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RESEARCH ARTICLE

New approach to the numerical solution of forward-backward equations*

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Abstract This paper is concerned with the approximate solution of functional differential equations having the form: $x'(t) = \alpha x(t) + \beta x(t-1) + \gamma x(t+1)$. We search for a solution x, defined for $t \in [-1, k]$, $k \in \mathbb{N}$, which takes given values on intervals $[-1, 0]$ and $(k-1, k]$. We introduce and analyse some new computational methods for the solution of this problem. Numerical results are presented and compared with the results obtained by other methods.

Keywords Mixed-type functional differential equations, collocation method, theta-method, method of steps
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1 Introduction

The present paper is devoted to the solution of linear functional differential equations with both delayed and advanced arguments with the form

$$x'(t) = \alpha x(t) + \beta x(t-1) + \gamma x(t+1). \tag{1}$$

Such equations are often referred to in the literature as mixed type functional differential equation (MTFDE) or forward-backward equations. The analysis of this type of equation has begun comparatively recently and is less developed compared with other classes of functional equations. Many important questions remain open. Interest in MTFDEs is motivated by problems in optimal control (see Ref. [9]) and applications also arise in nerve conduction

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[2], economic dynamics [10] and travelling waves in a spatial lattice [1]. Some problems about the decomposition of solutions of MTFDEs were investigated in the works of Mallet-Paret and Verduyn-Lunel [6,7].

In Refs. [3,4], a particular case of (1) is considered, with $\alpha = 0$. As remarked in that work, (1) can be reduced to an equation without the term $\alpha x(t)$ by means of a variable substitution. The authors searched for a solution of this equation which satisfies the boundary conditions

$$x(t) = \begin{cases} \varphi_1(t), & t \in [-1, 0], \\ f(t), & t \in (k-1, k], \end{cases} \tag{2}$$

where φ_1 and f are smooth real-valued functions, defined on $[-1, 0]$ and $(k-1, k]$, respectively $(1 < k \in \mathbb{N})$. In order to analyse and solve this boundary value problem (BVP) the authors considered an initial value problem (IVP), with the conditions

$$x(t) = \varphi(t), \quad t \in [-1, 1], \tag{3}$$

where the function φ is defined by

$$\varphi(t) = \begin{cases} \varphi_1(t), & t \in [-1, 0], \\ \varphi_2(t), & t \in (0, 1], \end{cases} \tag{4}$$

In this case, to solve problem (1)-(2) one needs to determine the function φ_2 such that the solution of (1) satisfies the second boundary condition of (2). This reformulation provides the basis for both analytical and numerical construction of solutions using ideas based on Bellman's method of steps for solving delay differential equations. One solves the equation over successive intervals of length unity. An initial value problem of the same type was previously investigated by Iakovleva and Vanegas [5] for a particular case of (1) (with $\alpha = 0,\ \beta = \gamma = 1$). Using the same approach, in Ref. [3], an algorithm was presented for constructing a smooth solution of (1), in the case $\alpha = 0$, on the interval $[1, k]$, for any k (provided that the initial function φ satisfies certain conditions). Noting that in this case (1) can be rewritten in the form

$$x(t+1) = ax'(t) + bx(t-1), \tag{5}$$

where $a = 1/\gamma,\ b = -\beta/\gamma$, the authors have shown that any solution of (5) which satisfies (4) can be computed using the following formulae:

$$\begin{aligned} x(t) = &\sum_{k=0}^{l-1} \gamma_{l,2k} a^{2k} b^{l-k} \varphi^{(2k)}(t-2l) \\ &+ \sum_{k=0}^{l-1} \gamma_{l,2k+1} a^{2k+1} b^{l-k-1} \varphi^{(2k+1)}(t-(2l-1)), \quad t \in (2l-1, 2l), \end{aligned} \tag{6}$$

or

$$x(t) = \sum_{k=0}^{l} \delta_{l,2k} a^{2k} b^{l-k} \varphi^{(2k)}(t-2l)$$

$$+\sum_{k=0}^{l-1}\delta_{l,2k+1}a^{2k+1}b^{l-k}\varphi^{(2k+1)}(t-(2l+1)), \quad t\in(2l,2l+1), \qquad (7)$$

$l=1,2,\ldots.$

Here, $\gamma_{i,j}$ and $\delta_{i,j}$ are real coefficients that can be computed recursively. One can use these formulae to provide basic existence theory.

