Front. Math. China 2010, 5(2): 191–209
DOI 10.1007/s11464-010-0001-0

RESEARCH ARTICLE

Finite element method for a nonsmooth elliptic equation*

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Abstract In this paper, we study the finite element method for a nonsmooth elliptic equation. Error analysis is presented, including a priori and a posteriori error estimates as well as superconvergence analysis. We also propose two algorithms for solving the underlying equation. Numerical experiments are employed to confirm our error estimations and the efficiency of our algorithms.

Keywords Finite element method, nonsmooth elliptic equation, a priori error estimate, a posteriori error estimate, superconvergence analysis, active set method
MSC 65N15, 65N30

1 Introduction

In this paper, we will consider the finite element method for a kind of nonsmooth elliptic equation. Specifically, we consider the following nonsmooth elliptic equation:

$$\begin{cases} -\Delta y + \lambda \max(0, y) = f & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ is a constant, $f \in L^2(\Omega)$, Ω is an open, bounded convex subset of $\mathbb{R}^N$, $N \leqslant 3$, with a Lipschitz continuous boundary $\partial\Omega$. The above nonsmooth elliptic equation can be found in equilibrium analysis of confined

* Received August 31, 2009; accepted November 30, 2009

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magnetohydrodynamics (MHD) plasmas [5,9], thin stretched membranes partially covered with water [10], or reaction-diffusion problems [1]. Some numerical methods on this kind of equation are discussed in, e.g., Refs. [1,4,10].

For the nonsmooth elliptic equation (1), it is not clear how to obtain the a priori and a posteriori error estimates using the standard linearization technique. Especially, it is impossible to prove the L^2-error estimate using the well-known Nitsche technique, because we cannot construct the conjugated equation for the nonlinear equation (1). In this paper, we apply some ideas used in error analysis of the finite element method for optimal control problems governed by partial differential equations (see, e.g., Ref. [13]) to complete our theoretical analysis for problem (1). Moreover, we utilize an active set method for solving optimal control problems (see e.g., Refs. [2,8]) in our numerical experiments to overcome the difficulty arising from very large parameter λ.

The outline of the paper is as follows. In Section 2, we will introduce the model problem and its finite element scheme. In Section 3, we derive the a priori error estimations of the finite element method for nonsmooth elliptic equation. We obtain the error estimate of order $O(h^2)$ in L^2-norm and $O(h)$ in H^1-norm when $\lambda = O(1)$. Under certain extra conditions, we also have superconvergence result, i.e.,

$$|y_I - y_h|_{1,\Omega} = O(h^2),$$

where y_h and y_I are the finite element approximation and the interpolant of the exact solution. In Section 4, we derive a posteriori error estimators with respect to H^1-norm and L^2-norm, respectively. Both reliability and efficiency of the a posteriori error estimates are proved. In Section 5, we present two algorithms for solving the underlying problem. When λ is very large, an essential difficulty arises for computation of the problem. We utilize an active set method for optimal control problems (see, e.g., Refs. [2,8]) to overcome this difficulty successfully. At the end of this paper, we present some numerical examples to confirm our theoretical results.

2 Model problem and its finite element approximation

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$,

$$\|\cdot\|_{m,2,\Omega} = \|\cdot\|_{m,\Omega}, \quad |\cdot|_{m,2,\Omega} = |\cdot|_{m,\Omega},$$

and set

$$H_0^1(\Omega) \equiv \{v \in H^1(\Omega)\colon\ v|_{\partial\Omega} = 0\}.$$

In the rest of the paper c or C denotes a general positive constant independent of λ and mesh size h.

The weak form of (1) reads as: find $y \in H_0^1(\Omega)$ such that

$$(\nabla y, \nabla v) + \lambda(\max(0, y), v) = (f, v), \quad \forall\ v \in H_0^1(\Omega). \tag{2}$$

It is well known that problem (2) admits a unique solution (see, e.g., Ref. [6]).

Now, let us consider the finite element approximation of problem (2). Here, we consider only n-simplex elements and conforming finite element. Let Ω^h be a polygonal approximation to Ω with a boundary $\partial\Omega^h$. For simplicity, we assume that $\Omega^h = \Omega$. Let $\mathscr{T}^h$ be a regular partitioning of Ω into disjoint n-simplices τ, so that

$$\overline{\Omega} = \bigcup_{\tau\in\mathscr{T}^h} \overline{\tau},$$

$\overline{\tau}$ and $\overline{\tau}'$ have either only one common vertex or a whole edge or face or $\overline{\tau}\cap\overline{\tau}' = \phi$. Let h_τ denote the diameter of τ, and let

$$h = \max_{\tau\in\mathscr{T}^h} h_\tau.$$

Associated with $\mathscr{T}^h$ is a finite dimensional subspace V^h of $C(\overline{\Omega})$, such that $\chi|_\tau$ are polynomials of m-order ($m \geqslant 1$), $\forall\ \chi \in V^h$ and $\tau \in \mathscr{T}^h$. Let

$$V_0^h = V^h \cap H_0^1(\Omega).$$

For simplicity, we will only consider the conforming piecewise linear finite element space here, i.e., $m = 1$ for V^h.

Then the standard FEM approximation of (2) is to look for $y_h \in V_0^h$ such that

$$(\nabla y_h, \nabla v_h) + \lambda(\max(0, y_h), v_h) = (f, v_h), \quad \forall\ v_h \in V_0^h(\Omega). \tag{3}$$

3 A priori error estimates

In this section, we will provide the error analysis of the finite element approximation introduced in the last section, including the error estimates with H^1- and L^2-norms and the superconvergence analysis.

In order to discuss the a priori error estimate of the nonsmooth elliptic equation (2), we should resort to an auxiliary problem: find $\widetilde{y}_h \in V_0^h$ such that

$$(\nabla \widetilde{y}_h, \nabla v_h) + \lambda(\max(0, y), v_h) = (f, v_h), \quad \forall\ v_h \in V_0^h(\Omega). \tag{4}$$

For the above auxiliary problem, we have the following error estimate that will be used for the error analysis in this section.

