SCIENCE CHINA
Physics, Mechanics & Astronomy

• Article •

June 2024 Vol. 67 No. 6: 260412
https://doi.org/10.1007/s11433-023-2354-1

Energy flux and waveforms by coalescing spinless binary system in effective one-body theory

Sheng Long[1†], Weike Deng[1†], and Jiliang Jing[1,2*]

[1] *Department of Physics, Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China;*
[2] *Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China*

Received November 28, 2023; accepted February 22, 2024; published online May 14, 2024

We present a study on the energy radiation rate and waveforms of the gravitational wave generated by coalescing spinless binary systems up to the third post-Minkowskian approximation in the effective one-body theory. To derive an analytical expansion of the null tetrad components of the gravitational perturbed Weyl tensor Ψ_4 in the effective spacetime, we utilize the method proposed by Sasaki et al. During this investigation, we discover more general integral formulas that provide a theoretical framework for computing the results in any order. Subsequently, we successfully compute the energy radiation rate and waveforms of the gravitational wave, which include the results of the Schwarzschild case and the correction terms resulting from the dimensionless parameters a_2 and a_3 in the effective metric.

post-Minkowskian approximation, effective one-body theory, gravitational waveform template

PACS number(s): 04.25.Nx, 04.30.Db, 04.20.Cv

Citation: S. Long, W. Deng, and J. Jing, Energy flux and waveforms by coalescing spinless binary system in effective one-body theory, Sci. China-Phys. Mech. Astron. **67**, 260412 (2024), https://doi.org/10.1007/s11433-023-2354-1

1 Introduction

Gravitational waveform templates play an important role in the detection of gravitational wave events generated by coalescing binary systems [1-10]. The foundation of gravitational waveform templates is the theoretical model of gravitational radiation, in which the key point is studying the late-stage dynamical evolution of a coalescing binary system.

Damour and Buonanno et al. [11,12] proposed an effective one-body (EOB) theory that maps the real two-body problem with masses m_1 and m_2 to a test particle of mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ moving around an effective spacetime of mass $M = m_1 + m_2$ (and we denote the symmetric mass ratio as $\nu = \mu/M$). This theory enables the study of gravitational radiation produced by merging binary systems. Based on the EOB theory with the post-Newtonian (PN) approximation, Damour et al. [13] provided an estimate of the gravitational waveforms emitted throughout the inspiral, plunge, and coalescence phases [14].

To release the assumption that v/c is a small quantity, in 2016, Damour et al. [15, 16] introduced another theoretical model by combining the EOB theory with the post-Minkowskian (PM) approximation. Damour and Rettegno [17] compared numerical relativistic (NR) data for equal-mass binary black hole scattering with analytical predictions based on the fourth PM (4PM) dynamics [18-25] and pointed out that the reconstruction of PM information in terms of EOB radial potentials leads to remarkable agreement with NR data, especially when using radiation-reacted 4PM infor-

*Corresponding author (email: jljing@hunnu.edu.cn)
† These authors contributed equally to this work.

mation. Therefore, this new model may lead to a theoretically improved version of the EOB conservative dynamics and may be useful in the upcoming era of high-signal-to-noise-ratio gravitational wave observations.

The dynamical evolution of a coalescing binary system for a spinless EOB theory can be described by the Hamilton equation [26], and the Hamiltonian $H[g_{\mu\nu}^{\rm eff}]$ is dependent on the effective metric. The radiation reaction forces $\mathcal{F}_R[g_{\mu\nu}^{\rm eff}]$ and $\mathcal{F}_\varphi[g_{\mu\nu}^{\rm eff}]$ in the Hamilton equation can be described by the energy radiation rate as follows: $\frac{{\rm d}E}{{\rm d}t} = \frac{1}{4\pi G^2\omega^2}\int|\Psi_4^B|^2 r^2{\rm d}\Omega$ [14, 27]. Furthermore, the "plus" and "cross" modes of gravitational waves are related to the null tetrad components of the gravitational perturbed Weyl tensor Ψ_4^B in the Newman-Penrose formalism as follows: $\Psi_4^B = \frac{1}{2}(\ddot{h}_+ - {\rm i}\ddot{h}_\times)$. Thus, as long as we obtain the effective metric and the solution of Ψ_4^B in the Newman-Penrose formalism, we can calculate the energy radiation rate and construct gravitational waveforms.