Theorem 1 [3] *The solution to problem* (5)-(3) *with* $\varphi\in C^{\infty}_{[-1,1]}$ *exists and is differentiable if and only if*

$$\varphi^{(n+1)}(0)=\gamma\varphi^{(n)}(1)+\beta\varphi^{(n)}(-1), \quad n=0,1,2,\ldots. \qquad (8)$$

The relationship between the IVP (5)-(3) and the BVP (1)-(2) is complex. While it is straightforward to determine the BVP corresponding to a given IVP, the inverse problem is both ill-posed and highly unstable. Indeed, it may not be possible to solve a given BVP using this method (or at all). In other words, existence and uniqueness results are available for problem (5)-(3) but not for problem (1)-(2). One must study the associated numerical methods with an awareness of the dangers inherent in this observation.

In this paper, we continue the study started in Ref. [3]. Our goal is to develop new numerical approaches to the solution of problem (1)-(2) and to provide a comparative analysis of the numerical results.

In Section 2, we revisit the numerical algorithms proposed in Ref. [3] and describe new approaches, based on collocation and least squares methods. We also show how to reduce the considered BVP for an MTFDE to a BVP for an ordinary differential equation (ODE). In Section 3, we give a preliminary error analysis for the algorithm based on the collocation method. In Section 4, we present and discuss numerical results obtained by the different methods. Finally, in Section 5 we summarize the conclusions of the work.

2 Numerical methods

2.1 Algorithm proposed by Ford and Lumb [3]

The approach adopted is to use the boundary conditions to provide *approximate* initial conditions and then solve the initial value problem to provide a solution on $[1,k-1]$.

We introduce the following notation. Fix $N\in\mathbb{N},\ h=1/N$ and define

$$y_{n+N}=(x_{n+N}\ x_{n+N-1}\ \cdots\ x_{n+1}\ x_n\ \cdots\ x_{n-N})^{\mathrm{T}},$$

where $x_j\approx x(jh),\ j=1,\ldots,N$. Here $x_{-N},\ldots,x_0$ can be obtained from the known values of $\varphi_1(t)$ while $x_1,\ldots,x_N$ would be obtained from the unknown function φ_2. $x_{(k-1)N+1},\ldots,x_{kN}$ may be obtained from values of the function f.

Discretization of (1) using a θ-method, with $h=1/N$, leads to an equation of the form

$$y_{n+N}=Ay_{n+N-1}, \qquad (9)$$

with the block-structured matrix

$$A = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where M_1 takes the form

$$\begin{pmatrix} -\frac{1-\theta}{\theta} & 0 & \cdots & 0 & \frac{1-h\theta\alpha}{h\theta\gamma} & \frac{-[1+h(1-\theta)\alpha]}{h\theta\gamma} & 0 & \cdots & 0 & \frac{-\beta}{\gamma} \end{pmatrix},$$

M_2 equals $\left(\frac{-\beta(1-\theta)}{\theta\gamma}\right)$, M_3 is the $2N$-dimensional identity matrix and $M_4 = (0 \ \cdots \ 0)^{\mathrm{T}}$.

Now we use the expression $y_{kN} = A^{(k-1)N} y_N$ and the known components of y_N (from φ_1) and y_{kN} (from f) to determine the unknown components of y_N.

Finally, we can use y_N and (9) to find the solution of (1) on $[1, k-1]$.

2.2 Algorithms based on collocation and least squares method

In this section we will use a generalization of formulae (6)-(7). These were obtained for the particular case of (1) with $\alpha = 0$; here we consider the general case. For the first three subintervals, the formulae for the case $\alpha \neq 0$ can be written as

$$x(t) = \begin{cases} a\varphi_2'(t-1) + b\varphi_1(t-2) + c\varphi_2(t-1), & t \in (1,2); \\ a^2\varphi_2''(t-2) + (b+c^2)\varphi_2(t-2) + ab\varphi_1'(t-3) & \\ \quad +2ac\varphi_2'(t-2) + cb\varphi_1(t-3), & t \in (2,3); \\ a^3\varphi_2'''(t-3) + 3a^2c\varphi_2''(t-3) + (2ab+3ac^2)\varphi_2'(t-3) & \\ \quad +a^2b\varphi_1''(t-4) + 2abc\varphi_1'(t-4) + (b^2+c^2b)\varphi_1(t-4) & \\ \quad +(2cb+c^3)\varphi_2(t-3), & t \in (3,4), \end{cases} \tag{10}$$

where

$$a = \frac{1}{\gamma}, \quad b = -\frac{\beta}{\gamma}, \quad c = -\frac{\alpha}{\gamma}. \tag{11}$$

Note that in this case formula (8) takes the form

$$\varphi^{(n+1)}(0) = \alpha\varphi^{(n)}(0) + \gamma\varphi^{(n)}(1) + \beta\varphi^{(n)}(-1). \tag{12}$$

Next we search for an approximate solution of (1) on $[-1, 1]$ in the form

$$\widetilde{x}_d(t) = x_0(t) + \sum_{j=0}^{d-1} C_j x_j(t), \quad t \in [-1, 1], \tag{13}$$

where x_0 is an initial approximation of the solution; $\{x_j\}_{0 \leqslant j \leqslant d-1}$ is a basis in the space of functions where the correction to the initial approximation is sought; d is the dimension of this space. The algorithm for computing $\widetilde{x}_d$ can be described as follows.

