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RESEARCH PAPER

Probabilistic elasto-plasticity: solution and verification in 1D

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Abstract In this paper, a solution is presented for evolution of probability density function (PDF) of elastic–plastic stress–strain relationship for material models with uncertain parameters. Developments in this paper are based on already derived general formulation presented in the companion paper. The solution presented is then specialized to a specific Drucker–Prager elastic–plastic material model. Three numerical problems are used to illustrate the developed solution. The stress–strain response (1D) is given as a PDF of stress as a function of strain. The presentation of the stress–strain response through the PDF differs significantly from the traditional presentation of such results, which are represented by a single, unique curve in stress–strain space. In addition to that the numerical solutions are verified against closed form solutions where available (elastic). In cases where the closed form solution does not exist (elastic–plastic), Monte Carlo simulations are used for verification.

Keywords Elasto-plasticity · Theory of probability

1 Introduction

An elastic–plastic constitutive law can be represented by a set of linear or non-linear ordinary differential equations (ODEs), which relate rate (increments) of stress with the rate of strain through linear or nonlinear material modulus:

$$\frac{d\sigma_{ij}(t)}{dt} = D^{ep}_{ijkl} \frac{d\epsilon_{kl}(t)}{dt} \quad (1)$$

where D^{ep}_{ijkl} could be linear or a non-linear function of stresses, strains and internal variables. If either the material modulus or the forcing term (strain rate) becomes random, this set of linear or nonlinear ODEs becomes a set of linear or non-linear stochastic differential equations (SDEs). The uncertainty associated with the coefficient (stiffness) term is generally attributed to the inherent variability of the material. Geomaterials are particularly notorious for their variability, sampling and testing errors and in general, uncertainty in their properties. The uncertainty in the forcing term arises when the material is subjected to uncertain loads (usually dynamic) like wind, waves or earthquakes. Due to randomness in the parameter and/or forcing term the response variable of the elastic–plastic constitutive rate equation (stress) will then be a random process. There exist several methods to estimate the probabilistic characteristics of the response variable [2].

For the case where the material modulus is linear and deterministic and the forcing is uncertain (Gaussian), the response is known to be Gaussian and can be estimated by standard methods [2]. General linear SDE with random forcing can be solved by cumulant expansion method [10]. One possible way to solve non-linear SDE with random forcing is to write its equivalent Fokker–Planck–Kolmogorov (FPK) form. The advantage of writing the FPK form is that it is linear and deterministic even though the original equation is non-linear and stochastic. The general solution method for FPK equation can be found in any standard textbook [2, 8]. A solution scheme for FPK equation with application to structural reliability was presented by Langtangen [5].

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For the particular case where the forcing is deterministic and the material linear elastic (but still uncertain), Eq. 1 simplifies to a linear set of algebraic equations of the form,

$$\sigma_{ij} = D^{\mathrm{el}}_{ijkl}\epsilon_{kl} \tag{2}$$

where D^{el}_{ijkl} is the elastic stochastic moduli tensor and hence the statistics of the response variable (stress) can be easily obtained by transformation method of random variable. For general linear SDEs with random coefficients cumulant expansion method could be used [10].

While the solution for the stochastic linear elastic stress–strain problem is readily available, the nonlinear (elastic–plastic) stochastic problem presents itself as much harder to solve. This problems involves finding the solution for a non-linear SDEs with random coefficients. Fortunately such solution was recently developed by Kavvas [4]. The developed solution is presented as a generic Eulerian–Lagrangian form of the Fokker–Planck–Kolmogorov equation. That probabilistic solution is second order exact for any stochastic nonlinear ODE or PDE with random coefficients and random forcing.

The probabilistic solution developed by Kavvas [4] was used in the companion paper by Jeremić et al. [3] to develop the probability density function (PDF) of a general local-average form of elastic–plastic constitutive rate equation. This Eulerian–Lagrangian FPK equation was then specialized to the particular cases of point-location scale linear elastic and Drucker–Prager associative linear hardening elastic–plastic constitutive rate equations to show the applicability of the general formulation. In this paper the solution process of those particular FPK equations will be presented.

2 Fokker–Planck–Kolmogorov equation for probabilistic elasticity and elasto-plasticity in 1D

In the companion paper, Jeremić et al. [3] applied the Eulerian–Lagrangian form of Fokker–Planck–Kolmogorov (FPK) equation to the description of the probabilistic behavior of elastic and elastic–plastic (Drucker–Prager associative linear hardening) 1D constitutive equations with random material parameters and random strain rate.

By focusing attention on the randomness of material properties only [i.e., assuming the forcing function (strain rate) as deterministic], partial differential equation (PDE) describing the evolution of probability density function (PDF) of stress can be simplified. In particular, for 1D case, and for linear elastic material (but still with probabilistic material properties, in this case shear modulus G) one can write the following PDE

$$\frac{\partial P(\sigma_{12}(t))}{\partial t} = -\left\langle G\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}\right\rangle \frac{\partial P(\sigma_{12}(t))}{\partial \sigma_{12}} + \left\{\int_0^t \mathrm{d}\tau\, \mathrm{Cov}_0\left[G\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}; G\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}\right]\right\}\frac{\partial^2 P(\sigma_{12}(t))}{\partial \sigma_{12}^2} \tag{3}$$

Similarly, for elastic–plastic state, again by neglecting the randomness in strain rate, one can write the PDE for evolution of PDF of stress in 1D as

$$\frac{\partial P(\sigma_{12}(t))}{\partial t} = -\left\langle (G^{\mathrm{ep}}(t))\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}\right\rangle \frac{\partial P(\sigma_{12}(t))}{\partial \sigma_{12}} + \left\{\int_0^t \mathrm{d}\tau\, \mathrm{Cov}_0\left[G^{\mathrm{ep}}(t)\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}; G^{\mathrm{ep}}(t-\tau)\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t}\right]\right\} \times \frac{\partial^2 P(\sigma_{12}(t))}{\partial \sigma_{12}^2} \tag{4}$$

where $G^{\mathrm{ep}}(a)$ is the probabilistic elastic–plastic tangent stiffness (given in Jeremić et al. [3])

$$G^{\mathrm{ep}}(a) = G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}}I_1(a)\alpha'} \tag{5}$$

where in (5), a assumes values t or $t-\tau$. With appropriate initial and boundary conditions as described in Jeremić et al. [3], one can solve Eqs. (3) and (4) for evolution of PDF of shear stress with shear strain.

