does not exist any nonabelian proper subalgebra of A containing A^1. Then the multiplication of A is completely determined by the left multiplication $\mathrm{ad}(e_4, \ldots, e_{n+2})$. And $\mathrm{ad}(e_4, \ldots, e_{n+2})|_{A^1}$ is nonsingular since $\dim A^1 = 3$. Therefore, we can choose a basis e_1, e_2, e_3 of A^1 such that the multiplication table of A in the basis $e_1, \ldots, e_{n+2}$ has the following possibilities:

$$(\mathrm{d}^{5\prime}) \quad \begin{cases} [e_1, e_4, \ldots, e_{n+2}] = \beta_1 e_1, \\ [e_2, e_4, \ldots, e_{n+2}] = \beta_2 e_2, \qquad 0 \neq \beta_i \in F, \ i = 1, 2, 3; \\ [e_3, e_4, \ldots, e_{n+2}] = \beta_3 e_3, \end{cases}$$

$$(\mathrm{d}^{6\prime}) \quad \begin{cases} [e_1, e_4, \ldots, e_{n+2}] = \alpha e_1 + e_2, \\ [e_2, e_4, \ldots, e_{n+2}] = \alpha e_2 + e_3, \qquad 0 \neq \alpha \in F; \\ [e_3, e_4, \ldots, e_{n+2}] = \alpha e_3, \end{cases}$$

$$(\mathrm{d}^{7\prime})\quad \begin{cases} [e_1, e_4, \ldots, e_{n+2}] = \gamma_1 e_1 + e_2, \\ [e_2, e_4, \ldots, e_{n+2}] = \gamma_1 e_2, & 0 \neq \gamma_j \in F,\ j = 1, 2. \\ [e_3, e_4, \ldots, e_{n+2}] = \gamma_2 e_3, \end{cases}$$

Fixing $e_5, \ldots, e_{n+2}$ in the n-ary multiplication of A, we get the solvable Lie algebra $A_1 = A$ (as vector spaces) with the Lie product:

$$[x, y]_1 = [x, y, e_5, \ldots, e_{n+2}], \quad x, y \in A_1.$$

Then the multiplication tables of A_1 with respect to $(\mathrm{d}^{5\prime})$, $(\mathrm{d}^{6\prime})$, $(\mathrm{d}^{7\prime})$ are

$$(\mathrm{d}^{5\prime\prime})\quad \begin{cases} [e_1, e_4]_1 = \beta_1 e_1, \\ [e_2, e_4]_1 = \beta_2 e_2, & 0 \neq \beta_i \in F,\ 1 = 1, 2, 3; \\ [e_3, e_4]_1 = \beta_3 e_3, \end{cases}$$

$$(\mathrm{d}^{6\prime\prime})\quad \begin{cases} [e_1, e_4]_1 = \alpha e_1 + e_2, \\ [e_2, e_4]_1 = \alpha e_2 + e_3, & 0 \neq \alpha \in F; \\ [e_3, e_4]_1 = \alpha e_3, \end{cases}$$

$$(\mathrm{d}^{7\prime\prime})\quad \begin{cases} [e_1, e_4]_1 = \gamma_1 e_1 + e_2, \\ [e_2, e_4]_1 = \gamma_1 e_2, & 0 \neq \gamma_j \in F,\ j = 1, 2. \\ [e_3, e_4]_1 = \gamma_2 e_3, \end{cases}$$

This implies that $(\mathrm{d}^{i\prime\prime})$ can be decomposed into the direct sum of ideals $Z(A_1)$ and B, where the center

$$Z(A_1) = Fe_5 + \cdots + Fe_{n+2},$$

and the ideal

$$B = Fe_1 + Fe_2 + Fe_3 + Fe_4, \quad i = 5, 6, 7.$$

By the classification of 4-dimensional solvable Lie algebras [12], we get (d^5), (d^6) and (d^7), and (d^i) is not isomorphic to (d^j) when $i \neq j$ for $5 \leqslant i, j \leqslant 7$. The n-Lie algebras corresponding to case (d^5) with coefficients β and β' are isomorphic if and only if $\beta = \beta'$. We also have the n-Lie algebras corresponding to case (d^7) with coefficients s, t, u and s', t', u' are isomorphic if and only if there exists a nonzero element $r \in F$ such that

$$s = r^3 s', \quad t = r^2 t', \quad u = r u', \quad s, s', t, t', u, u' \in F.$$

It is evident that (d^5), (d^6), and (d^7) are not isomorphic to other cases since (d^5), (d^6), and (d^7) have no nonabelian proper subalgebras containing A^1. Case (d^4) is not isomorphic to any cases of (d^1), (d^2), and (d^3), since (d^4) is decomposable. Because (d^1) has nontrivial center, (d^1) is not isomorphic to (d^2) and (d^3). By direct computation, we know that dimensions of the derivation algebras of cases (d^2) and (d^3) are $n^2 + 2$ and $n^2 + 3$ respectively. Therefore, (d^2) and (d^3) represent non-isomorphic classes.