Step 1 From formulae (6)–(8) it follows that if a solution, constructed by the method of steps, belongs to $C^n((l-1,l])$ (for certain $l \geqslant 1,\ n \geqslant 1$), then it also belongs to $C^{n-1}((l,l+1])$. Therefore, since we want $\widetilde{x}_d$ to be at least continuous on $[-1,k]$ (for a certain $k \geqslant 2$), we require that x_0 belong to $C^k((-1,l])$. With this in mind, we define x_0 on $[-1,1]$ in the following way:

$$x_0(t) = \begin{cases} \varphi_1(t), & t \in [-1,0]; \\ P_{2k}(t) = a_0 + a_1 t + \cdots + a_{2k}t^{2k}, & t \in [0,1]. \end{cases} \tag{14}$$

Since x_0 must be k times continuously differentiable on $(-1,1]$, it must satisfy the following conditions at $t = 0$:

$$P_{2k}(0) = \varphi_1(0); \quad P_{2k}^{(j)}(0) = \varphi_1^{(j)}(0), \quad j = 1,...,k. \tag{15}$$

On the other hand, in order to satisfy (12) with $n = 0, 1, \ldots, k-1$, the following equalities must hold:

$$\begin{gathered} P_{2k}(1) = a\varphi_1'(0) + b\varphi_1(-1) + c\varphi_1(0); \\ P_{2k}^{(j)}(1) = a\varphi_1^{(j+1)}(0) + b\varphi_1^{(j)}(-1) + c\varphi_1^{(j)}(0), \quad j = 1,...,k-1. \end{gathered} \tag{16}$$

Here,

$$a = \frac{1}{\alpha}, \quad b = -\frac{\beta}{\alpha}, \quad c = -\frac{\gamma}{\alpha}.$$

Conditions (15) and (16) define a linear system of $2k+1$ equations with $2k+1$ unknowns. It is possible to show that this system has a nonsingular matrix for any $k \geqslant 2$.

Further, x_0 is extended from $[-1,1]$ to $[-1,k]$ using the recurrence formulae (10). Let us denote this extension by $x_0^{[-1,k]}$.

Step 2 With the purpose of computing a correction to the initial approximation on $[0,k]$, we first consider this correction on $[0,1]$. Let us define a grid of stepsize h on this interval. Let $h = 1/N$ ($N \in \mathbb{N},\ N \geqslant k+1$) and $t_i = ih,\ i = 0, \ldots, N$. The correction $\widetilde{x}_d(t) - x_0(t)$ on $[0,1]$ will be sought as a k-th degree spline, $S_k(t)$, defined on this grid, which satisfies $S_k(0) = S_k(1) = 0$. As usual, we will use as basis functions $x_j(t)$, the so-called B-splines of degree k. Following the usual definition (for details see Ref. [8]), we have

$$x_j(t) = \frac{1}{h^k}\Delta^{k+1}(t-t_j)_+^k, \quad j = 0, \ldots, N-k-1, \tag{17}$$

where Δ^k represents a k-th order forward difference (with respect to t_j) and

$$(t-t_j)_+ = \begin{cases} 0, & t < t_j; \\ t - t_j, & t \geqslant t_j. \end{cases}$$

From the definition it follows that the basis functions have the following properties:

- $x_j \in C^{k-1}[0,1]$;
- $x_j(t)$ is different from zero only in (t_j, t_{j+k+1}).
- $x_j(t)$ is a polynomial of degree k on each interval $[t_i, t_{i+1}]$, $i = 0, \ldots, N-$
1.

Note that we have $N-k$ functions x_j with these properties; therefore, we set $d = N - k$.

The case of $k = 3$ (cubic B-splines) will be analysed in more detail in Section 3.

Next, the basis functions are extended to the interval $[0,k]$ using the method of steps. Let us denote the extended basis functions by $x_j^{[0,k]}$. Each time we extend the basis function to the next interval, the degree of the splines decreases by 1 unit. Therefore, on the interval $[k-1,k]$ the basis functions are 1st degree splines (continuous but not continuously differentiable functions). On the whole interval $[-1,k]$, the approximate solution is given by

$$\widetilde{x}_d(t) = x_0^{[-1,k]}(t) + \sum_{j=0}^{N-k-1} C_j x_j^{[0,k]}(t), \quad t \in [-1,k]. \tag{18}$$

Step 3 Finally, we compute the coefficients C_j, $j = 0, \ldots, N-k-1$, of expansion (18) from the condition that $\widetilde{x}_d$ approximates f on the interval $(k-1,k]$. Two alternative methods were used for this purpose.