3 Example problem statements

The applicability of proposed FPK equations [Eqs. (3) and (4)] in describing probabilistic elasto-plastic behavior, is verified using the following three example problems.

Problem I Assume the material is linear elastic, probabilistic, with probabilistic shear modulus (G) given by a normal distribution at a point-location scale with mean of 2.5 MPa and standard deviation of 0.707 MPa. The aim is to calculated the evolution of PDF of shear stress (σ_{12}) with shear strain (ε_{12}) for a displacement-controlled test with deterministic shear strain increment. The other parameters are considered deterministic and are as follows: Poisson's ratio $v = 0.2$, and confining pressure $I_1 = 0.03$ MPa.

Problem II Assume elastic–plastic material model, composed of linear elastic component and Drucker–Prager associative isotropic linear hardening elastic–plastic component. The probabilistic shear modulus (G) is given through a normal distribution at a point-location scale with mean of 2.5 MPa and standard deviation of 0.707 MPa. The aim is to calculate the evolution of the PDF of shear stress (σ_{12}) with shear strain (ε_{12}) for a displacement-controlled test with deterministic shear strain increment. The other parameters are considered deterministic and are as follows: Poisson's ratio $\nu = 0.2$, confining pressure $I_1 = 0.03$ MPa, yield parameter[1] $\alpha = 0.071$, plastic slope[2] $\alpha' = 5.5$.

Problem III Assume elastic–plastic material model, with linear elastic component and Drucker–Prager associative isotropic linear hardening elastic–plastic component. The probabilistic yield parameter (α) is given through a normal distribution at a point-location scale with mean of 0.52 and standard deviation of 0.1. The aim is to calculate the evolution of the PDF of shear stress (σ_{12}) with shear strain (ε_{12}) for a displacement-controlled test with deterministic shear strain increment. The other parameters are considered deterministic and are as follows: shear modulus $G = 2.5$ MPa, Poisson's ratio $\nu = 0.2$, confining pressure $I_1 = 0.03$ MPa, and the plastic slope $\alpha' = 5.5$.

The above three problems will be solved using the proposed FPK equation approach. In addition to that, the solution will verified using either variable transformation method, for linear elastic case or repetitive Monte Carlo type simulations for elastic–plastic case.

4 Numerical scheme for solving Fokker–Planck–Kolmogorov equation

For probabilistic elastic and elastic–plastic constitutive rate equations, the PDEs [Eqs. (3) and (4)] which describe the evolution of probability densities of σ_{12} have the following general form:

$$\frac{\partial P}{\partial t} = -N_{(1)}\frac{\partial P}{\partial \sigma_{12}} + N_{(2)}\frac{\partial^2 P}{\partial \sigma_{12}^2} \tag{6}$$

with an appropriate initial condition, which depends on the type of problem, and boundary conditions in the form

$$\zeta(-\infty, t) = \zeta(\infty, t) = 0 \tag{7}$$

where $N_{(1)}$ and $N_{(2)}$ are called advection and diffusion coefficients, respectively. The above PDE system [Eqs. (6) and (7) with appropriate initial condition] were solved numerically by method of lines Wolfram [11] using commercially available software Mathematica Wolfram Research Inc [12]. The stress (state) variable σ_{12} theoretically spans space from $-\infty$ to $+\infty$. However, for simulation (and practical) purposes, this theoretical domain is reduced to between –0.1 and +0.1 MPa. This reduction is based on the material properties of the example problems and span the practical range of shear stress, σ_{12}. The Fokker–Planck–Kolmogorov PDE was semi-discretized (Fig. 1) in stress (σ_{12}) domain by finite difference technique to obtain a set of linear simultaneous ODE systems.

This set of linear simultaneous ODEs is solved using central difference technique. By referring to Fig. 1 a semi-discretized form of Eq. (6) can be written at any intermediate node i as,

$$\frac{\partial P_i}{\partial t} = P_{i-1}\left(\frac{N_{(1)}}{2\Delta\sigma_{12}} + \frac{N_{(2)}}{\Delta\sigma_{12}^2}\right) - P_i\left(\frac{2N_{(2)}}{\Delta\sigma_{12}^2}\right) + P_{i+1}\left(-\frac{N_{(1)}}{2\Delta\sigma_{12}} + \frac{N_{(2)}}{\Delta\sigma_{12}^2}\right). \tag{8}$$

Previous discretized system of equations forms an initial value problem in the time dimension. By using forward difference technique, one can introduce the boundary condition at the left boundary (node 1 in Fig. 1) as,

$$N_{(1)}P_1 - N_{(2)}\frac{P_2 - P_1}{\Delta\sigma_{12}} = 0 \tag{9}$$

or, after rearranging,

$$P_1 = P_2\left(\frac{N_{(2)}/\Delta\sigma_{12}}{N_{(1)} + N_{(2)}/\Delta\sigma_{12}}\right) = 0. \tag{10}$$

Similarly, using backward difference technique, one can introduce the boundary condition at the right boundary (node n in Fig. 1) as,

$$N_{(1)}P_n - N_{(2)}\frac{P_n - P_{n-1}}{\Delta\sigma_{12}} = 0 \tag{11}$$

which, after rearranging becomes

$$P_n = P_{n-1}\left(\frac{N_{(2)}/\Delta\sigma_{12}}{N_{(2)}/\Delta\sigma_{12} - N_{(1)}}\right) = 0. \tag{12}$$

The initial condition depends on the type of the problem and it could be deterministic or random. For elastic constitutive rate equation with random shear modulus (Problem I) and for pre-yield elastic–plastic linear hardening constitutive rate equation with random shear modulus (elastic part of Problem II) the initial condition is deterministic. It will, therefore, be best represented as Dirac Delta function of the form,

$$P(\sigma_{12}) = \Delta(\sigma_{12}). \tag{13}$$

For simulation purpose the Dirac delta initial condition was approximated with a Gaussian function. That is, for Problem I, the initial condition was approximated with a Gaussian function with mean of 0 and standard deviation of 0.00001 MPa as shown in Fig. 2.

