Front. Math. China 2009, 4(1): 49–61
DOI 10.1007/s11464-009-0010-z

RESEARCH ARTICLE

Spectral methods for pantograph-type differential and integral equations with multiple delays*

Ishtiaq ALI[1,2], **Hermann BRUNNER**[3,4], **Tao TANG**[4]

1 Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
2 Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan
3 Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NL A1C 5S7, Canada
4 Department of Mathematics, Hong Kong Baptist University, Hong Kong, China

Abstract We analyze the convergence properties of the spectral method when used to approximate smooth solutions of delay differential or integral equations with two or more vanishing delays. It is shown that for the pantograph-type functional equations the spectral methods yield the familiar exponential order of convergence. Various numerical examples are used to illustrate these results.

Keywords Delay differential equation, Volterra functional integral equation, multiple vanishing delays, Legendre spectral method, convergence analysis
MSC 34K28, 65Q05, 65R20

1 Introduction

Consider the multiple-delay pantograph differential equation

$$\begin{cases} y'(t) = a(t)y(t) + \sum_{l=1}^{r} b_l(t)y(q_l t), \quad t \in I := [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

and the analogous multiple-delay Volterra integral equation

* Received April 28, 2008; accepted November 19, 2008

Corresponding author: Ishtiaq ALI
E-mail: ishtiaq@lsec.cc.ac.cn, ishtiaqali@comsats.edu.pk

$$y(t) = g(t) + \sum_{l=1}^{r} \int_0^{q_l t} K_l(t,s) y(s) \mathrm{d}s, \quad t \in I, \tag{1.2}$$

where $0 < q_1 < \cdots < q_r < 1$ $(r \geqslant 2)$. It will be assumed that the given functions in (1.1)-(1.2) are smooth on their respective domains. This implies that the solutions of (1.1)-(1.2) are (globally) smooth on I. Thus, it appears natural to employ spectral methods for the numerical solution of these functional equations since the resulting spectral approximations are globally smooth, too, and in addition exhibit high-order convergence.

For ease of exposition, and without essential loss of generality, we will describe and analyze the spectral methods for the case of two proportional delays; that is, for the functional equations

$$\begin{cases} y'(t) = a(t)y(t) + b_1(t)y(q_1 t) + b_2(t)y(q_2 t), \quad t \in I, \\ y(0) = y_0, \end{cases} \tag{1.3}$$

and the delay Volterra integral equation

$$y(t) = g(t) + \int_0^{q_1 t} K_1(t,s) y(s) \mathrm{d}s + \int_0^{q_2 t} K_2(t,s) y(s) \mathrm{d}s, \tag{1.4}$$

with $0 < q_1 < q_2 < 1$.

The (quantitative and qualitative) theory of linear pantograph-type delay differential and integro-differential equations with multiple delays have been analyzed by, e.g., Iserles [8], Iserles and Liu [9], Derfel and Vogl [5], Qiu, Mitsui and Kuang [14], and Liu and Li [12]. To our knowledge, the numerical solution of such functional equations by high-order methods, in particular by collocation and spectral methods, has not yet been studied in detail. Compare, however, Refs. [10,17] where the optimal order of the collocation error at $t = t_1 = h$ was analyzed, and Refs. [11,12,14] in which the asymptotic stability of numerical solutions was studied. We also note that the powerful software package RADAR5, based on the 3-stage (fifth-order) Radau IIA Runge-Kutta method and developed by Guglielmi and Hairer (Ref. [7]; see also Ref. [6]) is designed to handle multiple-delay delay differential equations (DDEs).

It is known that the Fredholm-type integral equations behave more or less like a boundary value problem. As a result, some efficient numerical methods useful for boundary value problems (such as spectral methods) can be used directly to handle the Fredholm-type integral equations However, the Volterra equations of the second kind behave similar to an initial value problem. Therefore, it is very difficult and seems inappropriate to apply the spectral approximations to the Volterra-type integral equations. The main reason is that the second-kind Volterra equations uses the information from $[0, t]$ while the spectral methods use global basis functions in $[0, T]$. The main difficulty is how to implement the method so that spectral accuracy can be eventually obtained. On the other hand, the numerical methods for the Volterra equations will differ from those for the standard initial value problems in the sense that the former requires storage of all values at grid points while the latter only requires information at a fixed number of previous grid points. The storage requirement also makes the use of the global basis functions of the spectral methods more acceptable. For (1.4) without delay (i.e., $q_1 = q_2 = 1$), a spectral-collocation method is proposed in Ref. [16]. It was shown that the errors decay exponentially provided that the kernel function and the source function are sufficiently smooth. In Ref. [1], this idea was extended to the pantograph DDEs with a single delay.