(i) Collocation Method.

In this case, the coefficients are obtained from the condition

$$\begin{aligned} \widetilde{x}_d(t_{(k-1)N+i}) &= x_0^{[-1,k]}(t_{(k-1)N+i}) + \sum_{j=0}^{N-k-1} C_j x_j^{[0,k]}(t_{(k-1)N+i}) \\ &= f(t_{(k-1)N+i}), \quad i = i_{\min}, \ldots, i_{\max}, \end{aligned} \tag{19}$$

where

$$i_{\min} = \begin{cases} (k+1)/2, & k \text{ is odd}, \\ k/2, & k \text{ is even}; \end{cases} \qquad i_{\max} = \begin{cases} N - \dfrac{k+1}{2}, & k \text{ is odd}, \\ N - \dfrac{k}{2} - 1, & k \text{ is even}. \end{cases}$$

Equations (19) form a linear system with an $(N-k) \times (N-k)$ band matrix. This system can be solved by standard methods.

(ii) Least Squares Method.

In this case, the coefficients C_j are obtained from the condition that the following integral is minimized:

$$\int_{k-1}^{k} \left(f(t) - x_0^{[-1,k]}(t) - \sum_{j=0}^{N-k-1} C_j x_j^{[0,k]}(t) \right)^2 \mathrm{d}t.$$

Given the form of the basis functions, this method leads us to the solution of a system of $N-k$ linear equations with a band matrix.

2.3 Reducing to an ODE

On the interval $[-1, 1]$ the solution of (1)-(2) can be written in the form

$$x(t) = x_0(t) + u(t), \quad t \in [-1, 1], \tag{20}$$

where x_0 is defined by (14); u is a correction that we want to compute.

First of all, note that $u(t) \equiv 0, \ \forall \ t \in [-1, 0]$ (otherwise, x does not satisfy the first boundary condition). Therefore, if we define u on $(0, 1]$, we can extend it to the whole interval $[-1, k]$ using the method of steps. Let us denote as $u^{[-1,k]}$ the extension of $u(t)$ to the interval $[-1, k]$. We shall now express $u^{[k-1,k]}(t)$ in terms of $u(t)$. From (1), taking into account that $u(t) \equiv 0$, for $t \in [-1, 0]$, we obtain

$$u^{[-1,k]}(t) = au'(t-1) + cu(t-1), \quad t \in (1, 2], \tag{21}$$

where a and b are defined by (11). In the same way, $u^{[k-1,k]}(t)$ can be extended to the interval $(2, 3]$:

$$u^{[-1,k]}(t) = a(u^{[-1,k]})'(t-1) + bu^{[-1,k]}(t-2) + cu^{[-1,k]}(t-1). \tag{22}$$

From (21) and (22) we conclude that

$$u^{[-1,k]}(t) = a^2u''(t-2) + 2acu'(t-2) + (b + c^2)u(t-2), \quad t \in (2, 3]. \tag{23}$$

Continuing this process, we can define $u^{[-1,k]}(t)$ through $u(t)$, on any interval $(l-1, l]$, by an equation of the form

$$\begin{aligned} u^{[-1,k]}(t) &= c_{l-1,k}u^{(l-1)}(t-l+1) + c_{l-2,k}u^{(l-2)}(t-l+1) \\ &\quad + \cdots + c_{0,k}u(t-l+1), \quad t \in (l-1, l]. \end{aligned} \tag{24}$$

Here c_{ij} are coefficients that can be computed recursively, just as the δ_{ij} and γ_{ij} coefficients in the right-hand side of (6) and (7). In particular, on the interval $(k-1, k]$, we obtain

$$\begin{aligned} u^{[-1,k]}(t) &= L^{k-1}u(t-k+1) \\ &:= c_{k-1,k}u^{(k-1)}(t-k+1) + \cdots + c_{0,k}u(t-k+1), \quad t \in (k-1, k]. \end{aligned} \tag{25}$$

Here, L^{k-1} denotes a linear differential operator of order $k-1$.