In previous study, we attempted to develop a self-consistent EOB theory for spinless and spinning binaries based on the PM approximation [28-31]. Furthermore, in a recent paper [32], we obtained the effective metric up to the 4PM order. We adopted the black hole perturbation method used by Teukolsky et al. [33, 34] and decomposed all quantities into background and perturbation (denoted with a superscript B) parts in the Newman-Penrose formalism. After choosing a shadow gauge [29, 35, 36] with Ψ_1 and Ψ_3 set to 0, we can decouple the equations for the null tetrad components of the gravitational perturbed Weyl tensor Ψ_4^B. Subsequently, upon separating the variables in the equations, we obtained a radial equation, which is the so-called Teukolsky-like equation, and an angular equation that features spin-weighted spherical harmonics.

This Teukolsky-like equation is more complex compared with the Teukolsky equation in Kerr and Schwarzschild spacetimes. We were unable to find a similar transformation to convert the homogeneous Teukolsky-like equation into hypergeometric or Heun equations; thus, we did not choose to adopt the so-called MST [37, 38] method or the Heun function [39-44]. Instead, we follow the approach used by Sasaki. Several researchers [45-57] have employed numerical methods to solve the Teukolsky equation and achieved significant success by combining the EOB theory with numerical relativity.

We initially applied the Sasaki-Nakamura-Chandrasekhar-like (S-N-C-like) transformation [35, 58] to convert the homogeneous Teukolsky-like equation into a homogeneous Sasaki-Nakamura-like (S-N-like) equation [27, 38, 58, 59]. In an asymptotically flat spacetime, this homogeneous S-N-like equation can be simplified to the Klein-Gordon equation. Subsequently, we performed a Taylor expansion with respect to $\eta = 2GM\omega$. The equation of the zeroth order is the spherical Bessel equation, and its solutions are linear combinations of the first and second kind spherical Bessel functions, denoted as j_ℓ and n_ℓ, respectively, allowing us to construct higher-order solutions based on the zeroth-order solution. By performing an inverse transformation, we can deduce the solutions of the homogeneous Teukolsky-like equation. This framework enables us to construct the solutions of the inhomogeneous Teukolsky equation, which includes a source term, using Green's function method.

However, due to the complexity of Green's function method and the fact that the integral formulas provided in the previous work of Sasaki et al. [59] were insufficient for our needs in computing higher-order solutions, we found some new integral formulas presented in Supplementary materials of this article, which are crucial for our journey toward calculating higher-order solutions.

Sect. 2 introduces the effective metric of 3PM, while sect. 3 discusses the solutions of the equation for Ψ_4^B in the effective spacetime. Specifically, sect. 3.1 summarizes the general structure of the solutions of the radial equation (Teukolsky-like equation) of Ψ_4^B. Sect. 3.2 provides a comprehensive explanation of the calculation of the homogeneous S-N-like equation. Subsequently, we employ boundary conditions to determine the amplitudes. Sect. 3.3 presents the source terms for quasi-circular orbits; by combining the homogeneous solutions provided in sect. 3.2 and utilizing eq. (17), we can obtain the solution of Ψ_4^B under quasi-circular orbits. In sect. 4, we present the energy radiation rate $\frac{{\rm d}E}{{\rm d}t}$ and the gravitational waveforms $h_{\ell m}$.