Lemma 3.1 *Let y, y_h and $\widetilde{y}_h$ be the solutions of problems* (2), (3) *and* (4), *respectively. Then we have*

$$|\widetilde{y}_h - y_h|_{1,\Omega} \leqslant c\sqrt{\lambda}\|y - \widetilde{y}_h\|_{0,\Omega}. \tag{5}$$

Proof First, we observe that

$$(\max(0, y) - \max(0, y_h), \max(0, y) - y)$$
$$+ (\max(0, y) - \max(0, y_h), y_h - \max(0, y_h))$$

$$
= \begin{cases} (y - y_h, 0) + (y - y_h, 0), & y \geqslant 0,\ y_h \geqslant 0, \\ (y, 0) + (y, y_h), & y \geqslant 0,\ y_h < 0, \\ (-y_h, -y) + (-y_h, 0), & y < 0,\ y_h \geqslant 0, \\ (0, -y) + (0, y_h), & y < 0,\ y_h < 0, \end{cases}
$$

$$
\leqslant 0. \tag{6}
$$

Thus, it follows from (3), (4) and (6) that

$$
\begin{aligned}
&|\widetilde{y}_h - y_h|_{1,\Omega}^2 + \lambda \| \max(0, y) - \max(0, y_h)\|_{0,\Omega}^2 \\
&\quad = (f - \lambda \max(0, y), \widetilde{y}_h - y_h) - (f - \lambda \max(0, y_h), \widetilde{y}_h - y_h) \\
&\qquad + \lambda \| \max(0, y) - \max(0, y_h)\|_{0,\Omega}^2 \\
&\quad = \lambda(\max(0, y) - \max(0, y_h), y_h - \widetilde{y}_h) \\
&\qquad + \lambda(\max(0, y) - \max(0, y_h), \max(0, y) - \max(0, y_h)) \\
&\quad = \lambda(\max(0, y) - \max(0, y_h), y - \widetilde{y}_h) \\
&\qquad + \lambda(\max(0, y) - \max(0, y_h), \max(0, y) - y) \\
&\qquad + \lambda(\max(0, y) - \max(0, y_h), y_h - \max(0, y_h)) \\
&\quad \leqslant \lambda \| \max(0, y) - \max(0, y_h)\|_{0,\Omega} \|y - \widetilde{y}_h\|_{0,\Omega} \\
&\quad \leqslant C(\delta)\lambda \|y - \widetilde{y}_h\|_{0,\Omega}^2 + C\delta\lambda \| \max(0, y) - \max(0, y_h)\|_{0,\Omega}^2,
\end{aligned} \tag{7}
$$

where δ is an arbitrary positive number. Setting δ to be small enough, we can deduce from (7) that

$$
|\widetilde{y}_h - y_h|_{1,\Omega}^2 \leqslant c\lambda \|y - \widetilde{y}_h\|_{0,\Omega}^2.
$$

Then inequality (5) follows. □

Using Lemma 3.1, we can present our main results of a priori error estimates for the finite element approximation of the nonsmooth elliptic equation (2) in the following theorem.

Theorem 3.2 *Let y and y_h be the solutions of problems* (2) *and* (3), *respectively. Assume that $y \in H^2(\Omega)$ and the domain Ω is convex. Then we have*

$$
|y - y_h|_{1,\Omega} \leqslant c(h + \sqrt{\lambda} h^2)|y|_{2,\Omega}, \tag{8}
$$

$$
\|y - y_h\|_{0,\Omega} \leqslant c(h^2 + \sqrt{\lambda} h^2)|y|_{2,\Omega}. \tag{9}
$$

Proof Setting $v = v_h$ in (2) and combining with (4), we have

$$
(\nabla y, \nabla v_h) - (\nabla \widetilde{y}_h, \nabla v_h) = 0, \quad \forall\ v_h \in V_0^h, \tag{10}
$$

where $\widetilde{y}_h$ is the solution of (4). Let $y_I \in V_0^h$ be the piecewise linear Lagrange interpolation of y. By the standard error estimate technique of the finite element method (see, e.g., Ref. [6]), (10) implies that

$$\begin{aligned}
|y-\widetilde{y}_h|_{1,\Omega}^2 &= (\nabla(y-\widetilde{y}_h),\nabla(y-\widetilde{y}_h))\\
&= (\nabla(y-\widetilde{y}_h),\nabla(y-y_I))+(\nabla(y-\widetilde{y}_h),\nabla(y_I-\widetilde{y}_h))\\
&= (\nabla(y-\widetilde{y}_h),\nabla(y-y_I))+0\\
&\leqslant |y-\widetilde{y}_h|_{1,\Omega}\,|y-y_I|_{1,\Omega}\\
&\leqslant ch|y|_{2,\Omega}\,|y-\widetilde{y}_h|_{1,\Omega}.
\end{aligned}$$

Therefore, we have

$$|y-\widetilde{y}_h|_{1,\Omega}\leqslant ch|y|_{2,\Omega}. \tag{11}$$

Moreover, note that Ω is convex. Then for each $\varphi\in L^2(\Omega)$, there exists $\psi\in H_0^1(\Omega)\cap H^2(\Omega)$ such that

$$(\nabla\psi,\nabla v)=(\varphi,v),\quad \forall\, v\in H_0^1(\Omega),$$

$$|\psi|_{2,\Omega}\leqslant c\|\varphi\|_{0,\Omega}.$$

By using the well-known Aubin-Nitsche technique (see, e.g., Ref. [6]), setting

$$\varphi=y-\widetilde{y}_h,\quad v=y-\widetilde{y}_h$$

and using (10), then there exists ψ such that

$$\begin{aligned}
\|y-\widetilde{y}_h\|_{0,\Omega}^2 &= (y-\widetilde{y}_h,y-\widetilde{y}_h)\\
&= (\nabla\psi,\nabla(y-\widetilde{y}_h))\\
&= (\nabla(\psi-\psi_I),\nabla(y-\widetilde{y}_h))\\
&\leqslant |\psi-\psi_I|_{1,\Omega}\,|y-\widetilde{y}_h|_{1,\Omega}\\
&\leqslant ch^2|\psi|_{2,\Omega}\,|y|_{2,\Omega}\\
&\leqslant ch^2\|y-\widetilde{y}_h\|_{0,\Omega}\,|y|_{2,\Omega},
\end{aligned}$$

where $\psi_I\in V_0^h$ is the piecewise linear Lagrange interpolation of ψ. Thus, we obtain

$$\|y-\widetilde{y}_h\|_{0,\Omega}\leqslant ch^2|y|_{2,\Omega}. \tag{12}$$

Using triangle inequality, it follows from (11), (12) and Lemma 3.1 that

$$\begin{aligned}
|y-y_h|_{1,\Omega} &\leqslant |y-\widetilde{y}_h|_{1,\Omega}+|\widetilde{y}_h-y_h|_{1,\Omega}\\
&\leqslant ch|y|_{2,\Omega}+c\sqrt{\lambda}\|y-\widetilde{y}_h\|_{0,\Omega}\\
&\leqslant ch|y|_{2,\Omega}+c\sqrt{\lambda}h^2|y|_{2,\Omega}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\|y-y_h\|_{0,\Omega} &\leqslant \|y-\widetilde{y}_h\|_{0,\Omega}+\|\widetilde{y}_h-y_h\|_{0,\Omega}\\
&\leqslant ch^2|y|_{2,\Omega}+c\sqrt{\lambda}\|y-\widetilde{y}_h\|_{0,\Omega}\\
&\leqslant ch^2|y|_{2,\Omega}+c\sqrt{\lambda}h^2|y|_{2,\Omega},
\end{aligned} \tag{14}$$

where we used the following well-known result (see, e.g., Ref. [6]):

$$\|v\|_{0,\Omega} \leqslant C|v|_{1,\Omega}, \quad \forall\, v \in H_0^1(\Omega).$$