Notice that x_0 can also be extended to the interval $[-1, k]$, using the method of steps. Let us denote by $x_0^{[-1,k]}$ the extension of x_0 to this interval. Then we conclude that x satisfies

$$\begin{aligned} x(t) &= x_0^{[-1,k]}(t) + u^{[-1,k]}(t) \\ &= L^{k-1}u(t-k+1) + x_0^{[-1,k]}(t), \quad t \in [k-1, k]. \end{aligned} \tag{26}$$

Now, since x must satisfy the second boundary condition in (2), we conclude that

$$L^{k-1}u(t-k+1) + x_0^{[-1,k]}(t) = f(t), \quad t \in [k-1, k], \tag{27}$$

or equivalently

$$L^{k-1}u(t) = f(t+k-1) - x_0^{[-1,k]}(t+k-1), \quad t \in [0,1]. \tag{28}$$

Moreover, since $u(t) = x(t) - x_0(t)$, $\forall\, t \in [0,1]$, and P_{2k} satisfies (15)-(16), we conclude that $u \in C^k([0,1)) \cap C^{k-1}(1)$ and the following boundary conditions must be satisfied:

$$\begin{aligned} u(0) = u'(0) = \cdots = u^{(k)}(0) = 0, \\ u(1) = u'(1) = \cdots = u^{(k-1)}(1) = 0. \end{aligned} \tag{29}$$

The number of boundary conditions in (29), $(2k+1)$, is higher than the order of the considered ODE (28). Therefore, there may not exist a solution of (28) which satisfies all the conditions (29). This is not surprising, since the existence of a solution to the original boundary value problem (1)-(2) is also not guaranteed (as discussed in Ref. [3]).

Hence, when solving problem (28)-(29), one has to keep only $k-1$ conditions and ignore the remaining ones. If $k-1$ is even, we consider $(k-1)/2$ boundary conditions at each end; if $k-1$ is odd, we consider $k/2$ conditions at $t=0$ and $\frac{k}{2}-1$ at $t=1$. Let us call the obtained boundary value problem (with $k-1$ boundary conditions) the *reduced BVP*.

For example, in the case $k=3$, (28) is a second order ODE and the reduced BVP has two boundary conditions: $u(0) = 1$, $u(1) = 0$. The obtained BVP can then be solved by standard numerical methods, for example, the collocation method with a basis of cubic B-splines.

2.4 Error analysis

2.4.1 ODE Approach

When the original problem (1)-(2) is reduced to a BVP by the method described in the previous subsection, we have to compute an approximate solution of (28) which satisfies certain boundary conditions on $[0,1]$. This can be otained, for example, by the collocation method, using a basis of splines of appropriate order. We shall restrict our error analysis to the case $k=3$. In this case, we have to solve a BVP for a second order ODE and, if f is at least 4 times continuously differentiable, we have the following estimate for the error norm (see, for example, Ref. [8]):

$$\|u^{(N)}(t) - u(t)\|_\infty \leqslant Bh^2,$$

where $u^{(N)}$ is the obtained approximate solution with N basis functions and B is a constant that does not depend on h. This result applies to the interval $[0,1]$, where u is defined. On the interval $[1, k-1]$ this estimate is not valid and the convergence order may in principle be lower. However, as we will see in Section 4, the numerical experiments indicate that second order convergence can be attained in all of the domain.

2.4.2 Method of steps approach

Let us now turn our attention to the collocation method, described in Sec. 2.2. In this case the error analysis is much more complicated. Focusing our attention on the case $k = 3$, we note that when we solve the linear system (19), we are approximating the solution x on the interval $[2, 3]$ by a certain function $\widetilde{x}_N$ which does not have the same properties as the cubic splines, defined on $[0, 1]$. Actually, taking into account formula (10), used to extend the basis functions, $\widetilde{x}_N$ is piecewise polynomial of the third degree and does not belong to C^2; it is just a continuous function. As far as we know, there are no available results on the convergence of collocation methods with basis functions of this kind. So the error analysis of the proposed method is still an open question that we intend to investigate in the future. The obtained numerical results, presented in Section 4, suggest that the method converges, though its convergence order is lower than in the case of the ODE approach.

3 Numerical results and comparison of methods

In order to analyse the performance of the described numerical methods, we have considered the following MTFDE:

$$x'(t) = (m - 0.5\mathrm{e}^{-m} - 0.5\mathrm{e}^{m})x(t) + 0.5x(t-1) + 0.5x(t+1), \tag{30}$$

with the boundary conditions (2):

$$\phi_1(t) = \mathrm{e}^{mt}, \quad t \in [-1, 0]; \quad f(t) = \mathrm{e}^{mt}, \quad t \in (k-1, k].$$

The exact solution is $x(t) = \mathrm{e}^{mt}$. This example was also considered in Ref. [3] for some different values of m.