2 Effective metric for the EOB theory

In the EOB theory, the main idea is to map the two-body problem onto an EOB problem, that is, a test particle orbits around a massive black hole described by an effective metric. With the help of the scattering angles, we found that the effective metric for spinless binaries with radiation reaction effects in the EOB theory, up to the 3PM approximation, can be expressed as follows [32]:

$$\begin{aligned}{\rm d}s_{\rm eff}^2 &= g_{\mu\nu}^{\rm eff}{\rm d}x^\mu{\rm d}x^\nu \\ &= \frac{\Delta_r}{r^2}{\rm d}t^2 - \frac{r^2}{\Delta_r}{\rm d}r^2 - r^2({\rm d}\theta^2 + \sin^2\theta{\rm d}\varphi^2),\end{aligned} \tag{1}$$

with

$$\begin{aligned}\Delta_r =& r^2 - 2GMr + a_2(GM)^2 + a_3\frac{(GM)^3}{r} \\ =& \frac{1}{r}(r - 2c_hGM)(r - 2c_1GM)(r - 2c_2GM),\end{aligned} \tag{2}$$

the definitions of c_1, c_2, and c_h are as follows:

$$c_1 = \frac{1}{3} - \frac{1}{2}\Big[(1-\mathrm{i}\sqrt{3})(Q+\sqrt{P^3+Q^2})^{\frac{1}{3}} + (1+\mathrm{i}\sqrt{3})(Q-\sqrt{P^3+Q^2})^{\frac{1}{3}}\Big], \tag{3}$$

$$c_2 = \frac{1}{3} - \frac{1}{2}\Big[(1+\mathrm{i}\sqrt{3})(Q+\sqrt{P^3+Q^2})^{\frac{1}{3}} + (1-\mathrm{i}\sqrt{3})(Q-\sqrt{P^3+Q^2})^{\frac{1}{3}}\Big], \tag{4}$$

$$c_h = \frac{1}{3} + (Q+\sqrt{P^3+Q^2})^{\frac{1}{3}} + (Q-\sqrt{P^3+Q^2})^{\frac{1}{3}}, \tag{5}$$

where $Q = \frac{1}{27} - \frac{a_2}{24} - \frac{a_3}{16}$, $P = \frac{1}{3}\left(\frac{a_2}{4} - \frac{1}{3}\right)$ and a_2 and a_3 are dimensionless parameters expressed as follows:

$$a_2 = \frac{3(1-\Gamma)(1-5\gamma^2)}{\Gamma(3\gamma^2-1)}, \tag{6}$$

$$a_3 = \frac{3}{2(4\gamma^2-1)}\left[\frac{3-2\Gamma-3(15-8\Gamma)\gamma^2+6(25-16\Gamma)\gamma^4}{\Gamma(3\gamma^2-1)} - 2P_{30} - \frac{2\chi_3^{rr}}{\sqrt{\gamma^2-1}}\right], \tag{7}$$

in which $\gamma = \frac{E}{\mu} = \frac{1}{2}\frac{\mathcal{E}^2-m_1^2-m_2^2}{m_1 m_2}$ is the Lorentz factor variable, $\mathcal{E}$ is the real two-body energy [17,32], E is the effective energy, $\Gamma = E/M = \sqrt{1+2\nu(\gamma-1)}$ is the rescaled energy, and

$$P_{30} = \frac{18\gamma^2-1}{2\Gamma^2} + \frac{8\nu(3+12\gamma^2-4\gamma^4)}{\Gamma^2\sqrt{\gamma^2-1}}\operatorname{arcsinh}\sqrt{\frac{\gamma-1}{2}} + \frac{\nu}{\Gamma^2}\Big(1 - \frac{103}{3}\gamma - 48\gamma^2 - \frac{2}{3}\gamma^3 + \frac{3\Gamma(1-2\gamma^2)(1-5\gamma^2)}{(1+\Gamma)(1+\gamma)}\Big), \tag{8}$$

$$\chi_3^{rr} = -\frac{2\nu}{\Gamma^2}\frac{\gamma(2\gamma^2-1)^2}{3(\gamma^2-1)^{3/2}}\Big\{\frac{(5\gamma^2-8)}{\gamma}\sqrt{\gamma^2-1} + 2(9-6\gamma^2)\operatorname{arcsinh}\sqrt{\frac{\gamma-1}{2}}\Big\}. \tag{9}$$

In eq. (7), the term χ_3^{rr}, described by eq. (9), represents the 3PM radiation reaction effects, which shows that the structure of the effective spacetime is affected by the radiation reaction effect.