Then (8) and (9) follow from (13) and (14), respectively. This completes the proof. □

Remark 3.3 In the case that $\sqrt{\lambda} = O(h^\alpha)$ $(\alpha \geqslant 0)$,

$$|y - y_h|_{1,\Omega} = O(h), \quad \|y - y_h\|_{0,\Omega} = O(h^2),$$

i.e., the convergence orders in $H^1(\Omega)$-norm and $L^2(\Omega)$-norm are optimal. On the other hand, the convergence order may be affected greatly when $\sqrt{\lambda} \gg h^{-1}$. Therefore, it is necessary that the mesh should be fine enough if λ is large. Our numerical examples in Section 4 illustrate these points well. In the rest of this paper, we set $\lambda = O(1)$ for simplicity.

At the end of this section, we consider the superconvergence property of the nonsmooth elliptic equation. Using the standard superconvergence analysis technique, the following superclose property can be proved.

Lemma 3.4 *Let y and $\widetilde{y}_h$ be the solutions of problem (2) and (4), respectively. Suppose that $y \in H^3(\Omega)$, and the mesh $\mathscr{T}^h$ is a uniform triangular mesh. Then we have*

$$|\widetilde{y}_h - y_I|_{1,\Omega} \leqslant ch^2|y|_{3,\Omega}, \tag{15}$$

where $y_I \in V_0^h$ is the standard Lagrange linear interpolation of y.

Proof Considering the orthogonal property (10) and noticing that $\widetilde{y}_h - y_I \in V_0^h$, we have

$$\begin{aligned}|\widetilde{y}_h - y_I|_{1,\Omega}^2 &= (\nabla(\widetilde{y}_h - y_I), \nabla(\widetilde{y}_h - y_I))\\ &= (\nabla(y - y_I), \nabla(\widetilde{y}_h - y_I)).\end{aligned} \tag{16}$$

Moreover, it has been proved that (see, e.g., Refs. [3,11,12,15,18]) for all $v_h \in V_0^h$,

$$(\nabla(y - y_I), \nabla v_h) \leqslant Ch^2|y|_{3,\Omega}\,|v_h|_{1,\Omega}. \tag{17}$$

Thus, (15) follows from (16) and (17). □

Now, we are in the position to construct a post-processing operator π_{2h} (see, e.g., Refs. [12,17], for more details). Let τ^* be a macro element which is the union of the four elements $\tau_i \in \mathscr{T}^h,\ i = 1,2,3,4$, where the intersection of $\overline{\tau}_i,\ i = 1,2,3,4$, is nonempty (see Fig. 1).

Let π_{2h} be a post-process operator such that $\pi_{2h}v \in P_2(\tau^*)$, where P_2 is the space of quadratic polynomials, and

$$\pi_{2h}v(z_i) = v(z_i), \quad i = 1,2,\ldots,6,$$

where $z_i,\ i = 1,2,\ldots,6$, are the vertices of $\tau_i,\ i = 1,2,3,4$ (see Fig. 1). By the definition of interpolation operator π_{2h}, standard interpolation error estimate and scaling argument, we can prove the following three important properties:

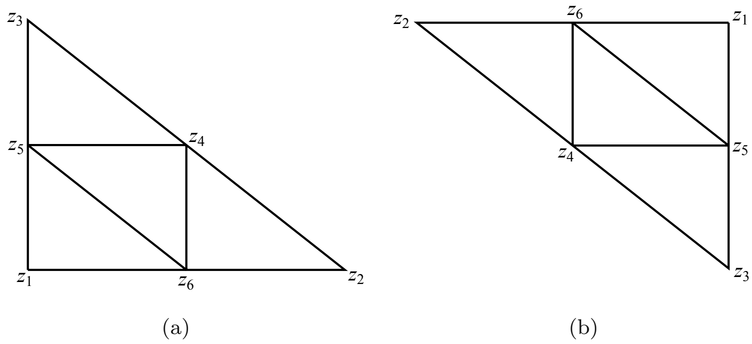

Fig. 1 (a) Macro element 1 and (b) macro element 2

$$\pi_{2h}v = \pi_{2h}v_I, \quad \forall\, v \in C(\Omega), \tag{18}$$

$$\|v - \pi_{2h}v\|_{1,\Omega} \leqslant ch^2|v|_{3,\Omega}, \quad \forall\, v \in H^3(\Omega), \tag{19}$$

$$\|\pi_{2h}v_h\|_{1,\Omega} \leqslant c\|v_h\|_{1,\Omega}, \quad \forall\, v_h \in V^h, \tag{20}$$

where $v_I \in V_0^h$ is the standard Lagrange linear interpolation of v.

Using the post-processing operator π_{2h} defined above and its properties (18)–(20) as well as Lemma 3.4, we can prove the following global superconvergence result.

Theorem 3.5 *Let y and y_h be the solutions of problem* (2) *and* (3), *respectively. If $y \in H^3(\Omega)$, Ω is convex, and the mesh $\mathscr{T}^h$ is a uniform triangular mesh, then*

$$|y - \pi_{2h}y_h|_{1,\Omega} \leqslant ch^2|y|_{3,\Omega}. \tag{21}$$

Proof It follows from (18)–(20) and Lemma 3.4 that

$$\begin{aligned} |y - \pi_{2h}y_h|_{1,\Omega} &\leqslant |y - \pi_{2h}y|_{1,\Omega} + |\pi_{2h}y - \pi_{2h}y_I|_{1,\Omega} + |\pi_{2h}y_I - \pi_{2h}y_h|_{1,\Omega} \\ &\leqslant ch^2|y|_{3,\Omega} + 0 + c|y_I - y_h|_{1,\Omega} \\ &\leqslant ch^2|y|_{3,\Omega} + c|y_I - \widetilde{y}_h|_{1,\Omega} + c|\widetilde{y}_h - y_h|_{1,\Omega} \\ &\leqslant ch^2|y|_{3,\Omega} + c|\widetilde{y}_h - y_h|_{1,\Omega}. \end{aligned} \tag{22}$$

Then (21) can be deduced from (22), Lemma 3.1 and (12). □

4 A posteriori error estimate

It is well known that adaptive finite element method is an efficient numerical algorithm for solving partial differential equations (PDEs). In order to have an efficient and reliable indicator for the adaptive mesh refinement, the a posteriori error estimator should be provided. In this section, we will discuss the a posteriori error estimates for the finite element approximation of the nonsmooth elliptic equation (1) and formulate a posteriori error estimator.