Now, fixing $e_4, \dots, e_{n+1}$ in the multiplication of A of case (d^2), and substituting $(-1)^{n-2}e_{n+2}$ for e_{n+2}, we get a solvable Lie algebra A_2 ($A_2 = A$ as vector spaces) with the operation

$$[x, y]_2 = [x, y, e_4, \dots, e_{n+1}],$$

and the multiplication table of A_2 in the basis $e_1, \dots, e_{n+2}$ is as follows:

$$\begin{cases} [e_2, e_3]_2 = e_1, \\ [e_3, e_{n+2}]_2 = e_3 + \alpha e_2, \\ [e_2, e_{n+2}]_2 = e_2, \\ [e_1, e_{n+2}]_2 = e_1. \end{cases}$$

Then A_2 has a decomposition

$$A_2 = B \oplus Z(A_2),$$

where the center

$$Z(A_2) = Fe_1 + \cdots + Fe_{n+1},$$

and

$$B = Fe_1 + Fe_2 + Fe_3 + Fe_{n+2}$$

is an ideal. By the classification of solvable Lie algebras [12], we get that n-Lie algebras of case (d^2) with coefficients α and α' are isomorphic if and only if $\alpha = \alpha'$.

5. By Lemma 3.3, we have $\dim A^1 = r,\ 4 \leqslant r \leqslant n+1$. Suppose

$$A^1 = Fe_1 + \ldots + Fe_r.$$

From Lemmas 3.1 and 3.2, the multiplication table of A in the basis $e_1, \dots, e_{n+2}$ has following possibilities:

$$(1) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(2) \quad \begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(3) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(4) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(5)\quad \begin{cases} [e_1, e_3, \ldots, e_{n+1}] = e_1, \\ [e_2, e_3, \ldots, e_{n+1}] = e_2, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(6)\quad \begin{cases} [e_1, \ldots, \hat{e}_i, \ldots, e_m, \ldots, e_r, \ldots, e_{n+1}] = e_i, \\ \qquad\qquad 1 \leqslant i \leqslant m,\ 3 \leqslant m \leqslant r-1, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

$$(7)\quad \begin{cases} [e_1, \ldots, \hat{e}_i, \ldots, e_r, e_{r+1}, \ldots, e_{n+1}] = e_i,\ 1 \leqslant i \leqslant r, \\ [e_1, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}] = \sum\limits_{k=1}^{r} b_{ij}^k e_k; \end{cases}$$

where $b_{ij} \in F,\ 1 \leqslant i < j \leqslant n+1$.

First, we study case (1). For substituting $[e_2, \ldots, e_{n+1}] = e_1$ into the other equations and by the Jacobi identities on

$$\begin{aligned} &\{[e_2, \ldots, e_{n+1}], e_2, \ldots, e_r, e_{r+1}, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\}, \\ &\qquad\qquad r+1 \leqslant i < j \leqslant n+1; \\ &\{[e_2, \ldots, e_{n+1}], e_2, \ldots, \hat{e}_i, \ldots, e_r, e_{r+1}, \ldots, \hat{e}_j, \ldots, e_{n+1}, e_{n+2}\}, \\ &\qquad\qquad 2 \leqslant i \leqslant r,\ r+1 \leqslant j \leqslant n+1; \\ &\{[e_2, \ldots, e_{n+1}], e_2, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_r, e_{r+1}, \ldots, e_{n+2}\}, \quad 2 \leqslant i < j \leqslant r, \end{aligned}$$

we get

$$b_{ij}^k = 0, \quad 1 \leqslant k \leqslant r,\ r+1 \leqslant i < j \leqslant n+1;$$

$$b_{ij}^1 = (-1)^{i-2} b_{1j}^i, \quad b_{ij}^2 = \ldots = b_{ij}^r = 0, \quad 2 \leqslant i \leqslant r,\ r+1 \leqslant j \leqslant n+1;$$

$$b_{ij}^1 = (-1)^{i-2} b_{1j}^i + (-1)^{j-3} b_{1i}^j, b_{ij}^2 = \ldots = b_{ij}^r = 0, \quad 2 \leqslant i < j \leqslant r,$$

respectively.