For post-yield, probabilistic elastic–plastic behavior (plastic part of Problem II), the initial condition is random

[1] The yield parameter α is an internal variable and is a function of the friction angle ϕ given by $\alpha = 2\sin\phi/(\sqrt{3}(3-\sin\phi))$ (e.g., Chen and Han [1]).

[2] The plastic slope α' is a rate of change of friction angle governing linear hardening.

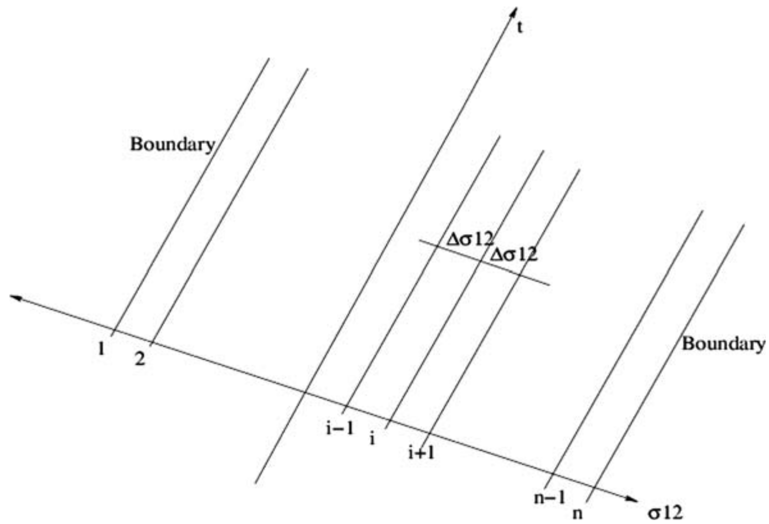

Fig. 1 Stress domain discretization of Fokker–Planck–Kolmogorov PDE

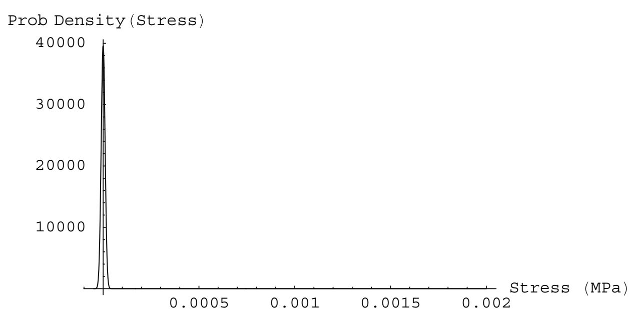

Fig. 2 Approximation of Dirac delta function, used as an initial condition for Problem I

and it corresponds to the probability density function of σ_{12} just prior to yielding, that is obtained from elastic part of Problem II. For Problem III, the pre-yield elastic part is deterministic but initial condition for the post-yield elastic–plastic response is random and corresponds to the assumed distribution in yield strength.

5 Determination of coefficients for Fokker–Planck–Kolmogorov equation

To solve Problems I, II, and III, the advection and diffusion coefficients $N_{(1)}$ and $N_{(2)}$ must be determined for all three problems. For sake of simplicity, a constant strain rate is assumed and hence, terms containing $d\varepsilon_{12}/\mathrm{d}t$ in coefficients of Eqs. (3) and (4) can be substituted by a constant numerical value for the entire simulation of the evolution of PDF. It should be noted that the FPK equation [Eqs. (3) or (4)] describes the evolution of PDFs of stress with time, while, similarly, strain rate describes the evolution of strain with time. Combining the two, the evolution of PDF of stress with strain can be obtained. Time has been brought in this simulation as an intermediate dimension to help in solution process, and hence, the numerical value of strain rate could be any arbitrary value, which will cancel out once the time evolution of PDF of stress is converted to strain evolution of PDF of stress. For simulation of all the three example problems, an arbitrary value of strain rate of $d\varepsilon_{12}/\mathrm{d}t = 0.054\ \mathrm{s}^{-1}$ is assumed.

It should also be noted that since the material properties are assumed as random variables at a point-location scale, the covariance terms appearing within the advection and diffusion coefficients become variances of random variables. For estimations of means and variances of functions of random variables (e.g., for Problems II and III) from basic random variables, commercially available statistical software mathStatica Rose and Smith [9] was used.

Substituting the values of deterministic and random material properties and the strain rate, coefficients $N_{(1)}$ and $N_{(2)}$ of the FPK equations can be obtained for all problems:

5.1 Problem I

$$
\begin{aligned}
N_{(1)} &= \left\langle G \frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}} t \right\rangle \\
&= 2 \frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t} \langle G \rangle \\
&= 0.27\,\mathrm{MPa/s}
\end{aligned}
$$

$$
\begin{aligned}
N_{(2)} &= \int_0^t \mathrm{d}\tau \mathrm{Var}\left[G \frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t} \right] \\
&= 4t \left(\frac{\mathrm{d}\epsilon_{12}}{\mathrm{d}t} \right)^2 \mathrm{Var}[G] \\
&= 0.0058t\,(\mathrm{MPa/s})^2
\end{aligned}
$$