First of all, we should introduce the well-known error estimates for the Clément type interpolation (see Refs. [7,14] for more details), which will be used in a posteriori error analysis in this section.

Lemma 4.1 *Let I_h be the Clément type interpolation operator defined in Ref.* [14]. *Then for any $v \in H_0^1(\Omega)$, we have $I_h v \in V_0^h$, and for all element $\tau \in \mathscr{T}^h$,*

$$\|v - I_h v\|_{L^2(\tau)} + h_\tau \|\nabla(v - I_h v)\|_{L^2(\tau)} \leqslant \sum_{\overline{\tau}' \cap \overline{\tau} \neq \emptyset} C h_\tau |\nabla v|_{L^2(\tau')}, \tag{23}$$

$$\|v - I_h v\|_{L^2(e)} \leqslant \sum_{e \subset \overline{\tau}'} C h_e^{1/2} |\nabla v|_{L^2(\tau')}, \tag{24}$$

where e is the edge or face of the element.

Moreover, we should introduce another auxiliary problem similar to (4). Let $\widetilde{y} \in H_0^1(\Omega)$ be the solution of the following equation:

$$(\nabla \widetilde{y}, \nabla v) + \lambda(\max(0, y_h), v) = (f, v), \quad \forall\, v \in H_0^1(\Omega), \tag{25}$$

where y_h is the solution of (3). Similar to Lemma 3.1, we have the following lemma.

Lemma 4.2 *Let y, y_h and $\widetilde{y}$ be the solutions of problem* (2), (3) *and* (25), *respectively. Then we have*

$$|y - \widetilde{y}|_{1,\Omega} \leqslant c\|\widetilde{y} - y_h\|_{0,\Omega}. \tag{26}$$

We omit the proof of Lemma 4.2 here because it is similar to Lemma 3.1.

In the following theorem, we prove a posteriori error estimator with respect to H^1-norm and its reliability and efficiency.

Theorem 4.3 *Let y and y_h be the solutions of* (2) *and* (3), *respectively. Then*

$$\|y - y_h\|_{1,\Omega}^2 \leqslant c\eta_1^2, \tag{27}$$

$$\eta_1^2 \leqslant c\|y - y_h\|_{1,\Omega}^2 + c\varepsilon^2, \tag{28}$$

where

$$\eta_1^2 = \sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h)\|_{0,\tau}^2 + \sum_{l \cap \partial\Omega = \phi} h_l \Big\| \Big[\frac{\partial y_h}{\partial n} \Big] \Big\|_{0,l}^2,$$

$$\varepsilon^2 = \sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h) - \overline{(f - \lambda \max(0, y_h))}\,\|_{0,\tau}^2,$$

$$\overline{v}|_\tau = \frac{\int_\tau v}{\int_\tau 1}, \quad \forall\, v \in L^2(\Omega),$$

and $l = \overline{\tau}_l^1 \cap \overline{\tau}_l^2$ *is an edge of the element,* $[v(x)]$ *is the jump of* $v(x)$ *across the element edge*:

$$[v(x)] = \lim_{\delta \to 0^+} (v(x + \delta n) - v(x - \delta n)), \quad \forall\, x \in l,$$

n *is the unit outward normal vector of* τ_l^1.

Proof Let $e = \widetilde{y} - y_h$, and $e_I \in V_0^h$ be the Clément-type interpolation of e defined in Lemma 4.1. Then we have

$$\begin{aligned}
|e|_{1,\Omega}^2 &= (\nabla(\widetilde{y} - y_h), \nabla(e - e_I)) \\
&= (\nabla \widetilde{y}, \nabla(e - e_I)) - (\nabla y_h, \nabla(e - e_I)) \\
&= (f - \lambda \max(0, y_h), e - e_I) - \sum_{\tau \in \mathscr{T}^h} \int_\tau \nabla y_h \cdot \nabla(e - e_I) \\
&= \sum_{\tau \in \mathscr{T}^h} \int_\tau (f - \lambda \max(0, y_h))(e - e_I) - \sum_{l \cap \partial\Omega = \phi} \int_l \Big[\frac{\partial y_h}{\partial n}\Big](e - e_I) \\
&\leqslant \Big(\sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h)\|_{0,\tau}^2\Big)^{1/2} \Big(\sum_{\tau \in \mathscr{T}^h} h_\tau^{-2} \|e - e_I\|_{0,\tau}^2\Big)^{1/2} \\
&\quad + \Big(\sum_{l \cap \partial\Omega = \phi} h_l \Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2\Big)^{1/2} \Big(\sum_{l \cap \partial\Omega = \phi} h_l^{-1} \|e - e_I\|_{0,l}^2\Big)^{1/2} \\
&\leqslant C\Big(\sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h)\|_{0,\tau}^2\Big)^{1/2} |e|_{1,\Omega} \\
&\quad + C\Big(\sum_{l \cap \partial\Omega = \phi} h_l \Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2\Big)^{1/2} |e|_{1,\Omega}.
\end{aligned}$$

In the last inequality we have used the error estimates of Clément-type interpolator presented in Lemma 4.1. Therefore, it is easy to see that

$$\begin{aligned}
|e|_{1,\Omega}^2 &= |\widetilde{y} - y_h|_{1,\Omega}^2 \\
&\leqslant C \sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h)\|_{0,\tau}^2 + C \sum_{l \cap \partial\Omega = \phi} h_l \Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2.
\end{aligned} \tag{29}$$

It is well known (see, e.g., Ref. [6]) that for all $v \in H_0^1(\Omega)$,

$$\|v\|_{0,\Omega} \leqslant C|v|_{1,\Omega}. \tag{30}$$

Thus, (27) follows (29), (30) and Lemma 4.2.