In Section 2 we describe the spectral method for the multiple-delay pantograph delay differential equation (1.3) and the multiple-delay pantograph integral equation (1.4). This is followed, in Section 3, by corresponding results on the attainable order of convergence of these spectral methods and by remarks on their extension to equations with nonlinear vanishing delays. Section 4 is used to illustrate the convergence results by numerical examples.

2 Spectral discretizations

Let $\{t_k\}_{k=0}^N$ be the set of the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points in $[-1,1]$, and denote by $\mathscr{P}_N$ the space of real polynomials of degree not exceeding N.

2.1 Multiple-delay pantograph differential equation (1.3)

We will describe the spectral method to the integrated form of (1.3). Integration of (1.3) from $[0,t_i]$ leads to

$$y(t_i) = y_0 + \int_0^{t_i} a(s)y(s)\mathrm{d}s + \int_0^{t_i} b_1(s)y(q_1 s)\mathrm{d}s + \int_0^{t_i} b_2(s)y(q_2 s)\mathrm{d}s. \quad (2.1)$$

Employ the linear transformation

$$s_\theta^i = \frac{t_i}{2}\,\theta + \frac{t_i}{2},$$

let $\{\omega_k\}_{i=0}^N$ be the corresponding weights, and suppose that the spectral approximation $Y(t)$ has the form

$$y(t) \approx Y(t) := \sum_{j=0}^{N} y(t_j)F_j(t), \quad t \in [0,T], \quad (2.2)$$

where $F_j(t)$ are the Lagrange canonical polynomials with respect to the set $\{\theta_k\}_{k=0}^N$. Then the spectral approximation to the transformed equation (2.1),

$$\begin{aligned} y(t_i) = y_0 &+ \frac{t_i}{2}\int_{-1}^{1} a(s_\theta^i)y(s_\theta^i)\mathrm{d}\theta + \frac{t_i}{2}\int_{-1}^{1} b_1(s_\theta^i)y(q_1 s_\theta^i)\mathrm{d}\theta \\ &+ \frac{t_i}{2}\int_{-1}^{1} b_2(s_\theta^i)y(q_2 s_\theta^i)\mathrm{d}\theta, \end{aligned} \quad (2.3)$$

is determined by the spectral approximations

$$Y(t_i) = y_0 + \frac{t_i}{2}\sum_{k=0}^{N} a(s_{ik})Y(s_{ik})w_k + \frac{t_i}{2}\sum_{k=0}^{N} b_1(s_{ik})Y(q_1 s_{ik})w_k$$
$$+\frac{t_i}{2}\sum_{k=0}^{N} b_2(s_{ik})Y(q_2 s_{ik})w_k, \qquad (2.4)$$

where

$$Y(s_{ik}) := \sum_{j=0}^{N} y(t_j)F_j(s_{ik}), \quad s_{ik} := s^i_{\theta_k} = \frac{t_i}{2}(\theta_k + 1).$$

Setting

$$Y_N := [Y(t_0), \ldots, Y(t_N)]^{\mathrm{T}}, \quad b := [y_0, \ldots, y_0]^{\mathrm{T}}$$

and

$$A_{ij} := \frac{t_i}{2}\sum_{k=0}^{N}(a(s_{ik})F_j(s_{ik}) + b_1(s_{ik})F_j(q_1 s_{ik}) + b_2(s_{ik})F_j(q_2 s_{ik}))w_k$$

$(i, j = 0, 1, \ldots, N)$, we can write the spectral equations (2.4) in the form $(I - A_N)Y_N = b$, where $A_N := [A_{ij}] \in \mathbb{R}^{(N+1)\times(N+1)}$.

2.2 Multiple-delay Volterra integral equation (1.4)

Since the delay Volterra integral equation (1.4) is a generalization of the integrated form (2.1) of the delay differential equation (1.3), the previous section contains all the essential ingredients for describing the spectral approximation for (1.4) (transformed on the interval $[-1, 1]$). Thus, the spectral equations for the transformed equation (1.4) with respect to the points $\{t_i\}_{i=0}^N$ are given by

$$y(t_i) = g(t_i) + \int_0^{q_1 t_i} K_1(t_i, s)Y(s)\mathrm{d}s + \int_0^{q_2 t_i} K_2(t_i, s)Y(s)\mathrm{d}s$$
$$= g(t_i) + \frac{q_1 t_i}{2}\int_{-1}^{1} K_1(t_i, \theta_1)Y(\theta_1)\mathrm{d}\theta + \frac{q_2 t_i}{2}\int_{-1}^{1} K_2(t_i, \theta_2)y(\theta_2)\mathrm{d}\theta, \quad (2.5)$$