3 Solutions of equation for Ψ_4^B in effective spacetime

In this section, we first present the formal solution of the radial equation for (Ψ_4^B). Then, we transform the radial equation without source to the corresponding S-N-like equation, and we look for its solution. At last, we work out the solution of the radial equation of (Ψ_4^B) with the source, which describes the gravitational radiation induced by the motion of an effective particle in an effective background.

3.1 Formal solution of the radial equation of Ψ_4^B in effective spacetime

In the EOB theory for the spinless real two-body system, we have found a decoupled equation of Ψ_4^B for the gravitational perturbation in the effective spacetime (1) using the gauge transform property of the tetrad components of the perturbed Weyl tensors and separated the decoupled equation in the radial and angular parts, in which the radial part of Ψ_4^B is given by [28,29]

$$\left[\frac{\Delta_r^2}{\mathfrak{f}}\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{\mathfrak{f}}{\Delta_r}\frac{\mathrm{d}}{\mathrm{d}r}\Big) - V(r)\right]R_{lm\omega}(r) = T_{\ell m\omega}(r), \tag{10}$$

with

$$\begin{aligned}
\mathfrak{f} &= -\frac{3GM}{r^3F_4},\\
V(r) &= -\frac{r^2\omega(r^2\omega+2\mathrm{i}\Delta_r')}{\Delta_r} + \mathrm{i}r\omega\left(5+\frac{2r^3F_1}{\Delta_r}\right) - \frac{3\Delta_r}{r^2} - \frac{3\Delta_r'}{r} + 6rF_1 - 2r^2F_4 + \lambda F_2,\\
T_{\ell m\omega}(r) &= -\mu\int_{-\infty}^{\infty}\mathrm{d}t e^{\mathrm{i}\omega t-\mathrm{i}m\varphi(t)}\Delta_r^2\Big\{\mathbf{A}_0\delta(r-r(t)) + \Big[\mathbf{A}_1\delta(r-r(t))\Big]' + \Big[\mathbf{A}_2\delta(r-r(t))\Big]''\Big\},
\end{aligned} \tag{11}$$

where the prime $'$ denotes derivation with respect to r, and $\mathbf{A}_0 = A_{nn\,0}+A_{\overline{m}n\,0}+A_{\overline{m}\overline{m}\,0}$, $\mathbf{A}_1 = A_{\overline{m}n\,1}+A_{\overline{m}\overline{m}\,1}$, and $\mathbf{A}_2 = A_{\overline{m}\overline{m}\,2}$.