Replacing

$$e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j$$

for e_{n+2}, and fixing e_{n+2} in the multiplication of A, we get an $(n+2)$-dimensional $(n-1)$-Lie algebra $A_3 = A$ (as vector spaces) with the product

$$[x_1, \ldots, x_{n-1}]_3 = [x_1, \ldots, x_{n-1}, e_{n+2}], \quad \forall\ x_1, \ldots, x_{n-1} \in A_3,$$

and the multiplication table in the basis $e_1, \ldots, e_{n+2}$ is as follows:

$$\begin{cases} [e_2,\ldots,\hat{e}_j,\ldots,e_r,\ldots,e_{n+1}]_3 = \sum\limits_{k=2}^{r} b_{1j}^k e_k, \quad 2 \leqslant j \leqslant r, \\ [e_2,\ldots,e_r,e_{r+1},\ldots,\hat{e}_j,\ldots,e_{n+1}]_3 = \sum\limits_{k=2}^{r} b_{1j}^k e_k, \quad r+1 \leqslant j \leqslant n+1, \\ [e_1,\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_r,\ldots,e_{n+1}]_3 = b'_{ij} e_1, \quad 2 \leqslant i < j \leqslant r, \\ [e_1,\ldots,\hat{e}_i,\ldots,e_r,\ldots,\hat{e}_j,\ldots,e_{n+1}]_3 = (-1)^i b_{1j}^i e_1, \\ \qquad 2 \leqslant i \leqslant r < j \leqslant n+1, \end{cases}$$

where

$$b'_{ij} = (-1)^i b_{1j}^i + (-1)^{j-1} b_{1i}^j.$$

Set

$$B = Fe_2 + \cdots + Fe_{n+1}.$$

Then B is a subalgebra of A_3 with the multiplication table

$$(1_b) \quad \begin{cases} [e_2,\ldots,\hat{e_j},\ldots,e_r,\ldots,e_{n+1}]_3 = \sum\limits_{k=2}^{r} b_{1j}^k e_k, \quad 2 \leqslant j \leqslant r, \\ [e_2,\ldots,e_r,e_{r+1},\ldots,\hat{e_j},\ldots,e_{n+1}]_3 = \sum\limits_{k=2}^{r} b_{1j}^k e_k, \quad r+1 \leqslant j \leqslant n+1. \end{cases}$$

By similar discussion leading to Theorem 3 in [3], taking a suitable transformation of basis $e_1, e_2, \ldots, e_{n+1}$, (1) is isomorphic to ($r^1$).

Second, replacing e_{n+2} by

$$e_{n+2} - \sum_{i=1}^{n} (-1)^{n-i} b_{in+1}^1 e_i$$

in (2) and substituting $[e_1, \ldots, e_n] = e_1$ into other equations, we get

$$\begin{gathered} b_{in+1}^2 = \cdots = b_{in+1}^r = 0, b_{1n+1}^i = 0, \quad 2 \leqslant i \leqslant r; \\ b_{in+1}^2 = \cdots = b_{in+1}^r = 0, \quad r+1 \leqslant i \leqslant n; \\ b_{1j}^r = 0, \quad i = 1,\ 2 \leqslant j \leqslant n; \\ b_{ij}^r = 0, \quad 2 \leqslant i < j \leqslant n, \end{gathered}$$

respectively. Then $\dim A^1 = r - 1$, which is a contradiction. Therefore, case (2) is not realized. By similar arguments to above, cases (3)–(6) are not realized.

Lastly, we study case (7). For $r + 1 \leqslant i < j \leqslant n + 1$, imposing the Jacobi identities on

$$\{[e_2,\ldots,e_{n+1}], e_2,\ldots,e_r,e_{r+1},\ldots,\hat{e}_i,\ldots,\hat{e}_j,\ldots,e_{n+1},e_{n+2}\},$$

we get

$$b_{ij}^k = 0, \quad 1 \leqslant k \leqslant r.$$

For every $i \neq p$, $1 \leqslant i, p \leqslant r$, substituting

$$[e_1,\ldots,\hat{e}_i,\ldots,e_r,e_{r+1},\ldots,e_{n+1}] = e_i$$

into the other equations, we get

$$b_{pj}^{p} = b_{1j}^{1}, \quad b_{pj}^{k} = 0, \quad k \neq p, \ 1 \leqslant p, k \leqslant r < j \leqslant n+1.$$