where

$$\theta_1 = q_1 t_i(\theta + 1)/2, \quad \theta_2 = q_2 t_i(\theta + 1)/2.$$

We can approximate the integral by using Gauss quadrature and $y(t)$ by using $Y(t)$:

$$Y(t_i) = y_0 + \frac{q_1 t_i}{2}\sum_{k=0}^{N} K_1\Big(t_i, \frac{q_1 t_i}{2}(\theta_k + 1)\Big)Y_j F_j\Big(\frac{q_1 t_i}{2}(\theta_k + 1)\Big)w_k$$
$$+\frac{q_2 t_i}{2}\sum_{k=0}^{N} K_2\Big(t_i, \frac{q_2 t_i}{2}(\theta_k + 1)\Big)Y_j F_j\Big(\frac{q_2 t_i}{2}(\theta_k + 1)\Big)w_k, \qquad (2.6)$$

If we write it into matrix form, we obtain $(I-A_N)Y_N = g$, where the elements of the matrix $A_N := [A_{(i,j)}]$ are given by

$$A_{(i,j)} = \frac{q_1 t_i}{2}\sum_{k=0}^{N} K_1\Big(t_i, \frac{q_1 t_i}{2}(\theta_k+1)\Big)F_j\Big(\frac{q_1 t_i}{2}(\theta_k+1)\Big)w_k$$

$$+\frac{q_2 t_i}{2}\sum_{k=0}^{N} K_2\Big(t_i, \frac{q_2 t_i}{2}(\theta_k+1)\Big)F_j\Big(\frac{q_2 t_i}{2}(\theta_k+1)\Big)w_k. \tag{2.7}$$

3 Convergence analysis

3.1 Some auxiliary results

To carry out the convergence analysis of our spectral methods we first introduce some useful lemmas.

Lemma 3.1 [4] *Assume that an $(N+1)$-point Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight is used to integrate the product $y\phi$, where $y \in H^m(I)$ with $I := (-1,1)$ for some $m \geqslant 1$ and $\phi \in \mathscr{P}_N$. Then there exists a constant C independent of N such that*

$$\left|\int_{-1}^{1} y(x)\phi(x)\mathrm{d}x - (y,\phi)_N\right| \leqslant CN^{-m}|y|_{\widetilde{H}_{m,N}(I)}\|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$|y|_{\widetilde{H}_{m,N+1}(I)} = \Bigg(\sum_{k=\min(m,N)}^{m} \|y^{(k)}\|^2_{L^2(I)}\Bigg)^{1/2}, \quad (y,\phi)_N = \sum_{k=0}^{N}\omega_k y(x_k)\phi(x_k).$$

Lemma 3.2 *Assume that $y \in H^m(I)$ and denote by $I_N y$ the interpolation polynomial associated with the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points $\{x_k\}_{k=0}^N$. Then*

$$\|y - I_N y\|_{L^2(I)} \leqslant CN^{-m}|y|_{\widetilde{H}_{m,N}(I)}, \tag{3.2}$$

$$\|y - I_N y\|_{L^\infty(I)} \leqslant CN^{\frac{1}{2}-m}|y|_{\widetilde{H}_{m,N}(I)}. \tag{3.3}$$

Proof Estimate (3.2) is given on p. 289 of Ref. [4]. The estimate

$$\|y - I_N y\|_{H^s(I)} \leqslant CN^{2s-\frac{1}{2}-m}|y|_{\widetilde{H}_{m,N}(I)}, \quad 1 \leqslant s \leqslant m,$$

is also given in Ref. [4]. Using the above estimate and the inequality

$$\|v\|_{L^\infty(a,b)} \leqslant \sqrt{\frac{1}{b-a}+2}\ \|v\|^{1/2}_{L^2(a,b)}\ \|v\|^{1/2}_{H^1(a,b)}, \quad \forall\ v \in H^1(a,b),$$

we obtain (3.3). □

From Ref. [13], we have the following result on the Lebesgue constant for Lagrange interpolation based on the zeros of the Legendre polynomials.

Lemma 3.3 *Assume that $F_j(t)$ is the j-th Lagrange interpolation polynomial with respect to the $(N+1)$ Gauss-Legendre, or Gauss-Radau, or Gauss-Lobatto points $\{t_k\}_{k=0}^{N}$. Then*

$$\max_{t\in I}\sum_{j=0}^{N}|F_j(t)| \leqslant C\sqrt{N}. \tag{3.4}$$

The next lemma presents the extension of the classical Gronwall lemma to integral equations with two proportional delays; its version for an arbitrary number of such delays is obvious.