$$\begin{aligned}
\mathbf{A}_{nn\,0} &= -\frac{2r^4}{\sqrt{2\pi}\Delta_r^2}C_{nn}F_2\mathscr{L}_1^+\Big[\mathscr{L}_2^+\Big({}_{-2}Y_{\ell m}(\theta)\Big)\Big],\\
\mathbf{A}_{\overline{m}n\,0} &= \frac{r^3}{\sqrt{\pi}\Delta_r}C_{\overline{m}n}\Big[(1+F_2)\frac{\mathrm{i}r^2\omega}{\Delta_r} + \frac{F_2}{r} - F_2' - \frac{F_4'}{F_4}\Big] \times\mathscr{L}_2^\dagger\Big({}_{-2}Y_{\ell m}(\theta)\Big),\\
\mathbf{A}_{\overline{m}\overline{m}\,0} &= \frac{r^2}{\sqrt{2\pi}}C_{\overline{m}\overline{m}}\,{}_{-2}Y_{\ell m}(\theta)\Big[\mathrm{i}\Big(\frac{r^2\omega}{\Delta_r}\Big)' + \frac{r^4\omega^2}{\Delta_r^2} + \frac{\mathrm{i}r^2\omega}{\Delta_r}\Big(\frac{1}{r}+\frac{F_4'}{F_4}\Big) + \frac{F_4'}{rF_4} + \Big(\frac{F_4'}{F_4}\Big)'\Big],\\
\mathbf{A}_{\overline{m}n\,1} &= \frac{r^3}{\sqrt{\pi}\Delta_r}C_{\overline{m}n}(1+F_2)\mathscr{L}_2^\dagger\Big({}_{-2}Y_{\ell m}(\theta)\Big),\\
\mathbf{A}_{\overline{m}\overline{m}\,1} &= \frac{r^2}{\sqrt{2\pi}}C_{\overline{m}\overline{m}}\,{}_{-2}Y_{\ell m}(\theta)\Big(-2\mathrm{i}\frac{r^2\omega}{\Delta_r} + \frac{1}{r} + \frac{F_4'}{F_4}\Big),\\
\mathbf{A}_{\overline{m}\overline{m}\,2} &= -\frac{r^2}{\sqrt{2\pi}}C_{\overline{m}\overline{m}}\,{}_{-2}Y_{\ell m}(\theta),
\end{aligned} \tag{12}$$

the explicitly definitions of F_a ($a = 1,2,3,4$), $\mathscr{L}_n$ (or $\mathscr{L}_n^\dagger$), and C_b (b is nn or $\overline{m}n$ or $\overline{m}\overline{m}$) can be found in ref. [29], and ${}_{-2}Y_{\ell m}(\theta)$ is the spin-weighted spherical harmonics [38,60].

The radial equation, i.e., eq. (10), can be solved using Green's function method. That is, based on the homogeneous solutions of eq. (10):

$$R^{\rm in}_{\ell m\omega} \to \begin{cases} B^{\rm trans}_{\ell m\omega}\Delta_r^2 {\rm e}^{-{\rm i}\omega r^*}, & \text{for } r \to r_+, \\ r^3 B^{\rm ref}_{\ell m\omega}{\rm e}^{{\rm i}\omega r^*} + r^{-1}B^{\rm in}_{\ell m\omega}{\rm e}^{-{\rm i}\omega r^*}, & \text{for } r \to +\infty, \end{cases} \tag{13}$$

$$R^{\rm up}_{\ell m\omega} \to \begin{cases} C^{\rm up}_{\ell m\omega}{\rm e}^{{\rm i}\omega r^*} + \Delta_r^2 C^{\rm ref}_{\ell m\omega}{\rm e}^{-{\rm i}\omega r^*}, & \text{for } r \to r_+, \\ C^{\rm trans}_{\ell m\omega} r^3 {\rm e}^{{\rm i}\omega r^*}, & \text{for } r \to +\infty, \end{cases} \tag{14}$$

where r^* denotes the tortoise coordinate defined by $r^* = \int \frac{r^2}{\Delta_r}{\rm d}r$. The inhomogeneous solution of the radial eq. (10) can be expressed as follows:

$$R_{\ell m\omega} = \frac{1}{2{\rm i}\omega C^{\rm trans}_{\ell m\omega}B^{\rm inc}_{\ell m\omega}}\left\{R^{\rm up}_{\ell m\omega}\int_{r_+}^{r}{\rm d}\tilde{r}\frac{\mathfrak{f}R^{\rm in}_{\ell m\omega}(\tilde{r})T_{\ell m\omega}(\tilde{r})}{\Delta_r^2(\tilde{r})}\right.$$
$$\left. + R^{\rm in}_{\ell m\omega}\int_{r}^{\infty}{\rm d}\tilde{r}\frac{\mathfrak{f}R^{\rm up}_{\ell m\omega}(\tilde{r})T_{\ell m\omega}(\tilde{r})}{\Delta_r^2(\tilde{r})}\right\}, \tag{15}$$