The choice of basis functions depends on k. As an example, let us describe the case $k = 3$. In this case, we search for an approximate solution in form (13), where x_0 satisfies

$$x_0(t) = \begin{cases} \mathrm{e}^{mt}, & t \in [-1, 0]; \\ P_6(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_5 t^5 + a_6 t^6, & t \in [0, 1]. \end{cases}$$

Since x_0 and its first three derivatives must be continuous at $t = 0$, we obtain

$$a_0 = 1, \quad a_1 = m, \quad a_2 = m^2/2, \quad a_3 = m^3/6.$$

The remaining coefficients are obtained from conditions

$$P_6(1) = a\phi_1'(0) + b\phi_1(-1) + c\phi_1(0) = a + b\,\mathrm{e}^{-1} + c,$$
$$P_6'(1) = a\phi_1''(0) + b\phi_1'(-1) + c\phi_1'^{-1} + c,$$
$$P_6''(1) = a\phi_1'''(0) + b\phi_1''(-1) + c\phi_1''^{-1} + c.$$

The basis functions $x_i(t),\ i = 0, \ldots, N-k-1$, in this case are the well-known cubic B-splines.

Next we present the results of some numerical experiments. In Table 1 we present the results obtained by the least squares method in the case $k = 4$. The numerical results suggest that the convergence order of the method is $p = 0.5$, for different values of m (in the 2-norm).

Table 1 Error ε on [0,2] and convergence order p obtained by the least squares method for $k = 3$ and $m = -0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x - \widetilde{x}_d\|_2/(k-1)$)

i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/8	9.5247×10^{-9}		5.2519×10^{-8}		2.8279×10^{-2}	
2	1/16	6.7022×10^{-9}	0.51	3.7075×10^{-8}	0.50	1.8448×10^{-2}	0.62
3	1/32	4.7348×10^{-9}	0.50	2.6214×10^{-8}	0.50	1.2804×10^{-2}	0.53
4	1/64	3.3473×10^{-9}	0.50	1.8536×10^{-8}	0.50	9.0123×10^{-3}	0.51
5	1/128	2.3668×10^{-9}	0.50	1.3107×10^{-8}	0.50	6.3654×10^{-3}	0.50

Tables 1 and 2 illustrate the application of the collocation and least squares methods, described in Section 2.2, to equation (30), in the case $k = 3$. In both cases, the estimated convergence order is close to 0.5. The error norm is lower in the case of the collocation method.

Table 2 Error ε on [0,2] and convergence order p obtained by the collocation method for $k = 3$ and $m = -0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x - \widetilde{x}_d\|_2/(k-1)$)

i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/8	2.0367×10^{-9}		1.2200×10^{-8}		2.3550×10^{-3}	
2	1/16	7.8305×10^{-10}	1.38	4.6304×10^{-9}	1.40	1.1040×10^{-3}	1.09
3	1/32	5.2024×10^{-10}	0.59	3.0491×10^{-9}	0.60	7.6923×10^{-4}	0.52
4	1/64	3.6656×10^{-10}	0.51	2.1467×10^{-9}	0.51	5.4371×10^{-4}	0.50
5	1/128	2.5914×10^{-10}	0.50	1.5176×10^{-9}	0.50	3.8445×10^{-4}	0.50

The results obtained by the same methods in the case when $k = 4$ are displayed in Tables 3 and 4. It is interesting to remark that in the case of

Table 3 Error ε on [0,3] and estimated convergence order p obtained by the least squares method for $k = 4$ and $m = -0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x - \widetilde{x}_d\|_2/(k-1)$)

i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/8	1.3361e − 7		6.9579e − 7		2.2567e − 1	
2	1/16	7.0260e − 8	0.93	3.8791e − 7	0.84	2.1589e − 1	0.06
3	1/32	4.5664e − 8	0.62	2.5652e − 7	0.60	1.5093e − 1	0.51
4	1/64	3.1657e − 8	0.53	1.7863e − 7	0.52	1.0587e − 1	0.51
5	1/128	2.2332e − 8	0.50	1.2611e − 7	0.50	7.4702e − 1	0.50

Table 4 Error ε on [0,3] and estimated convergence order p obtained by the collocation method of Sec. 2.2 for $k = 4$ and $m = -0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x - \widetilde{x}_d\|_2/(k-1)$)

i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/8	5.7351e − 8		3.3729e − 7		1.1459e − 1	
2	1/16	2.2176e − 8	1.37	1.3422e − 7	1.33	2.0082e − 2	1.54
3	1/32	7.9488e − 9	1.48	4.8700e − 8	1.46	1.3571e − 2	1.54
4	1/64	2.8136e − 9	1.50	1.7338e − 8	1.49	4.6936e − 3	1.53
5	1/128	1.0031e − 9	1.49	6.1928e − 9	1.49	1.6495e − 3	1.51

the collocation method the estimated convergence order is higher in the case when $k = 4$ (about 1.5) than when $k = 3$. The same does not happen with the least squares method.