Lemma 3.4 (Gronwall inequality) *Let $T>0$ and $C_1, C_2, C_3 \geqslant 0$. If a non-negative continuous function $E(t)$ satisfies*

$$E(t) \leqslant C_1\int_0^t E(q_1 s)\mathrm{d}s + C_2\int_0^t E(q_2 s)\mathrm{d}s + C_3\int_0^t E(s)\mathrm{d}s + G(t), \quad \forall\ t\in[0,T], \tag{3.5}$$

where $q_1, q_2 \in (0,1)$ are constants and $G(t)$ is a continuous function. Then

$$\|E\|_{L^\infty(I)} \leqslant C\|G\|_{L^\infty(I)}. \tag{3.6}$$

Proof For $E \geqslant 0$, since $q_1 \in (0,1)$ and $G(s) \geqslant 0$, it follows from (3.5) and a simple change of variables that

$$\int_0^{q_1 t} E(s)\mathrm{d}s = q_1^{-1}\int_0^{q_1 t} E(s)\mathrm{d}s \leqslant C_1 q_1^{-1}\int_0^t E(s)\mathrm{d}s.$$

Similarly

$$\int_0^{q_2 t} E(s)\mathrm{d}s = q_2^{-1}\int_0^{q_2 t} E(s)\mathrm{d}s \leqslant C_2 q_2^{-1}\int_0^t E(s)\mathrm{d}s,$$

which implies that

$$E(t) \leqslant C\int_0^t E(s)\mathrm{d}s + G(t),$$

which is a standard Gronwall inequality. This leads to estimate (3.6). □

We are now ready to state and prove our two main theorems on the convergence of the spectral method when applied to multiple-delay pantograph DDE (1.3) and the multiple-delay Volterra integral equation (1.4).

3.2 Convergence of spectral approximations for (1.3)

Theorem 3.1 *Consider the pantograph differential equation* (1.3) *and its spectral approximation* (2.4). *If functions a, b_1 and b_2 are sufficiently smooth* (*which implies that the solution of* (1.1) *is similarly smooth*), *then*

$$\begin{aligned}\|Y-y\|_{L^\infty(I)} \leqslant\ & CN^{-m-\frac{1}{2}}|ay|_{\widetilde{H}_{m,N}(I)} + CN^{-m-\frac{1}{2}}|b_1 y(q_1 t)|_{\widetilde{H}_{m,N}(I)} \\ & + CN^{\frac{1}{2}-m}(|a|+|b_1|+|b_2|)_{\widetilde{H}_{m,N}(I)}\|y\|_{L^2(I)} \\ & + CN^{-m-\frac{1}{2}}|b_2 y(q_2 t)|_{\widetilde{H}_{m,N}(I)}, \end{aligned} \tag{3.7}$$

where Y *is the polynomial of degree* N *associated with the spectral approximation* (2.4), *and* C *is a constant independent of* N.

Remark 3.1 We point out that

(i) Theorem 3.1 remains true for any finite number of proportional delays, and

(ii) for $r=1$, Theorem 3.1 reduces to the result in Ref. [1].

Proof of Theorem 3.1 Following the notations of (2.4), let

$$[Y]_{N,s} = \frac{t_i}{2}\sum_{k=0}^{N} a(s_{ik})Y(s_{ik})w_k.$$

Then the second and third term on the right-hand side of (2.4) can be written as $[Y]_{N,q_1 s}$ and $[Y]_{N,q_2 s}$, respectively. It follows from the numerical scheme (2.4) that

$$Y = y_0 + [aY]_{N,s} + [b_1 Y]_{N,q_1 s} + [b_2 Y]_{N,q_2 s}, \tag{3.8}$$

which gives

$$\begin{aligned} Y(t_i) = y_0 &+ \frac{t_i}{2}\int_{-1}^{1} a(s_\theta^i)Y(s_\theta^i)\mathrm{d}\theta + \frac{t_i}{2}\int_{-1}^{1} b_1(s_\theta^i)Y(q_1 s_\theta^i)\mathrm{d}\theta \\ &+ \frac{t_i}{2}\int_{-1}^{1} b_2(s_\theta^i)Y(q_2 s_\theta^i)\mathrm{d}\theta + I_{i,1} + I_{i,2} + I_{i,3}, \quad 0 \leqslant i \leqslant N, \end{aligned} \tag{3.9}$$