Next, let us consider (28). Let τ be an arbitrary element and $b_\tau = 27\lambda_1\lambda_2\lambda_3$ be a bubble function (see, e.g., Ref. [16]) on τ, where λ_i, $i = 1, 2, 3$, are three basis functions on three vertices of element τ. Setting

$$\omega_\tau = (\overline{f - \lambda \max(0, y_h)})b_\tau,$$

then it has been proved that (see, e.g., Ref. [16])

$$\int_\tau (\overline{f-\lambda\max(0,y_h)})\omega_\tau = c\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau}^2, \tag{31}$$

$$ch_\tau|\omega_\tau|_{1,\tau} \leqslant \|\omega_\tau\|_{0,\tau} \leqslant c\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau}. \tag{32}$$

On the other hand,

$$\begin{aligned}
&\int_\tau (\overline{f-\lambda\max(0,y_h)})\omega_\tau \\
&= \int_\tau (f-\lambda\max(0,y_h))\omega_\tau \\
&\quad + \int_\tau (\overline{f-\lambda\max(0,y_h)} - (f-\lambda\max(0,y_h)))\omega_\tau \\
&= \int_\tau (-\Delta y + \lambda\max(0,y) - \lambda\max(0,y_h) + \Delta y_h)\omega_\tau \\
&\quad + \int_\tau (\overline{f-\lambda\max(0,y_h)} - (f-\lambda\max(0,y_h)))\omega_\tau \\
&= \lambda\int_\tau (\max(0,y) - \max(0,y_h))\omega_\tau + \int_\tau \nabla(y-y_h)\cdot\nabla\omega_\tau \\
&\quad + \int_\tau (\overline{f-\lambda\max(0,y_h)} - (f-\lambda\max(0,y_h)))\omega_\tau \\
&\leqslant \lambda\|y-y_h\|_{0,\tau}\,\|\omega_\tau\|_{0,\tau} + |y-y_h|_{1,\tau}\,|\omega_\tau|_{1,\tau} \\
&\quad + \|f-\lambda\max(0,y_h) - \overline{f-\lambda\max(0,y_h)}\|_{0,\tau}\,\|\omega_\tau\|_{0,\tau},
\end{aligned}$$

where we have used the fact that the operator $\max(\cdot)$ is Lipschitz continuous. It is deduced from (31), (32) and the above inequality that

$$\begin{aligned}
&\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau}^2 \\
&\leqslant ch_\tau^{-1}\|y-y_h\|_{1,\tau}\,\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau} \\
&\quad + c\|f-\lambda\max(0,y_h) - \overline{f-\lambda\max(0,y_h)}\|_{0,\tau}\,\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau},
\end{aligned}$$

that is,

$$\begin{aligned}
&\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau} \\
&\leqslant ch_\tau^{-1}\|y-y_h\|_{1,\tau} + c\|f-\lambda\max(0,y_h) - \overline{f-\lambda\max(0,y_h)}\|_{0,\tau}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\tau\in\mathscr{T}^h} h_\tau^2\|f-\lambda\max(0,y_h)\|_{0,\tau}^2 \\
&\leqslant \sum_{\tau\in\mathscr{T}^h} h_\tau^2\|\overline{f-\lambda\max(0,y_h)}\|_{0,\tau}^2 \\
&\quad + \sum_{\tau\in\mathscr{T}^h} h_\tau^2\|f-\lambda\max(0,y_h) - \overline{f-\lambda\max(0,y_h)}\|_{0,\tau}^2
\end{aligned}$$

$$\leqslant c \sum_{\tau \in \mathscr{T}^h} \|y - y_h\|_{1,\tau}^2 + c \sum_{\tau \in \mathscr{T}^h} h_\tau^2 \|f - \lambda \max(0, y_h) - \overline{f - \lambda \max(0, y_h)}\|_{0,\tau}^2$$
$$= c\|y - y_h\|_{1,\Omega}^2 + c\varepsilon^2. \tag{33}$$

Similarly, let $b_l = 4\lambda_{l_1}\lambda_{l_2}$ be the bubble function (see, e.g., Ref. [16]) on the edge $l = \overline{\tau}_l^1 \cap \overline{\tau}_l^2$, where λ_{l_1} and λ_{l_2} are basis functions on the two end points of l. Setting

$$\omega_l = \Big[\frac{\partial y_h}{\partial n}\Big] b_l.$$

Again, it is easy to see that

$$\int_l \Big[\frac{\partial y_h}{\partial n}\Big]\omega_l = c\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2, \tag{34}$$

$$ch_l|\omega_l|_{1,\tau_l^1\cup\tau_l^2} \leqslant \|\omega_l\|_{0,\tau_l^1\cup\tau_l^2} \leqslant ch_l^{1/2}\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}. \tag{35}$$

On the other hand,

$$\begin{aligned}
\int_l \Big[\frac{\partial y_h}{\partial n}\Big]\omega_l &= \int_l \Big[\frac{\partial (y_h - y)}{\partial n}\Big]\omega_l \\
&= \int_{\partial\tau_l^1} \frac{\partial (y_h - y)}{\partial n}\,\omega_l + \int_{\partial\tau_l^2} \frac{\partial (y_h - y)}{\partial n}\,\omega_l \\
&= \int_{\tau_l^1\cup\tau_l^2} \nabla(y_h - y)\cdot\nabla\omega_l + \int_{\tau_l^1\cup\tau_l^2} \Delta(y_h - y)\omega_l \\
&= \int_{\tau_l^1\cup\tau_l^2} \nabla(y_h - y)\cdot\nabla\omega_l + \int_{\tau_l^1\cup\tau_l^2} (f - \lambda\max(0, y))\omega_l \\
&= \int_{\tau_l^1\cup\tau_l^2} \nabla(y_h - y)\cdot\nabla\omega_l + \int_{\tau_l^1\cup\tau_l^2} \lambda(\max(0, y_h) - \max(0, y))\omega_l \\
&\quad + \int_{\tau_l^1\cup\tau_l^2} (f - \lambda\max(0, y_h))\omega_l \\
&\leqslant |y - y_h|_{1,\tau_l^1\cup\tau_l^2}\, |\omega_l|_{1,\tau_l^1\cup\tau_l^2} + \lambda\|y - y_h\|_{0,\tau_l^1\cup\tau_l^2}\, \|\omega_l\|_{0,\tau_l^1\cup\tau_l^2} \\
&\quad + \|f - \lambda\max(0, y_h)\|_{0,\tau_l^1\cup\tau_l^2}\, \|\omega_l\|_{0,\tau_l^1\cup\tau_l^2}.
\end{aligned}$$

It is deduced from (34), (35) and the above inequality that

$$\begin{aligned}
\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2 &\leqslant c(h_l^{-1/2}\|y - y_h\|_{1,\tau_l^1\cup\tau_l^2} + h_l^{1/2}\|f - \lambda\max(0, y_h)\|_{0,\tau_l^1\cup\tau_l^2}) \\
&\quad \cdot \Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l},
\end{aligned}$$

that is,

$$\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l} \leqslant ch_l^{-1/2}\|y - y_h\|_{1,\tau_l^1\cup\tau_l^2} + ch_l^{1/2}\|f - \lambda\max(0, y_h)\|_{0,\tau_l^1\cup\tau_l^2}.$$