5.2 Problem II

For pre-yield linear elastic case, the coefficients $N_{(1)}$ and $N_{(2)}$ will be the same as those for Problem I. For post-yield elastic–plastic case the coefficients are

$$N_{(1)} = \left\langle \left(G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right) \frac{d\epsilon_{12}}{dt} \right\rangle = \frac{d\epsilon_{12}}{dt} \left\langle G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right\rangle = 0.147\,\text{MPa/s}$$

$$N_{(2)} = t \left(\frac{d\epsilon_{12}}{dt} \right)^2 \text{Var} \left[G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right] = 0.00074t\,(\text{MPa/s})^2$$

5.3 Problem III

For post-yield elastic–plastic simulation the coefficients $N_{(1)}$ and $N_{(2)}$ are

$$N_{(1)} = \left\langle \left(G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right) \frac{d\epsilon_{12}}{dt} \right\rangle = \frac{d\epsilon_{12}}{dt} \left\langle G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right\rangle = 0.2365\,\text{MPa/s}$$

$$N_{(2)} = t \left(\frac{d\epsilon_{12}}{dt} \right)^2 \text{Var} \left[G - \frac{G^2}{G + 9K\alpha^2 + \frac{1}{\sqrt{3}} I_1 \alpha'} \right] = 0.0001t\,(\text{MPa/s})^2$$

It should be noted that for Problem III, since the shear modulus is deterministic, the pre-yield elastic case is deterministic.

6 Results and verifications of example problems

In this section results are presented for elastic and elastic–plastic probabilistic 1D problem. The results are obtained by using FPK equation approach described in previous sections and in the companion paper (Jeremić et al. [3]). In addition to that, the Monte Carlo based verification of developed solutions (results) is presented. The effort to verify developed solutions (that are based on FPK approach) plays a crucial role in presented development of probabilistic elasto-plasticity as there are no previously published solutions which could have been used for verification. In addition to that, verification and validation efforts should always be included in any modeling and simulations work (Oberkampf et al. [7]).

For linear elastic constitutive rate equations (Problem I and pre-yield case of Problem II) the verification is performed by comparing solutions obtained through the use of FPK equation approach with high accuracy (exact) solution, using a transformation method of random variables (Montgomery and Runger [6]). This method is applicable as for rate-independent linear elastic case the 1D shear constitutive equation simplify to a linear algebraic equation of the form,

$$\sigma_{12} = G\epsilon_{12} = u(G, \epsilon_{12}) \tag{14}$$

Using the definition of strain rate, the above equation can be written in terms of time t as,

$$\sigma_{12} = G(0.054t) = v(G, t) \tag{15}$$

where 0.054 s^{-1} is the arbitrary strain rate assumed for this example problem. According to the transformation method of random variables (Montgomery and Runger [6]), and, given the continuous random variable (shear modulus) G, with PDF $g(G)$ and Eqs. (14) or (15) as one-to-one transformations between the values of random variables of G and σ_{12}, one can obtain the PDF of shear stress (σ_{12}), $P(\sigma_{12})$ as,

$$P(\sigma_{12}) = g(u^{-1}(\sigma_{12}, \epsilon_{12}))|J| \tag{16}$$

which will allow for predicting the evolution of PDF of σ_{12} with ε_{12} or,

$$P(\sigma_{12}) = g(v^{-1}(\sigma_{12}, t))|J| \tag{17}$$

Equation (17) will predict the evolution of PDF of σ_{12} with t. In Eqs. (16) and (17), functions $G = u^{-1}(\sigma_{12}, \varepsilon_{12})$ or $G = u^{-1}(\sigma_{12}, t)$ are the inverse of functions $\sigma_{12} = u(G, \varepsilon_{12})$ or $\sigma_{12} = v(G, t)$ respectively and $J = du^{-1}(\sigma_{12}, \varepsilon_{12})/d\sigma_{12}$ and $J = dv^{-1}(\sigma_{12}, t)/d\sigma_{12}$ are their respective Jacobians of transformations.

For non-linear elastic–plastic constitutive rate equations (post-yield cases of Problems II and III) the verification is done using Monte Carlo simulation technique by generating sample data for material properties from standard normal distribution and by repeating solution of the deterministic elastic–plastic constitutive rate equation for each data generated above. The probabilistic characteristics of resulting random stress variable for each time (or strain) step are then easily computed. A relatively large number of data points (1,000,000) were generated for each material constant random variable for this simulation purpose.

6.1 Problem I

The evolution of PDF of shear stress with time and shear strain is shown in Figs. 3 and 4. Presented PDFs are for linear elastic material with random shear modulus, and were obtained using FPE approach (Figs. 3, 4: ElasticPDF) and transformation method (Fig. 4).

The contours of evolution of PDFs are compared in Fig. 5. Similarly, comparison of the evolution of mean and standard deviations are shown in Fig. 6. It can be seen from the comparison figure that even though the FPK approach predicted the mean behavior exactly, it slightly over-predicted the standard deviation. This is because of the approximation used to represent the Dirac delta function, which was used as the initial condition for the FPK. One may note that at $\varepsilon_{12} = 0$, the probability of shear stress σ_{12} should theoretically be 1, i.e., all the probability mass should theoretically be concentrated at $\sigma_{12} = 0$. As such, it would be best described by the Dirac delta function. However, for numerical simulation of FPK, Dirac delta function as initial condition was approximated with a Gaussian function of mean zero and standard deviation of 0.00001 MPa, as shown in Fig. 2. This error in the initial condition advected and diffused into the domain with the simulation of the evolution process. This error could be minimized by better approximating the Dirac delta initial condition (but at higher computational cost). The effect of approximating the initial condition of the PDF of shear stress at $\varepsilon_{12} = 0.0426\%$ is shown in Fig. 7. In this figure the actual PDF at $\varepsilon_{12} = 0.0426\%$ obtained using the transformation method was compared with the PDFs at $\varepsilon_{12} = 0.0426\%$ obtained using the FPK approach with three different approximate initial conditions—all having zero mean but standard deviations of 0.01, 0.005 and 0.00001 MPa.