where

$$I_{i,1} := \frac{t_i}{2}\int_{-1}^{1} a(s_\theta^i)Y(s_\theta^i)\mathrm{d}\theta - [aY]_{N,s},$$

$$I_{i,2} := \frac{t_i}{2}\int_{-1}^{1} b_1(s_\theta^i)Y(q_1 s_\theta^i)\mathrm{d}\theta - [b_1 Y]_{N,q_1 s},$$

$$I_{i,3} := \frac{t_i}{2}\int_{-1}^{1} b_2(s_\theta^i)Y(q_2 s_\theta^i)\mathrm{d}\theta - [b_2 Y]_{N,q_2 s}.$$

Employing Lemma 3.1 we then obtain the estimates

$$|I_{i,1}| \leqslant C_1 N^{-m}|a|_{H_{m,N}}\,\|Y\|_{L^2(I)}, \tag{3.10}$$

$$|I_{i,2}| \leqslant C_2 N^{-m}|b_1|_{H_{m,N}}\,\|Y\|_{L^2(I)}, \tag{3.11}$$

$$|I_{i,3}| \leqslant C_3 N^{-m}|b_2|_{H_{m,N}}\,\|Y\|_{L^2(I)}. \tag{3.12}$$

Multiplying $F_j(t)$ on both sides of (3.9) and summing up from 0 to N yields

$$Y(t) = I_N\left(\int_0^t aY(s)\mathrm{d}s\right) + I_N\left(\int_0^t b_1Y(q_1s)\mathrm{d}s\right)$$
$$+I_N\left(\int_0^t b_2Y(q_2s)\mathrm{d}s\right) + y_0 + J_1(t), \tag{3.13}$$

where we have used the fact that

$$\sum_{j=0}^{N} F_j(t) \equiv 1$$

and defined

$$J_1(t) := \sum_{j=0}^{N}(I_{i,1} + I_{i,2} + I_{i,3})F_j(t).$$

Let

$$e(t) := Y(t) - y(t)$$

(with

$$e(q_1t) = Y(q_1t) - y(q_1t), \quad e(q_2t) = Y(q_2t) - y(q_2t))$$

denote the spectral error. It follows from (3.13) that

$$Y(t) = I_N\left(\int_0^t ay(s)\mathrm{d}s\right) + I_N\left(\int_0^t b_1y(q_1s)\mathrm{d}s\right)$$
$$+ I_N\left(\int_0^t b_2y(q_2s)\mathrm{d}s\right) + J_1(t) + I_N\left(\int_0^t ae(s)\mathrm{d}s\right)$$
$$+I_N\left(\int_0^t b_1e(q_1s)\mathrm{d}s\right) + I_N\left(\int_0^t b_2e(q_2s)\mathrm{d}s\right). \tag{3.14}$$

Thus, combining (3.14) with (2.1) gives

$$e(t) = \int_0^t ae(s)\mathrm{d}s + \int_0^t b_1e(q_1s)\mathrm{d}s + \int_0^t b_2e(q_2s)\mathrm{d}s + J_2(t) + J_3(t), \tag{3.15}$$

where

$$J_2(t) := I_N\left(\int_0^t ay(s)\mathrm{d}s\right) - \int_0^t ay(s)\mathrm{d}s + I_N\left(\int_0^t b_1y(q_1s)\mathrm{d}s\right)$$
$$- \int_0^t b_1y(q_1s)\mathrm{d}s + I_N\left(\int_0^t b_2y(q_2s)\mathrm{d}s\right) - \int_0^t b_2y(q_2s)\mathrm{d}s,$$

$$J_3(t) := I_N\left(\int_0^t ae(s)\mathrm{d}s\right) - \int_0^t ae(s)\mathrm{d}s + I_N\left(\int_0^t b_1e(q_1s)\mathrm{d}s\right)$$
$$- \int_0^t b_1e(q_1s)\mathrm{d}s + I_N\left(\int_0^t b_2e(q_2s)\mathrm{d}s\right) - \int_0^t b_2e(q_2s)\mathrm{d}s.$$

According to the Gronwall estimate of Lemma 3.4, we find

$$\|e\|_{L^\infty(I)} \leqslant C(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)} + \|J_3\|_{L^\infty(I)}). \tag{3.16}$$