whereas the counterpart at infinity can be expressed as follows:

$$R_{\ell m\omega}(r\to\infty) \to \frac{r^3{\rm e}^{{\rm i}\omega r^*}}{2{\rm i}\omega B^{\rm inc}_{\ell m\omega}}\int_{r_+}^{\infty}{\rm d}\tilde{r}\frac{T_{\ell m\omega}(\tilde{r})R^{\rm in}_{\ell m\omega}(\tilde{r})}{\tilde{r}^3F_4(\tilde{r})\Delta_r^2(\tilde{r})}$$
$$\equiv \hat{Z}_{\ell m\omega}r^3{\rm e}^{{\rm i}\omega r^*}. \tag{16}$$

As discussed in ref. [29], for the point source case, after a lengthy calculation, we can obtain the expression for $\hat{Z}_{\ell m\omega}$. If we focus our attention just on the quasi-circular orbit, we have $\hat{Z}_{\ell m\omega_n} = Z_{\ell m\omega}\,\delta(\omega-\omega_n)$, in which

$$Z_{\ell m\omega} = \frac{\pi\nu GM}{{\rm i}\omega B^{\rm inc}_{\ell m\omega}}\left[\mathbf{A}_0\mathfrak{f}R^{\rm in}_{\ell m\omega} - \mathbf{A}_1\frac{\rm d}{{\rm d}r}\left(\mathfrak{f}R^{\rm in}_{\ell m\omega}\right) + \mathbf{A}_2\frac{{\rm d}^2}{{\rm d}r^2}\left(\mathfrak{f}R^{\rm in}_{\ell m\omega}\right)\right]. \tag{17}$$

Then, the solution of Ψ_4^B is described by

$$\Psi_4^B = \frac{1}{\mathcal{R}}\sum_{\ell mn}\hat{Z}_{\ell m\omega_n}\frac{{}_{-2}Y_{\ell m}}{\sqrt{2\pi}}{\rm e}^{{\rm i}\omega_n(r^*-t)+{\rm i}m\varphi}. \tag{18}$$

Eqs. (18) and (17) show that to get the explicit expression for Ψ_4^B, we should work out $\mathbf{A}_i(i=1,2,3)$, $B^{\rm inc}_{\ell m\omega}$, and $R^{\rm in}_{\ell m\omega}$.

3.2 S-N-like equation and its solution of the third PM approximation

In eq. (17), $R^{\rm in}_{\ell m\omega}$ is the solution of the homogeneous equation without the source term. To get the solution, we do not treat the Teukolsky-like equation directly because the potential function in the equation is a long-range potential. Instead, we transform the radial equation, without source into the S-N-like equation with a new function $X_{\ell m\omega}$, which has a short-range potential. Then, using the solution of $X^{\rm in}_{\ell m\omega}$, we find out $B^{\rm in}_{\ell m\omega}$ and $R^{\rm in}_{\ell m\omega}$.

3.2.1 S-N-like equation and relation between $R^{\rm in}_{\ell m\omega}$ and $X^{\rm in}_{\ell m\omega}$

Taking an S-N-C-like transformation as[1)]:

$$X(r) = \frac{rf^{1/2}}{\Delta_r}\left(\alpha(r)R(r) + \frac{\beta(r)}{\Delta_r}R'(r)\right), \tag{19}$$