Therefore, this inequality and (33) imply that

$$\begin{aligned}
&\sum_{l\cap\partial\Omega=\phi} h_l\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2\\
&\leqslant c\sum_{l\cap\partial\Omega=\phi}(\|y-y_h\|_{1,\tau_l^1\cup\tau_l^2}^2+h_l^2\|f-\lambda\max(0,y_h)\|_{0,\tau_l^1\cup\tau_l^2}^2)\\
&\leqslant c\|y-y_h\|_{1,\Omega}^2+c\sum_{\tau\in\mathscr{T}^h}h_\tau^2\|f-\lambda\max(0,y_h)\|_{0,\tau}^2\\
&\leqslant c\|y-y_h\|_{1,\Omega}^2+c\varepsilon^2. \qquad (36)
\end{aligned}$$

Thus, (28) follows from (33) and (36). □

Theorem 4.4 *Let y and y_h be the solutions of* (2) *and* (3), *respectively. Then*

$$\|y-y_h\|_{0,\Omega}^2\leqslant c\eta_2^2, \qquad (37)$$

where

$$\eta_2^2=\sum_{\tau\in\mathscr{T}^h}h_\tau^4\|f-\lambda\max(0,y_h)\|_{0,\tau}^2+\sum_{l\cap\partial\Omega=\phi}h_l^3\Big\|\Big[\frac{\partial y_h}{\partial n}\Big]\Big\|_{0,l}^2. \qquad (38)$$

Proof The proof is similar to (27), we omit it here. □

5 Numerical algorithms and numerical examples

5.1 Numerical algorithms

In this subsection, we present two algorithms for discretized nonsmooth elliptic equation expressed in (3). The first one is simple and natural, it is actually a general iteration algorithm for nonlinear problems.

Algorithm 1

1. Initialization: choose initial value $y_h^0\in V_0^h$, iteration termination tolerance Tol, and set $n=1$.

2. Solve PDE

$$(\nabla y_h^n,\nabla v_h)=(f,v_h)-\lambda(\max(0,y_h^{n-1}),v_h),\quad \forall\ v_h\in V_0^h.$$

3. If $\|y_h^n-y_h^{n-1}\|_{0,\Omega}<Tol$, stop and output y_h^n.

4. Else, update $n=n+1$ and goto **2.**

Although Algorithm 1 is simple and can be implemented easily, it has a defect, i.e., the algorithm cannot converge when λ is very large. Our numerical results indicate that the algorithm performs well if $\lambda\leqslant 10$, but cannot converge if $\lambda\geqslant 10^2$. To overcome the difficulty, we use the idea of a primal-dual active set method for constrained optimal control problems (see Refs. [2,8]).

Note that the nonsmooth elliptic equation is equivalent to the following problem: find $u \in \{v \in L^2(\Omega) \mid v(x) \leqslant 0 \text{ a.e. in } \Omega\}$ such that

$$\begin{cases} -\Delta y = f + u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ \left(y + \dfrac{1}{\lambda}u, v - u\right) \geqslant 0, & \forall\, v \in \{v \in L^2(\Omega) \mid v(x) \leqslant 0 \text{ a.e. in } \Omega\}. \end{cases} \tag{39}$$

Let us introduce the active and inactive sets for the solution u of problem (39) and define

$$\mathscr{A} = \{x \mid u(x) = 0 \text{ a.e.}\}, \quad \mathscr{I} = \{x \mid u(x) < 0 \text{ a.e.}\}.$$

The inequality in (39) implies that

$$u = -\lambda \max(0, y) = \min(0, -\lambda y).$$

Now, we introduce a Lagrange multiplier $\mu \geqslant 0$ such that

$$u = -\lambda(\mu + y).$$

It is easy to see that

$$\mu \geqslant 0 \ \text{ in } \mathscr{A}, \quad \mu = 0 \ \text{ in } \mathscr{I}.$$

Therefore, system (39) can be expressed as

$$\begin{cases} -\Delta y = f + u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ u = -\lambda(\mu + y), & \\ \mu = c \max\left(0, u + \dfrac{\mu}{c}\right) & \end{cases} \tag{40}$$

for every $c > 0$. Setting $c = 1/\lambda$, the third and the fourth equalities in (40) imply that

$$\mu = \max\{0, -y\}. \tag{41}$$

Then the second algorithm based on (41) can be stated as:

Algorithm 2

1. Initialization: choose initial value $y_h^0 \in V_0^h$ and set $n = 1$.

2. Determine the following subsets of Ω:

$$\mathscr{A}_n = \{x \in \Omega \mid -y_h^{n-1}(x) > 0\},$$

$$\mathscr{I}_n = \{x \in \Omega \mid -y_h^{n-1}(x) \leqslant 0\}.$$

3. If $n \geqslant 2$ and $\mathscr{A}_n = \mathscr{A}_{n-1}$, stop and output y_h^n.

4. Else, solve PDE: $\forall\, v_h \in V_0^h$,

$$(\nabla y_h^n, \nabla v_h) = (f, v_h) \quad \text{in } \mathscr{A}_n,$$

$$(\nabla y_h^n, \nabla v_h) + \lambda(y_h^n, v_h) = (f, v_h) \quad \text{in } \mathscr{I}_n,$$

update $n = n + 1$ and goto **2.**

5.2 Numerical examples

In this subsection, we carry out some numerical experiments with the help of AFEPack package[1)] to confirm the theoretical results provided in this paper and the efficiency of the algorithms introduced in the last subsection.

In our numerical examples, we consider problem (2) with the domain $\Omega = [0, 1] \times [0, 1]$. We compute examples by using Algorithm 1 if $\lambda = 1$ or $\lambda = 10$, otherwise we use Algorithm 2, because Algorithm 1 cannot converge well when $\lambda \geq 10^2$. It should be pointed out that the two algorithms obtain the same results when $\lambda = 1$ or $\lambda = 10$. In the numerical computing, the initial value of the iteration is chosen to be the solution of the following equation:

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Example 5.1 We use the following data:

$$y = (x_1 - x_1^2)(x_2 - x_2^2) + \frac{1}{4}\sin(2\pi x_1)\sin(2\pi x_2),$$

$$f = -\Delta y + \lambda \max(0, y).$$

It is easy to see that the solution of Example 5.1 is very smooth. We compute it on uniform meshes. Then it is shown in Tables 1–4 that the convergence orders are $O(h^2)$ in $L^2(\Omega)$-norm and $O(h)$ in $H^1(\Omega)$-norm, respectively. Superconvergence property can be found from Tables 1 and 2, where Dofs is the number of the degree of freedoms. These numerical results are consistent with our theoretical analysis. It should be pointed that for $\lambda = 10^3$ and $\lambda = 10^4$, the convergence order is not stable when the mesh is not refined enough.