We will now derive estimates for $\|J_1\|_{L^\infty(I)}$, $\|J_2\|_{L^\infty(I)}$ and $\|J_3\|_{L^\infty(I)}$. First,

$$\begin{aligned}\|J_1\|_{L^\infty(I)} &\leqslant C(\|I_{i,1}\|_{L^\infty(I)} + \|I_{i,2}\|_{L^\infty(I)} + \|I_{i,3}\|_{L^\infty(I)}) \max_{t\in I} \sum_{j=0}^{N} F_j(t) \\ &\leqslant CN^{\frac{1}{2}-m}(|a|_{H_{m,N}} + |b_1|_{H_{m,N}} + |b_2|_{H_{m,N}})\|Y\|_{L^2(I)} \\ &\leqslant CN^{\frac{1}{2}-m}(|a|_{H_{m,N}} + |b_1|_{H_{m,N}} + |b_2|_{H_{m,N}}) \\ &\quad \times(\|e\|_{L^\infty} + \|y\|_{L^2(I)}),\end{aligned} \tag{3.17}$$

where we have used Lemma 3.3. Next, we obtain

$$\begin{aligned}\|J_2\|_{L^\infty(I)} &\leqslant CN^{-m-\frac{1}{2}}|y|_{\widetilde{H}_{m,N}(I)} + CN^{-m-\frac{1}{2}}|y(q_1t)|_{\widetilde{H}_{m,N}(I)} \\ &\quad +CN^{-m-\frac{1}{2}}|y(q_2t)|_{\widetilde{H}_{m,N}(I)}.\end{aligned} \tag{3.18}$$

To derive the final estimate we employ Lemma 3.2 and find

$$\|J_3\|_{L^\infty(I)} \leqslant CN^{-1/2}\|e\|_{L^\infty}. \tag{3.19}$$

The above three estimates, together with (3.16), yield

$$\begin{aligned}\|e\|_{L^\infty(I)} &\leqslant CN^{\frac{1}{2}-m}(\|e\|_{L^\infty(I)} + \|y\|_{L^2(I)})CN^{-m-\frac{1}{2}}|y(q_1t)|_{\widetilde{H}_{m,N}(I)} \\ &\quad + CN^{-m-\frac{1}{2}}|y|_{\widetilde{H}_{m,N}(I)} + CN^{-m-\frac{1}{2}}|y(q_2t)|_{\widetilde{H}_{m,N}(I)} \\ &\quad + CN^{-\frac{1}{2}}\|e\|_{L^\infty},\end{aligned}$$

from which the result of Theorem 3.1 follows. □

3.3 Convergence of spectral approximations for (1.4)

The analogue of Theorem 3.1, namely the following theorem for the pantograph-type Volterra integral equation (1.2), can be proved by using an obvious adaptation of the analysis in Section 3.2.

Theorem 3.2 *Consider the pantograph differential equation* (1.2) *and its spectral approximation. If the function $g = g(t)$ and the kernels $K_1(t,s)$ and $K_2(t,s)$ in* (1.4) *are smooth* (*implying that the solution of* (1.4) *is smooth on I*), *then*

$$\begin{aligned}\|Y-y\|_{L^\infty(I)} &\leqslant CN^{-m-\frac{1}{2}}|y|_{\widetilde{H}_{m,N}(I)} + CN^{\frac{1}{2}-m} \max_{1\leqslant i\leqslant N}(|K_1(t_i,s(t_i,.))|_{\widetilde{H}_{m,N}(I)} \\ &\quad +|K_2(t_i,s(t_i,.))|_{\widetilde{H}_{m,N}(I)})\|y\|_{L^2(I)},\end{aligned} \tag{3.20}$$

where Y is the polynomial of degree N associated with the spectral approximation, and C is a constant independent of N.

Remark 3.2 Again we point out that

(i) Theorem 3.2 remains true for any finite number of proportional delays, and

(ii) for $r = 1$, Theorem 3.2 reduces to the result in Ref. [1].

3.4 Nonlinear vanishing delays

In the preceding analysis we considered the multi-delay differential equation (1.1) and the multiple-delay Volterra integral equation (1.2) with *linear* (pantograph-type) delay functions $\theta_k(t) = q_k t$ $(0 < q_1 < \cdots < q_r < 1,\ r \geqslant 2)$. A close look at the proofs of the convergence theorems in this section reveals that the analysis and hence the spectral convergence results in Theorems 3.1 and 3.2 remain valid for smooth *nonlinear* delay functions $\theta_k = \theta_k(t)$ $(k = 1, \ldots, r,\ r \geqslant 2)$ that are subject to the following assumptions:

(D1) $\theta_k(0) = 0$ and θ_k is strictly increasing on I $(k = 1, \ldots, r)$;

(D2) $0 < \theta_1(t) < \cdots < \theta_r(t) \leqslant \bar{q}t$ $(t \in (0, T])$, for some $\bar{q} \in (0, 1)$;

(D3) $\theta_k \in C^d(I)$ $(k = 1, \ldots, r)$ for some $d \geqslant 1$.