where

$$\begin{aligned}\alpha(r) &= 6\frac{\Delta_r}{r^2} + V(r) + 2{\rm i}r\omega + {\rm i}r^2\omega\frac{\Delta_r{}'}{\Delta_r} - \frac{r^4\omega^2}{\Delta_r},\\ \beta(r) &= \Delta_r(-2{\rm i}r^2\omega - 4\frac{\Delta_r}{r} - \Delta_r\frac{\mathfrak{f}'}{\mathfrak{f}} + \Delta_r),\end{aligned} \tag{20}$$

and considering the coordinate transformation $r \to r_*$, the radial equation (eq. (10)) without the source term ($T_{\ell m\omega}=0$) can be rewritten as the so-called S-N-like equation, as follows:

$$X_{,r^*r^*} - \mathscr{F}X_{,r^*} - \mathscr{U}X = 0, \tag{21}$$

with

$$\begin{aligned}\mathscr{F} &= \frac{\Delta_r}{r^2}\frac{\gamma'}{\gamma},\\ \mathscr{U} &= \frac{\Delta_r}{r^4}U + G^2 + G_{,r^*} - \frac{\Delta_r}{r^2}\frac{\gamma'}{\gamma}G,\end{aligned} \tag{22}$$

where

$$\begin{aligned}\gamma &= \alpha\left(\alpha + \frac{\beta'}{\Delta_r} - \frac{\beta}{\Delta_r}\frac{\mathfrak{f}'}{\mathfrak{f}}\right) - \frac{\beta}{\Delta_r}\left(\alpha' + \frac{\beta}{\Delta_r^2}V(r)\right),\\ U &= \frac{\Delta_r^2}{\beta}\left(\left(2\alpha+\frac{\beta'}{\Delta_r}-\frac{\beta}{\Delta_r}\frac{\mathfrak{f}'}{\mathfrak{f}}\right)' - \frac{\gamma'}{\gamma}\left(\alpha+\frac{\beta'}{\Delta_r}-\frac{\beta}{\Delta_r}\frac{\mathfrak{f}'}{\mathfrak{f}}\right)\right) + V(r),\\ G &= \frac{\Delta_r}{r^3} + \frac{\Delta_r\mathfrak{f}'}{2r^2\mathfrak{f}} - \frac{\Delta_r{}'}{r^2}.\end{aligned} \tag{23}$$

The asymptotic solution of $X^{\rm in}_\ell$ can be expressed as follows:

$$X^{\rm in}_\ell(r) = \begin{cases} A^{\rm trans}_\ell {\rm e}^{-{\rm i}\omega r^*}, & r^* \to -\infty, \\ A^{\rm out}_\ell{\rm e}^{{\rm i}\omega r^*} + A^{\rm in}_\ell{\rm e}^{-{\rm i}\omega r^*}, & r^* \to +\infty. \end{cases} \tag{24}$$

Meanwhile, the inverse transformation is described by the following expression:

$$R(r) = \frac{1}{\gamma}\left[\left(\alpha + \frac{\beta'}{\Delta_r} - \frac{\beta}{\Delta_r}\frac{\mathfrak{f}'}{\mathfrak{f}}\right)\frac{\Delta_r}{r\mathfrak{f}^{1/2}}X(r) - \frac{\beta}{\Delta_r}\left(\frac{\Delta_r}{r\mathfrak{f}^{1/2}}X(r)\right)'\right]. \tag{25}$$

1) Because the physical quantities we are concerned with are related to the ℓ, m, and ω, for any ℓ, ω depends on m and $m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell$. Thus, we shall henceforth represent the subscript labels $\ell m\omega$ of these physical quantities simply as ℓ or drop the subscript for the sake of brevity and clarity.