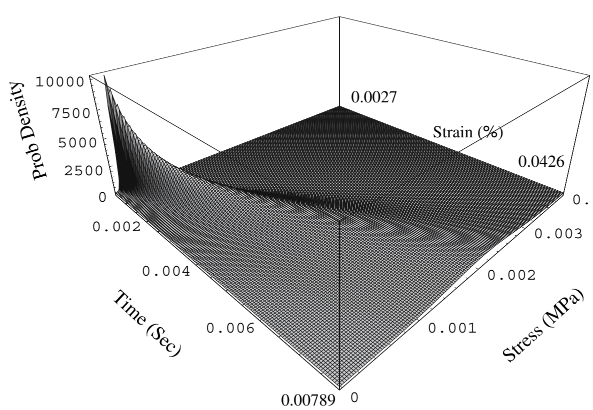

Fig. 3 Evolution of PDF of shear stress versus strain (or time) for linear elastic material model with random shear modulus (Problem I) obtained using FPK equation approach

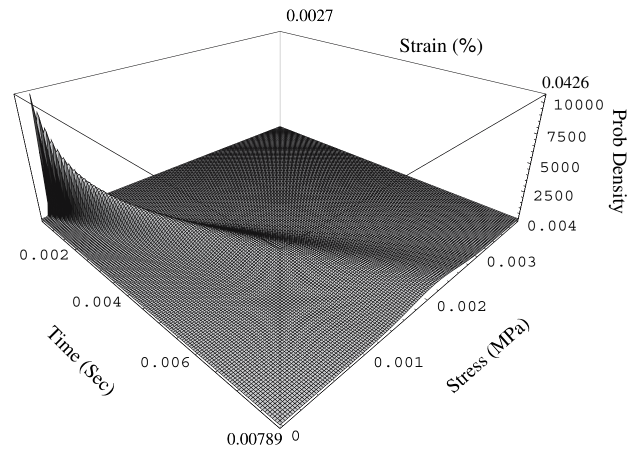

Fig. 4 Evolution of PDF of shear stress versus strain (or time) for linear elastic material model with random shear modulus (Problem I) obtained using transformation method

One may also note that finer approximation of initial condition necessitates finer discretization of stress domain close to (or at) $\sigma_{12} = 0$. The finite difference discretization scheme adopted here uses the same fine discretization uniformly all throughout the entire domain. It is noted that fine, uniform discretization is not needed (and is quite expensive) in later stages of calculation of evolution of PDF, but is kept the same for simplicity sake. In presented examples, to properly capture the approximate initial condition (as shown in Fig. 2), the stress domain between –0.1 and +0.1 MPa was discretized with a uniform step size of 0.000005 MPa and hence there is a total of 40,000 nodes. This not only requires large computational effort but is also very memory sensitive. An adaptive discretization technique will be a much better approach to solving this problem. Current work is going on in formulating an adaptive algorithm for the solution of this type of problem.

6.2 Problem II

The solution to this problem involves the solving two FPK equations, one corresponding to the pre-yield elastic part and the other corresponding to the post-yield elastic–plastic part. The elastic part of this problem is identical to Problem I. The initial condition for the post-yield elastic–plastic part of the problem is random and is shown in Fig. 8. It may be noted that this initial condition corresponds to the PDF of shear stress $[P(\sigma_{12})]$ at yield obtained from the solution of FPK equation of the pre-yield elastic part.

A view of the surface of evolution of the PDF of shear stress versus shear strain (time) is shown in Fig. 9. Another view to the PDF of stress–strain surface is shown in Fig. 10. It is noted that the yielding of this material

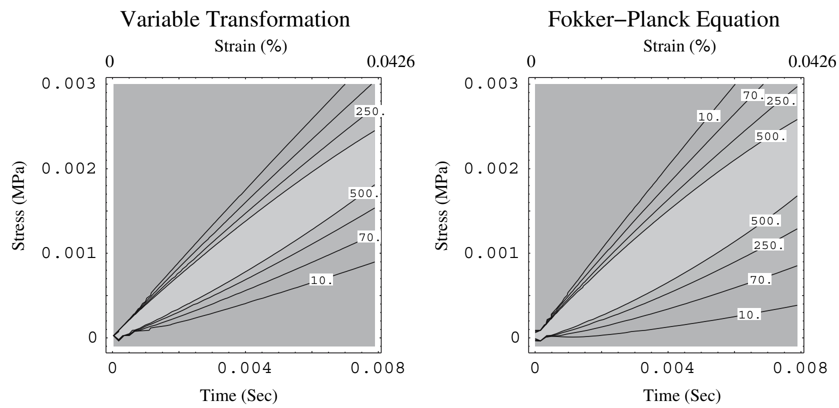

Fig. 5 Comparison of contours of time (or Strain) evolution of probability density function for shear stress for elastic constitutive rate equation with random shear modulus (Problem I) for FPE solution and variable transformation method solution

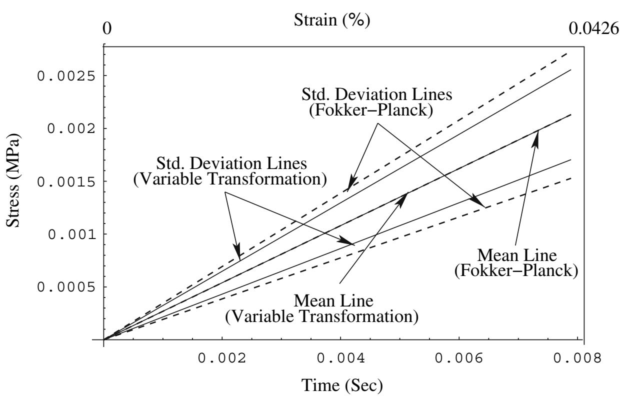

Fig. 6 Comparison of mean and standard deviation of shear stress for elastic constitutive rate equation with random shear modulus (Problem I) for FPE solution and variable transformation method solution