The proofs of Theorems 3.1 and 3.2 are then readily adapted to deal with pantograph-type functional differential and integral equations (1.3) and (1.4), respectively, containing these more general vanishing nonlinear delays: the convergence estimates of Theorems 3.1 and 3.2 remain valid since—by assumption (D2)—the Gronwall estimate in Lemma 3.4 carries over if the right-hand side of (3.5) contains r terms corresponding to the nonlinear delays θ_k. We leave the details of the proof to the reader.

4 Numerical examples

In the following, we use numerical examples to illustrate the accuracy and efficiency of the spectral methods (2.4) and (2.6). In our computations, we use the Legendre-Gauss quadrature with weights

$$\omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 < j \leqslant N.$$

Example 4.1 Let $b_1(t) = \cos t,\ b_2(t) = \sin t$ and $a(t) = 0$ in (1.3). Choose $g(t)$ such that the exact solution is then given by

$$y(t) = \sin(tq_1^{-1}) + \cos(tq_2^{-1}).$$

In Table 1, we list errors in various norms with $q_1 = 0.05$ and $q_2 = 0.95$ for Example 4.1. This is a quite extreme case with a very small value of the delay parameter q_1. For the polynomial collocation methods, it will require a few hundred collocation points to reach the errors of about 10^{-7}; while with the spectral approach only 20 points are needed. For another extreme case, $q_1 = 0.5$ and $q_2 = 0.99$, it is seen in Fig. 1 that machine accuracy is obtained with only 14 collocation points.

Table 1 Example 4.1: point-wise error using (2.4). $q_1 = 0.05,\ q_2 = 0.95$

N	L^∞	L^1	L^2
6	9.305e−002	8.120e−002	7.374e−002
8	2.586e−001	2.435e−001	2.119e−001
10	7.635e−002	9.814e−002	7.911e−002
12	1.139e−002	1.311e−002	1.071e−002
14	1.658e−003	1.506e−003	1.248e−003
16	2.073e−004	1.925e−004	1.634e−004
18	1.370e−005	1.506e−005	1.227e−005
20	7.220e−007	6.953e−007	5.908e−007

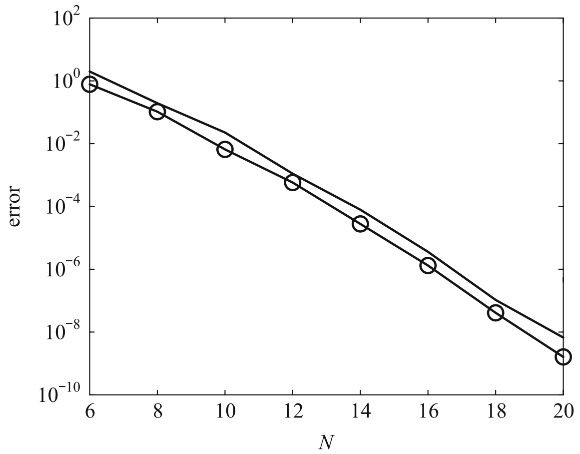

Fig. 1 Example 4.1: L^∞ (solid line) and L^2 (circle) errors obtained by using (2.4) for $q_1 = 0.5$ and $q_2 = 0.99$

Example 4.2 Let $K_1(t,s) = \cos(t-s)$, and $K_2(t) = \sin(t-s)$ in (1.4). The exact solution is given by $y(t) = \cos t$.

In Table 2, we list errors in various norms with $q_1 = 0.05$ and $q_2 = 0.95$ for Example 4.2. With a very small value of the delay parameter q_1, machine accuracy is achieved with about 10 spectral collocation points. For another extreme case, $q_1 = 0.5$ and $q_2 = 0.99$, it is seen in Fig. 2 that machine accuracy is obtained with about 10 collocation points.