Table 1 Error of y on uniform mesh ($\lambda = 1$)

Dofs	$\|y - y_h\|_{0,\Omega}$		$\|y - y_h\|_{1,\Omega}$		$\|y - \pi_{2h} y_h\|_{1,\Omega}$	
	error	order	error	order	error	order
81	0.0212104	\	0.425258	\	0.245984	\
289	0.00566407	1.9049	0.219264	0.9557	0.0674946	1.8657
1089	0.00144041	1.9754	0.110497	0.9887	0.0172891	1.9649
4225	0.000361654	1.9938	0.0553578	0.9971	0.00434903	1.9911
16641	9.05121e-5	1.9984	0.0276926	0.9993	0.00108895	1.9978
66049	2.26351e-5	1.9995	0.0138480	0.9998	2.72349e-4	1.9994

1) Li R, Liu W B. http://circus.math.pku.edu.cn/AFEPack

Table 2 Error of y on uniform mesh ($\lambda = 10$)

Dofs	$\|y - y_h\|_{0,\Omega}$		$|y - y_h|_{1,\Omega}$		$|y - \pi_{2h} y_h|_{1,\Omega}$	
	error	order	error	order	error	order
81	0.0198295	\	0.425730	\	0.244644	\
289	0.00526401	1.9134	0.219333	0.9568	0.0669529	1.8695
1089	0.00133642	1.9778	0.110506	0.9890	0.0171362	1.9661
4225	0.000335397	1.9944	0.0553590	0.9972	0.00430963	1.9914
16641	8.39346e-5	1.9985	0.0276928	0.9993	0.00107904	1.9978
66049	2.09896e-5	1.9996	0.0138480	0.9998	2.69866e-4	1.9994

Table 3 Error of y on uniform mesh ($\lambda = 10^3$)

Dofs	$\|y - y_h\|_{0,\Omega}$		$|y - y_h|_{1,\Omega}$	
	error	order	error	order
81	0.0463161	\	0.552544	\
289	0.0165054	1.4886	0.271942	1.0228
1089	0.00665862	1.3096	0.131186	1.0517
4225	7.94557e-4	3.0670	0.0549743	1.2548
16641	2.16051e-4	1.8788	0.0270996	1.0205
66049	4.73881e-5	2.1888	0.0134889	1.0065

Table 4 Error of y on uniform mesh ($\lambda = 10^4$)

Dofs	$\|y - y_h\|_{0,\Omega}$		$|y - y_h|_{1,\Omega}$	
	error	order	error	order
81	0.0707078	\	0.736673	\
289	0.0306332	1.2068	0.422835	0.8009
1089	0.0150571	1.0247	0.248599	0.7663
4225	0.00543572	1.4699	0.112680	1.1416
16641	7.88491e-4	2.7853	0.0330531	1.7694
66049	1.52631e-4	2.3690	0.0140885	1.2303
263169	3.30633e-5	2.2067	0.00679589	1.0518

Example 5.2 We use the following data:

$$r = \sqrt{x_1^2 + x_2^2}, \quad \forall\ (x_1, x_2) \in \Omega,$$

$$y(r) = -r^{3/5}\gamma(r),$$

$$f = -\Delta y + \lambda \max(0, y),$$

where

$$\gamma(r) = \begin{cases} 1, & r < 0.25, \\ -192(r-0.25)^5 + 240(r-0.25)^4 & \\ \quad -80(r-0.25)^3 + 1, & 0.25 < r < 0.75, \\ 0, & \text{otherwise}. \end{cases}$$

The profile of the solution y is shown in Fig. 2. The solution has a singular point at $r = 0$, and is zero when $x_1^2 + x_2^2 \geqslant 0.75$. We compute Example 5.2