By similar discussion, we have

$$b_{1j}^{k} = 0, \ b_{1j}^{1} = \cdots = b_{j-1j}^{j-1} = -b_{jj+1}^{j+1} = \cdots = -b_{jr}^{r}, \ b_{1j}^{j} = b_{12}^{2}, \ 2 \leqslant k \neq j \leqslant r;$$
$$b_{ij}^{k} = 0, \quad 1 \leqslant k \leqslant r, \ k \neq i, j, \ 2 \leqslant i < j \leqslant r.$$

Replacing e_{n+2} by

$$e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^{1} e_j - (-1)^{n-1} b_{12}^{2} e_1$$

in (7), we get (r^2). It is evident that (r^1) is not isomorphic to (r^2). $\square$

Acknowledgements The authors would like to thank the anonymous referees for many valuable corrections and suggestions. This work was supported in part by the National Natural Science Foundation of China (Grant No. 10871192), the Natural Science Foundation of Hebei Province (No. A2010000194), and the third author acknowledges the support of the Australian Research Council.

References

1. Bagger J, Lambert N. Gauge symmetry and supersymmetry of multiple M2-branes. Phys Rev D, 2008, 77: 065008
2. Bai R, Wang X, Xiao W, An H. The structure of low dimensional n-Lie algebras over a field of characteristic 2. Linear Algebra Appl, 2008, 428: 1912–1920
3. Bai R, Song G. The classification of six-dimensional 4-Lie algebras. J Phys A: Math Theor, 2009, 42: 035207
4. Benvenuti S, Rodríguez-Gómez D, Tonni E, Verlinde H. $N = 8$ superconformal gauge theories and M2 branes. J High Energy Phys, 2009, 01: 078
5. Cantarini N, Kac V. Classification of simple linearly compact n-Lie superalgebras. Comm Math Phys, 2010, 298: 833–853
6. Figueroa-O'Farrill J. Lorentzian Lie n-algebras. J Math Phys, 2008, 49: 113509
7. Figueroa-O'Farrill J. Metric Lie n-algebras and double extensions. 2008, arXiv: 0806.3534 [math.RT]
8. Figueroa-O'Farrill J, Papadopoulos G. Plücker-type relations for orthogonal planes. J Geom Phys, 2004, 49: 294–331
9. Filippov V. n-Lie algebras. Siberian Math J, 1985, 26(6): 126–140
10. Gauntlett J, Gutowski J. Constraining maximally supersymmetric membrane actions. J High Energy Phys, 2008, 06: 053
11. Gomis J, Milanesi G, Russo J. Bagger-Lambert theory for general Lie algebras. J High Energy Phys, 2008, 06: 075
12. Graaf W. Classification of solvable Lie algebras. Experimental Math, 2005, 14: 15–25
13. Gustavsson A. Algebraic structures on parallel M2-branes. Nuclear Phys B, 2009, 811(1-2): 66–76
14. Ho P, Imamura Y, Matsuo Y. M2 to D2 revisited. J High Energy Phys, 2008, 07: 003

15. Ho P, Matsuo Y, Shiba S. Lorentzian Lie (3-)algebra and toroidal compactification of M/string theory. J High Energy Phys, 2009, 03: 045
16. Kasymov S. Theory of n-Lie algebras. Algebra Logika, 1987, 26(3): 277–297
17. Ling W. On the structure of n-Lie algebras. Dissertation, University-GHS-Siegen, 1993
18. Medeiros P, Figueroa-O'Farrill J, Méndez-Escobar E. Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space. J High Energy Phys, 2008, 07: 111
19. Medeiros P, Figueroa-O'Farrill J, Méndez-Escobar E. Metric Lie 3-algebras in Bagger-Lambert theory. J High Energy Phys, 2008, 08: 045
20. Medeiros P, Figueroa-O'Farrill J, Méndez-Escobar E, Ritter P. Metric 3-Lie algebras for unitary Bagger-Lambert theories. J High Energy Phys, 2009, 04: 037
21. Nagy P. Prolongations of Lie algebras and applications. arXiv: 0712.1398 [math.DG]
22. Nambu Y. Generalized Hamiltonian dynamics. Phys Rev D, 1973, 7: 2405–2412
23. Papadopoulos G. M2-branes, 3-Lie algebras and Plucker relations. J High Energy Phys, 2008, 05: 054
24. Papadopoulos G. On the structure of k-Lie algebras. Classical Quantum Gravity, 2008, 25: 142002
25. Pozhidaev A. Simple quotient algebras and subalgebras of Jacobian algebras. Siberian Math J, 1998, 39: 512–517
26. Pozhidaev A. Two classes of central simple n-Lie algebras. Siberian Math J, 1999, 40: 1112–1118
27. Takhtajan L. On foundation of the generalized Nambu mechanics. Comm Math Phys, 1994, 160: 295–315