Table 2 Example 4.2: point-wise error using (2.6). $q_1 = 0.05,\ q_2 = 0.95$

N	L^∞	L^1	L^2
6	2.534e−010	2.427e−010	2.125e−010
8	1.908e−013	2.105e−013	1.731e−013
10	2.220e−016	2.762e−016	2.373e−016
12	4.441e−016	3.636e−016	3.454e−013
14	3.331e−016	2.550e−016	2.088e−016
16	4.441e−016	4.317e−016	3.628e−016
18	4.441e−016	3.042e−016	2.593e−016
20	4.441e−016	2.727e−016	2.606e−016

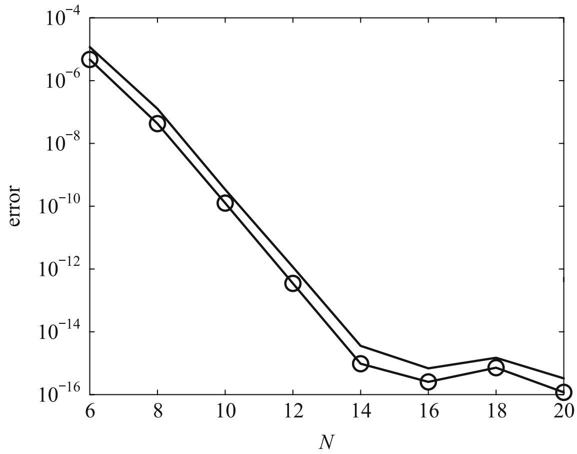

Fig. 2 Example 4.2: L^∞ (solid line) and L^2 (circle) errors obtained by using (2.6) for $q_1 = 0.5$ and $q_2 = 0.99$

5 Concluding remarks

We have shown that the spectral method yields an efficient and very accurate numerical method for the approximation of solutions to pantograph-type delay differential and Volterra integral equations with multiple proportional delays. In the quite difficult cases with small q_1 or q_2 close to 1, the proposed method can yield very accurate approximations with a small number of spectral collocation points. The method is readily extended to equations with nonlinear multiple vanishing delays.

Acknowledgements The research of Ali and Tang was supported in part by the Hong Kong Research Grants Council and Hong Kong Baptist University. The research of Brunner was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC Discovery Grant No. 9406) and Hong Kong Research Grants Council's CERG Grant (CERG HKBU 200207). Tang's research was also supported by a Joint Research Fund for Hong Kong and Macau Young Scholars awarded by the National Natural Science Foundarion of China.

References

1. Ali I, Brunner H, Tang T. A spectral method for pantograph-type delay differential equations and its convergence analysis
2. Bellen A, Zennaro M. Numerical Methods for Delay Differentials Equations. Oxford: Oxford University Press, 2003
3. Brunner H. Collocation Methods for Volterra Integral and Related Functional Equations. Cambridge: Cambridge University Press, 2004
4. Canuto C, Hussaini M, Quarteroni A, Zang T A. Spectral Methods: Fundamentals in Single Domains. Berlin: Springer, 2006

5. Derfel G A, Vogl F. On the asymptotics of solutions of a class of linear functional-differential equations. Europ J Appl Math, 1996, 7: 511–518
6. Guglielmi N. Short proofs and a counterexample for analytical and numerical stability of delay equations with infinite memory. IMA J Numer Anal, 2006, 26: 60–77
7. Guglielmi N, Hairer E. Implementing Radau IIA methods for stiff delay differential equations. Computing, 2001, 67: 1–12
8. Iserles A. On the generalized pantograph functional-differential equation. Europ J Appl Math, 1993, 4: 1–38
9. Iserles A, Liu Y -K. On pantograph integro-differential equations. J Integral Equations Appl, 1994, 6: 213–237
10. Ishiwata E. On the attainable order of collocation methods for the neutral functional-differential equations with proportional delays. Computing, 2000, 64: 207–222
11. Li D, Liu M Z. Asymptotic stability of numerical solution of pantograph delay differential equations. J Harbin Inst Tech, 1999, 31: 57–59 (in Chinese)
12. Liu M Z, Li D. Properties of analytic solution and numerical solution of multi-pantograph equation. Appl Math Comput, 2004, 155: 853–871
13. Mastroianni G, Occorsio D. Optimal systems of nodes for Lagrange interpolation on bounded intervals: A survey. J Comput Appl Math, 2001, 134: 325–341
14. Qiu L, Mitsui T, Kuang J X. The numerical stability of the θ-method for delay differential equations with many variable delays. J Comput Math, 1999, 17: 523–532
15. Shen J, Tang T. Spectral and High-Order Methods with Applications. Beijing: Science Press, 2006
16. Tang T, Xu X, Cheng J. On spectral methods for Volterra type integral equations and the convergence analysis. J Comput Math, 2008, 26: 825–837
17. Zhao J, Xu Y, Qiao Y. The attainable order of the collocation method for double-pantograph delay differential equation. Numer Math J Chinese Univ, 2005, 27: 297–308 (in Chinese)