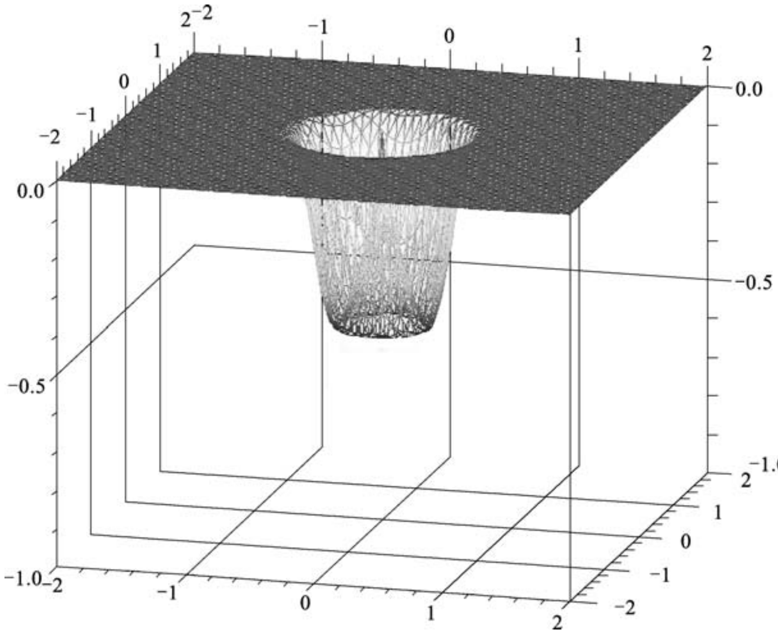

Fig. 2 Profile of y

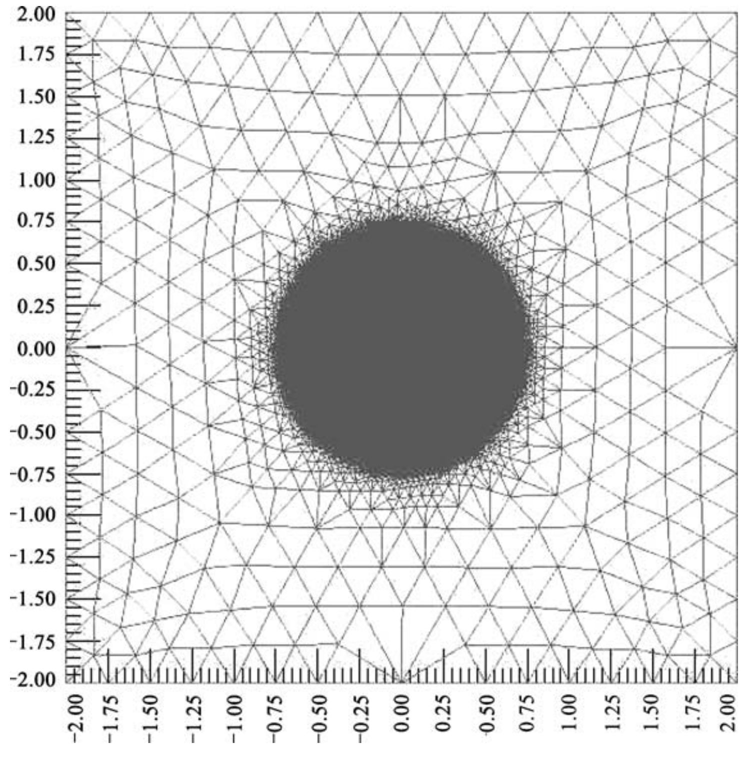

Fig. 3 Mesh of local refinement ($\lambda = 1$)

on uniform refined mesh and adaptive locally refined mesh, respectively. Fig. 3 illustrates that the meshes in $x_1^2 + x_2^2 \leqslant 0.75$ are much finer than the rest of the domains. The degrees of freedom and corresponding numerical results are presented in Tables 5 and 6. It can be found that the adaptive meshes can reduce Dofs greatly. Moreover, it can be seen clearly from Figs. 4 and 5 that the convergence order is $O(N^{-1/2})$, i.e., it is optimal, on the adaptive locally refined mesh, while the convergence order on the uniform refined mesh is not optimal. Thus, it demonstrates that the adaptive finite element methods can indeed save substantial computational work.

Table 5 Comparison of error $|y - y_h|_{1,\Omega}$ on uniform refined mesh and adaptive locally refined mesh ($\lambda = 1$)

uniform refined mesh		adaptive refined mesh	
Dofs	$\|y - y_h\|_{1,\Omega}$	Dofs	$\|y - y_h\|_{1,\Omega}$
4225	0.348999	1174	0.537986
16641	0.195401	3300	0.192077
66049	0.113061	11088	0.117229
263169	0.0680959	38365	0.0604397
1050625	0.0427120	115301	0.0362206

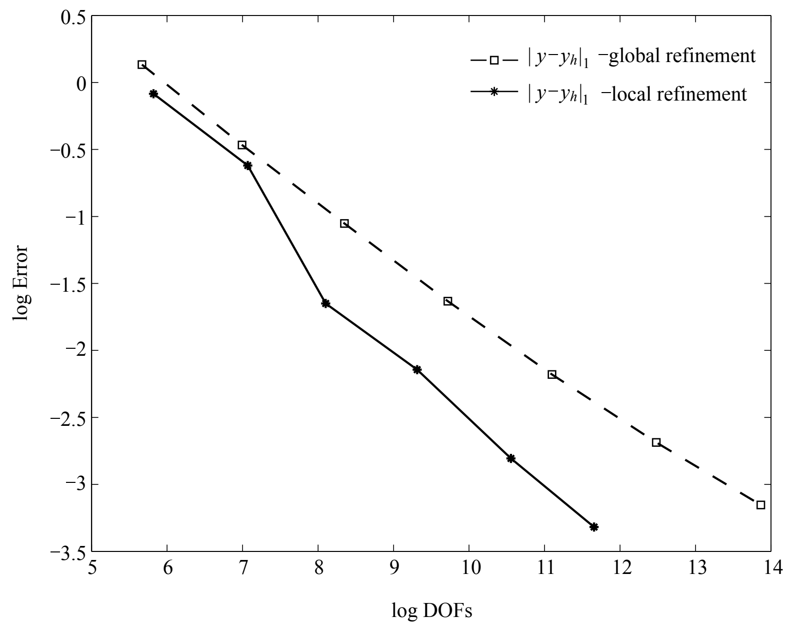

Fig. 4 Convergence history of adaptive refinement process and uniform refinement in case of $\lambda = 1$

Table 6 Comparison of error $|y-y_h|_{1,\Omega}$ on uniform refined mesh and adaptive locally refined mesh ($\lambda = 10^4$)

uniform refined mesh		adaptive refined mesh	
Dofs	$\|y-y_h\|_{1,\Omega}$	Dofs	$\|y-y_h\|_{1,\Omega}$
4225	0.751899	1245	0.537672
16641	0.195369	3839	0.191908
66049	0.113053	12517	0.117174
263169	0.0680945	39277	0.0604136
1050625	0.0426096	115819	0.0361846

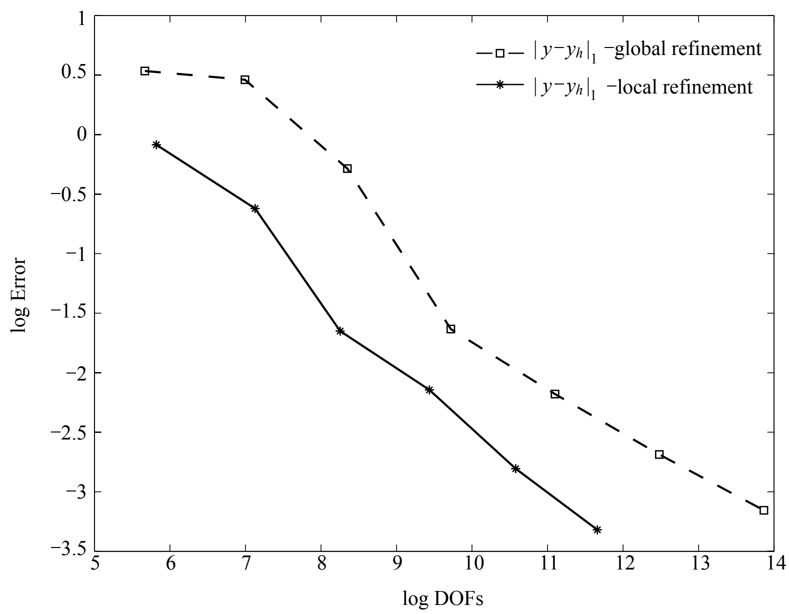

Fig. 5 Convergence history of adaptive refinement process and uniform refinement in case of $\lambda = 10^4$

Acknowledgements The authors thank the referees for their very helpful comments and suggestions, which improved the quality of this paper. This work was supported by the National Basic Research Program (Grant Nos. 2005CB321701, 2010CB731505) and the National Natural Science Foundation of China (Grant No. 10771211).

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