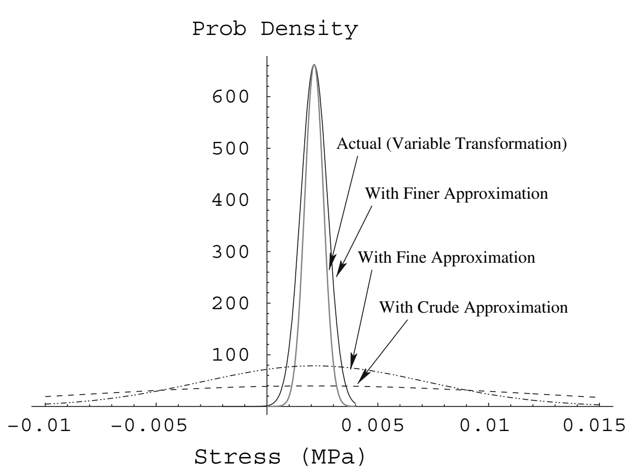

Fig. 7 Effect of approximating function of Dirac delta initial condition: PDF of stress at yield for different approximation of initial condition with actual (variable transformation method) solution)

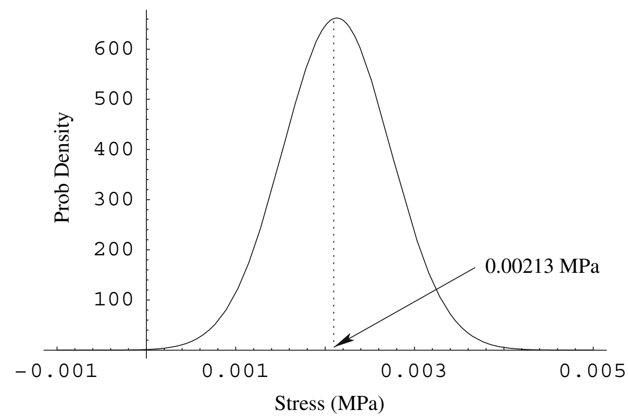

Fig. 8 Initial condition for FPK equation for elastic–plastic zone (Problem II)

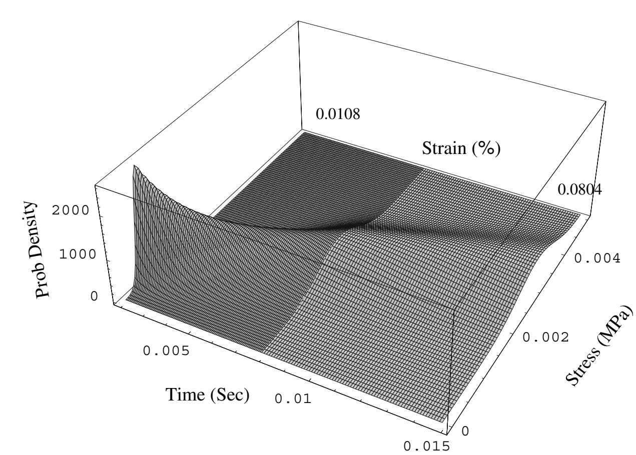

Fig. 9 Evolution of PDF of shear stress versus strain (time) for elastic–plastic material with random shear modulus (Problem II). View 1

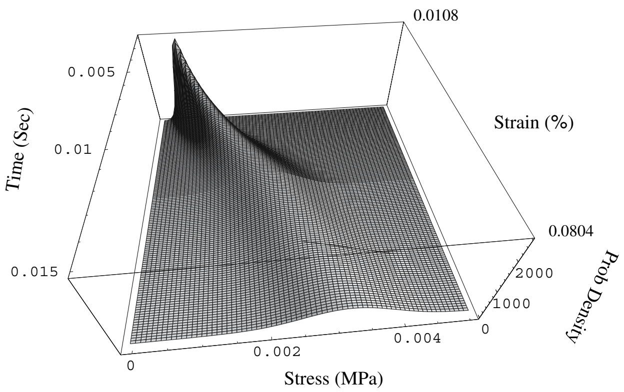

Fig. 10 Evolution of PDF of shear stress versus strain (time) for elastic–plastic material with random shear modulus (Problem II). View 2

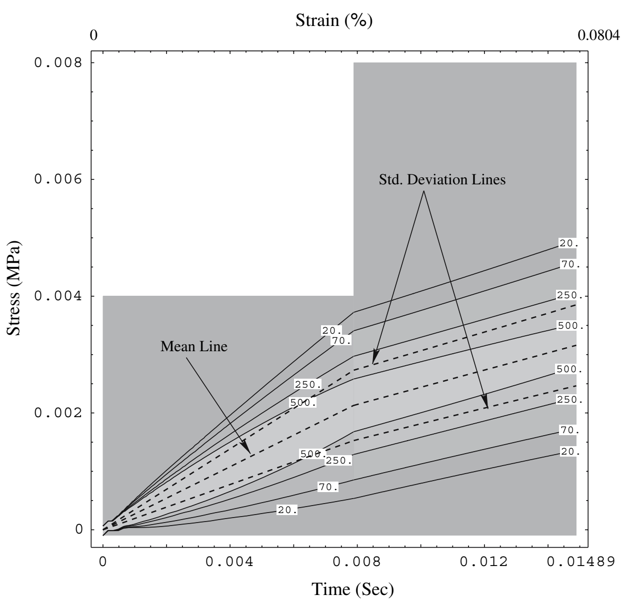

Fig. 11 Contour of evolution of PDF for shear stress versus strain (time) for elastic–plastic material with random shear modulus (Problem II)

occurred at $t = 0.00789$ s (which is equivalent to $\varepsilon_{12} = 0.0426\%$).