Some of the results obtained by the method described in Sec. 2.3 are displayed in Table 5. The convergence order is $p = 2$ (in the maximum norm), which is in agreement with the theoretical estimates given in Section 2.4.1.

Table 5 Error ε on intervals $[j, j+1]$ and estimated convergence order p obtained by the collocation method of Sec. 2.3 for $k = 3$ and $m = 1$ ($\varepsilon = h\|x - \widetilde{x}_d\|_\infty/(k-1)$)

i	step h_i	$[0,1]$		$[1,2]$	
		ε	p	ε	p
1	1/8	5.1105e − 7		4.6405e − 6	
2	1/16	1.2436e − 7	2.04	9.9905e − 7	2.22
3	1/32	3.0106e − 8	2.05	2.3445e − 7	2.09
4	1/64	7.3718e − 9	2.03	5.7268e − 8	2.03
5	1/128	1.8223e − 9	2.02	1.4156e − 8	2.02

Finally, in Tables 6 and 7, numerical results obtained by different methods are displayed, for comparison. The collocation methods of Sections 2.2 and 2.3 are compared with the θ-method in Table 6 (case $k = 3$); the least squares method and the collocation method of Section 2.3 are compared with the θ-method in Table 7 (case $k = 4$). In these tables, we have considered only the error norm $\|\cdot\|_2$ on $[0,1]$, so that the results could be compared with those in Ref. [3]. In Table 6 (case $k = 3$), the collocation method of Section 2.3 has

Table 6 Error ε on $[0,1]$ and estimated convergence order p obtained by three methods: collocation method of Sec. 2.2, collocation method of Sec. 2.3 and θ-method, for $k = 3$ and $m = -0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x - \widetilde{x}_d\|_2$)

		Collocation (Sec. 2.2)					
i	step h_i	$m = -0.5$		$m = 0.6$		$m = 3$	
		ε	p	ε	p	ε	p
1	1/16	7.830e − 10	1.38	4.630e − 9	1.40	1.104e − 3	1.09
2	1/32	5.202e − 10	0.59	3.049e − 9	0.61	7.692e − 4	0.52
3	1/64	3.666e − 10	0.50	2.147e − 9	0.51	5.437e − 4	0.50
		Collocation (Sec. 2.3)					
i	step h_i	$m = -0.5$		$m = 0.6$		$m = 3$	
		ε	p	ε	p	ε	p
1	1/16	6.291e − 11	2.61	4.156e − 10	2.53	2.216e − 5	3.26
2	1/32	1.041e − 11	2.60	7.109e − 11	2.55	2.585e − 6	3.10
3	1/64	1.767e − 12	2.56	1.230e − 11	2.53	3.627e − 7	2.83
		θ-Method (Sec. 2.1)					
i	step h_i	$m = -0.5$		$m = 0.6$		$m = 3$	
		ε	p	ε	p	ε	p
1	1/16	1.483e-5	2.04	8.896e − 5	2.08	1.568e − 2	2.12
2	1/32	3.654e − 6	2.02	2.164e − 5	2.04	3.752e − 3	2.06
3	1/64	9.067e − 7	2.01	5.328e − 6	2.02	9.168e − 4	2.03

Table 7 Error ε on $[0,1]$ and estimated convergence order p obtained by three methods: least squares method, collocation method of Sec. 2.3 and θ-method, for $k=4$ and $m=-0.5,\ 0.6,\ 3$ ($\varepsilon = h\|x-\widetilde{x}_d\|_2$)

		Least squares (Sec. 2.2)					
i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/16	1.466e − 9	0.50	8.585e − 9	0.50	2.175e − 3	0.50
2	1/32	1.037e − 9	0.50	6.070e − 9	0.50	1.538e − 3	0.50
3	1/64	7.330e − 10	0.50	4.292e − 9	0.50	1.087e − 3	0.50
		Collocation (Sec. 2.3)					
i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/16	2.040e − 10	6.30	1.9221e − 9	1.02	2.1035e − 4	1.83
2	1/32	9.3835e − 11	1.12	8.7412e − 10	1.31	5.9285e − 5	1.83
3	1/64	3.7585e − 11	1.41	3.5275e − 10	1.31	1.7756e − 5	1.74
		θ-Method (Sec. 2.1)					
i	step h_i	$m=-0.5$		$m=0.6$		$m=3$	
		ε	p	ε	p	ε	p
1	1/16	1.026e − 5	2.04	1.266e − 4	2.08	2.364e − 2	2.12
2	1/32	2.532e − 6	2.02	3.075e − 5	2.04	5.656e − 3	2.06
3	1/64	6.356e − 7	2.00	7.577e − 6	2.02	1.382e − 3	2.03

the highest convergence order (the estimates suggest that $p=1.5$). In Table 7 (case $k=4$), the estimated convergence order of the collocation method, described in Sec. 2.3, becomes less than the order of the θ-method (which is 2 in both cases). In spite of this, the error norm of the collocation method, for the considered stepsizes ($1/64 \leqslant h \leqslant 1/16$), is the lowest.