The evolution contours for PDF of shear stress versus strain (time) along with the mean and standard deviations are shown in Fig. 11. It can be seen from that figure that, as expected, the evolution of mean of shear stress changes slope after the material yielded. Another interesting aspect to note is the relative slope of the evolution of standard deviation with respect to the evolution of mean. The relative slope in the pre-yield elastic zone increases at a higher rate during the evolution process when compared with that in the post-yield elastic–plastic zone. In other words, in the evolution process the post-yield elastic–plastic constitutive rate equation did not amplify the initial uncertainty as

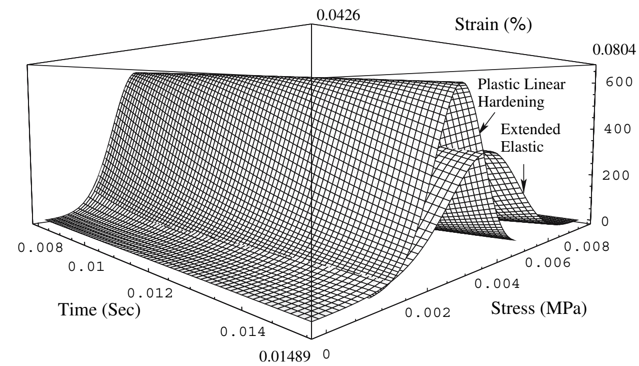

Fig. 12 Comparison of evolution of PDF for elastic–plastic material and extended elastic material cases for random shear modulus

much as the pre-yield elastic constitutive rate equation did. This can be easily viewed from Fig. 12 where the post-yield elastic–plastic evolution of PDF of shear stress was compared with fictitious extension of elastic evolution of PDF. Comparing the PDF of shear stress at $\varepsilon_{12} = 0.0804\%$ (which is equivalent to $t = 0.01489$ s), one can conclude that the variance of predicted elastic–plastic shear stress is much smaller (i.e., prediction is less uncertain) as compared to the same if the material were modeled as completely elastic.

Figure 13 compares the evolution of means and standard deviations of predicted shear stress obtained using FPK equation approach and transformation method (pre-yield behavior)/Monte Carlo approach (post-yield behavior). Although in the pre-yield response the FPK equation approach over-predicted the evolution of standard deviations because of reasons discussed earlier, in the post-yield response it matched closely at regions further from the yielding region. The somewhat larger difference between FPK equation solution and the verification one (Monte Carlo solution) close to the yielding region is attributed to the fact that the initial condition for solution of post-yield elastic–plastic FPK equation was obtained from the solution of pre-yield elastic FPK equation. One way to better predict the overall probabilistic elastic–plastic behavior, would probably be to obtain the pre-yield elastic behavior through the transformation method and then use the FPK approach to predict post-yield elastic–plastic behavior.

6.3 Problem III

In this problem, the pre-yield linear elastic part is deterministic, however, at yield there is a distribution (with very small standard deviation) in shear stress due to assumed distribution in yield parameter α. The distribution in shear stress corresponds to the PDF of the random variable αI_1 (first invariant of the stress tensor or mean confining stress)

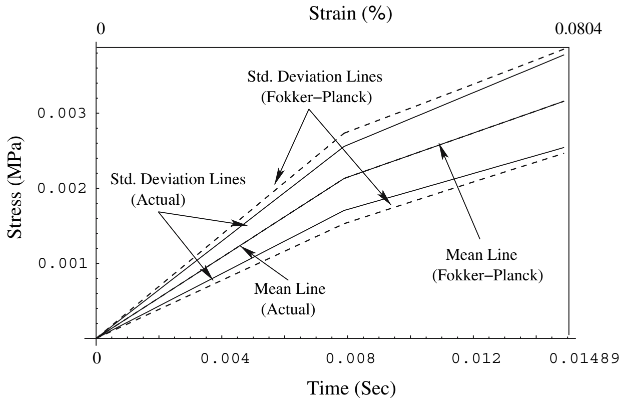

Fig. 13 Comparison of mean and standard deviation of shear stress for plastic constitutive rate equation with random shear modulus (Problem II) for FPK equation solution and Monte Carlo simulation solution

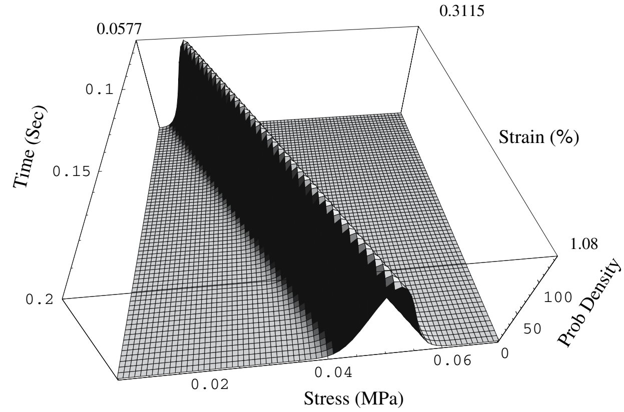

Fig. 15 Evolution of PDF for shear stress for elastic–plastic material with random yield strength (Problem III) (only plastic zone is shown)

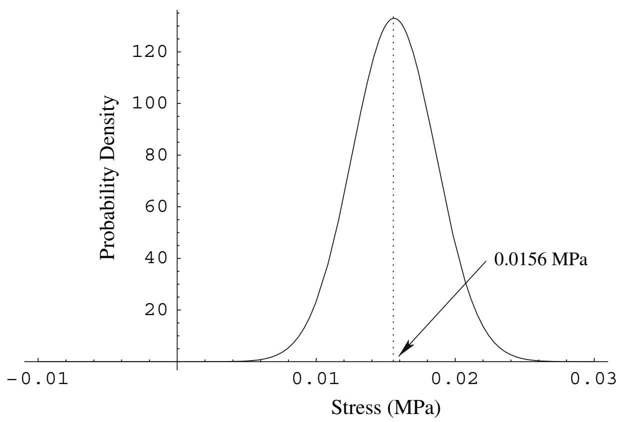

Fig. 14 Initial condition for FPK equation for elastic–plastic material with random yield strength (Problem III)

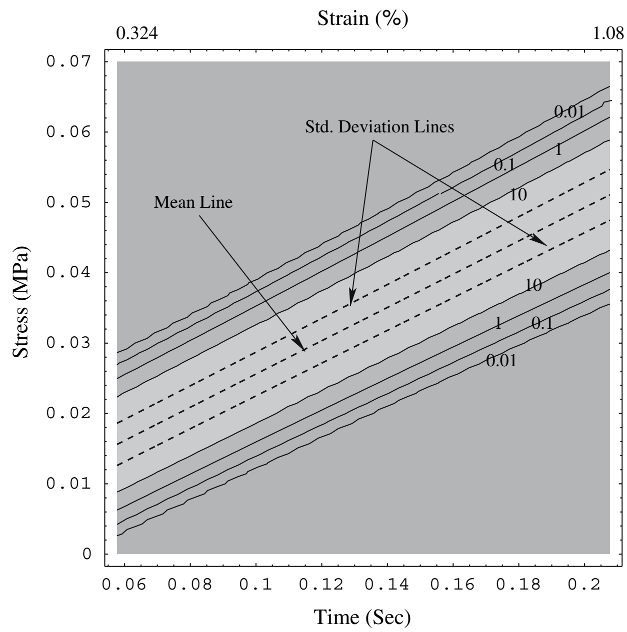

Fig. 16 Contour PDF for shear stress versus strain (time) for elastic–plastic material with random yield strength (Problem III)

and is assumed to be deterministic. This PDF of shear stress at yield was assumed to be the initial condition for the solution of post-yield elastic–plastic FPK equation and is shown in Fig. 14.

The evolution of PDF for shear stress versus strain (time) is shown in Fig. 15. In addition to that the contours (including mean and standard deviation) of the evolution of PDF for shear stress versus strain (time) are shown in Fig. 16.

Looking at Fig. 16 and comparing the slopes of evolution of mean and standard deviation, one can conclude that the elastic–plastic evolution process did not amplify the initial uncertainty in yield strength significantly. The initial (at yield) probability density function of shear stress just advected into the domain during the elastic–plastic evolution process without diffusing much. Figure 15 clearly shows this advection process. The evolution of mean and standard deviations of shear stress obtained from the FPK equation approach was compared with those obtained from the Monte Carlo simulation and is shown in Fig. 17.

7 Conclusions

In this paper a solution was presented for the evolution of the probability density function (PDF) of elastic–plastic stress–strain relationship, in 1D. The solution was based on expressions developed in a companion paper and specialized to the Drucker–Prager elasto–plastic material model with linear isotropic hardening.

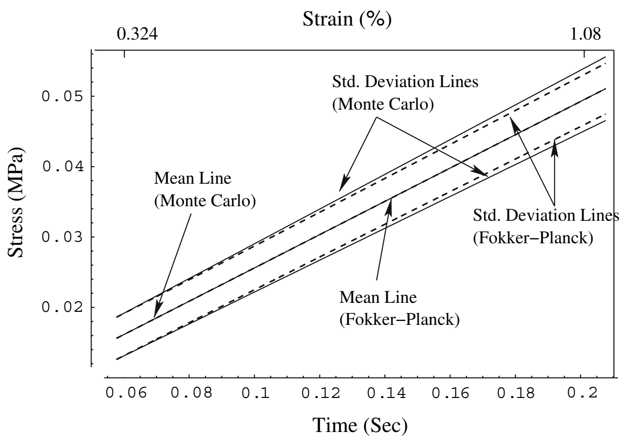

Fig. 17 Comparison of mean and standard deviation of shear stress for elastic–plastic constitutive rate equation with random yield strength (Problem III) for FPE solution and Monte Carlo simulation solution

Three numerical problems were used to illustrate the developed solution and discuss the general behavior of elastic–plastic materials which exhibit uncertainty in material parameters. The solutions to numerical problems were verified against closed, analytical forms, where available, while Monte Carlo simulations were used for all other verifications.

The approach to solving probabilistic elastic–plastic problems presented here is quite unique and shows great promise in dealing with general 3D probabilistic constitutive problems. Subsequently, the developed methodology is to be used in solving general, probabilistic elastic–plastic boundary value problems using the finite element method. Current work is progressing in that direction.

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References

1. Chen WF, Han DJ (1988) Plasticity for structural engineers. Springer, Heidelberg
2. Gardiner CW (2004) Handbook of stochastic methods for physics, chemistry and the natural science. Complexity, 3rd edn. Springer, Heidelberg
3. Jeremić B, Sett K, Kavvas ML (2007) Probabilistic elasto-plasticity: formulation in 1D. Acta Geotech doi:10.1007/s11440-007-0036-x
4. Kavvas ML (2003) Nonlinear hydrologic processes: conservation equations for determining their means and probability distributions. J Hydrol Eng 8(2):44–53
5. Langtangen H (1991) A general numerical solution method for Fokker-Planck equations with application to structural reliability. Probab Eng Mech 6(1):33–48
6. Montgomery DC, Runger GC (2003) Applied statistics and probability for engineers, 3rd edn. Wiley, New York, 10158 pp
7. Oberkampf WL, Trucano TG, Hirsch C (2002) Verification, validation and predictive capability in computational engineering and physics. In: Proceedings of the foundations for verification and validation on the 21st century workshop. Johns Hopkins University/Applied Physics Laboratory, Laurel, pp 1–74
8. Risken H (1989) The Fokker–Planck equation: methods of solution applications, 2nd edn. Springer, Heidelberg
9. Rose C, Smith MD (2002) Mathematical statistics with mathematica. Springer texts in statistics. Springer, New York
10. Van Kampen NG (1976) Stochastic differential equations. Phys Rep 24:171–228
11. Wolfram S (1991) Mathematica: a system for doing mathematics by computer, 2nd edn. Addison-Wesley, Redwood City
12. Wolfram Research Inc (2003) Mathematica Version 5.0. Wolfram Research, Champaign