The graphs of the absolute error for the case $k=3,\ m=3$ are displayed in Fig. 1 (by the collocation method of Sec. 2.2) and in Fig. 2 (by the collocation method of Sec. 2.3). The graphs of the absolute error for the case $k=4,\ m=1$ are displayed in Fig. 3 (by the collocation method of Sec. 2.2) and in Fig. 4 (by the collocation method of Sec. 2.3).

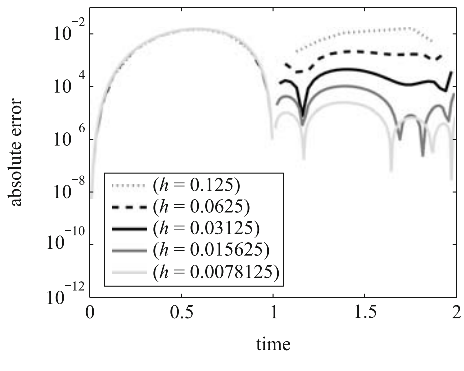

Fig. 1 Collocation (Sec. 2.2) $k=3,\ m=3$

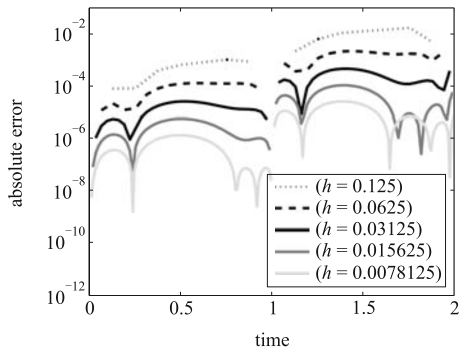

Fig. 2 Collocation (Sec. 2.3) $k = 3,\ m = 3$

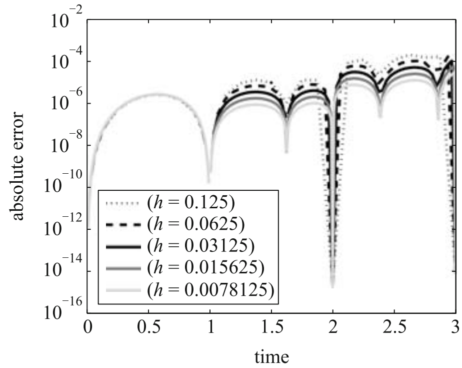

Fig. 3 Collocation (Sec. 2.2) $k = 4,\ m = 1$

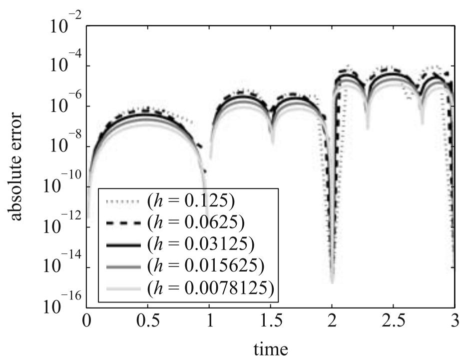

Fig. 4 Collocation (Sec. 2.3) $k = 4,\ m = 1$

As expected, the graphs show that the errors grow on each subinterval $[l-1, l]$, compared with the previous one. Moreover, at each integer point the error is zero, since the numerical methods produce the exact solution at these points.

4 Conclusions and future work

The proposed computational methods have produced accurate numerical results when applied to the solution of equation (30), with different values of m. Concerning the methods as described in Section 2.2, the algorithm based on collocation has a higher convergence order than the one based on the least squares. The θ-method has greater absolute error (in the 2-norm), when applied with stepsize greater than or equal to $h = 1/64$. However, it has a higher order of convergence ($p = 2$). Finally, the method described in Section 2.3, based on the reduction of the MTFDE to an ODE, seems to be the most efficient, providing second order convergence and giving results with absolute error, less than 10^{-7}, for $h = 1/64$, when $k = 3$ and $m = 1$.

In the future, we intend to carry out a more detailed numerical analysis of the presented methods. We also intend to use the ODE approach described in Section 2.3, to investigate the existence and uniqueness of the solution of the considered problem.

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