Using a method similar to that used in refs. [38, 61], the coefficient $A_\ell^{\rm in}$ in eq. (24) is related to $B_\ell^{\rm in}$ in eq. (13) as follows:

$$B_\ell^{\rm in} = -\frac{1}{4\omega^2}A_\ell^{\rm in}. \tag{26}$$

3.2.2 Solution of $X_\ell^{\rm in}$

We now look for the solution of $X_\ell^{\rm in}$ and amplitude $A_\ell^{\rm in}$ of the S-N-like equation. The method employed in this subsection is based on the work of Sasaki et al. [38] and Mino et al. [59]. We first take the following ansatz:

$$X_\ell^{\rm in} = {\rm e}^{-{\rm i}\phi(z)}z\xi_\ell(z), \tag{27}$$

where $z = \omega r$, $\eta = 2GM\omega$, $b_1 = \frac{c_1^3}{(c_1-c_2)(c_1-c_h)}$, $b_2 = \frac{c_2^3}{(c_2-c_1)(c_2-c_h)}$, $b_h = \frac{c_h^3}{(c_1-c_h)(c_h-c_2)}$, and

$$\begin{aligned}\phi(z) &= \int\left(\frac{r^2\omega}{\Delta_r}-\omega\right){\rm d}r\\ &= \eta(b_1\ln(z-c_1\eta)+b_2\ln(z-c_2\eta)-b_h\ln(z-c_h\eta)).\end{aligned} \tag{28}$$

With this choice of the phase function, $\xi_{\ell m}$ is regular and finite at $z = \eta c_h$. Then, we determine that eq. (21) can be expressed as follows:

$$L^{(0)}[\xi_\ell] = \eta L^{(1)}[\xi_\ell] + \eta^2 L^{(2)}[\xi_\ell] + \eta^3 L^{(3)}[\xi_\ell] + O(\eta^4), \tag{29}$$

with

$$\begin{aligned}L^{(0)} =& \frac{{\rm d}^2}{{\rm d}z^2} + \frac{2}{z}\frac{\rm d}{{\rm d}z} + \left[1 - \frac{\ell(\ell+1)}{z^2}\right],\\ L^{(1)} =& \frac{1}{z}\frac{{\rm d}^2}{{\rm d}z^2} + \frac{1+\frac{2}{3}a_2+2{\rm i}z}{z^2}\frac{\rm d}{{\rm d}z} - \frac{4+z^2+\frac{a_2(\ell^2+\ell+2)}{3}-{\rm i}z}{z^3},\\ L^{(2)} =& -\frac{a_2}{4z^2}\frac{{\rm d}^2}{{\rm d}z^2} + \left(\frac{\rm i}{z^2}\mathfrak{a}_\ell^{(2)} + \frac{\mathfrak{b}_\ell^{(2)}}{z^3}\right)\frac{\rm d}{{\rm d}z}\\ &+ \left(\frac{\mathfrak{c}_\ell^{(2)}}{z^2} + \frac{\rm i}{z^3}\mathfrak{d}_\ell^{(2)} + \frac{\mathfrak{e}_\ell^{(2)}}{z^4}\right),\\ L^{(3)} =& -\frac{a_3}{8z^3}\frac{{\rm d}^2}{{\rm d}z^2} + \left(\frac{\mathfrak{a}_\ell^{(3)}}{z^2} + \frac{\rm i}{z^3}\mathfrak{b}_\ell^{(3)} + \frac{\mathfrak{c}_\ell^{(3)}}{z^4}\right)\frac{\rm d}{{\rm d}z}\\ &+ \left(\frac{\rm i}{z^2}\mathfrak{d}_\ell^{(3)} + \frac{\mathfrak{e}_\ell^{(3)}}{z^3} + \frac{\rm i}{z^4}\mathfrak{f}_\ell^{(3)} + \frac{\mathfrak{g}_\ell^{(3)}}{z^5}\right),\end{aligned} \tag{30}$$

where the definitions of $\mathfrak{a}_\ell^{(n)}$ and other terms are shown in Supplementary materials.

In the low-frequency limit and noting that $\eta = 2GM\omega$ only appears on the right-hand side of eq. (29), we may look for the solution of the $\xi_\ell(z)$ perturbative in terms of η, i.e.,

$$\xi_\ell(z) = \sum_{n=0}^{\infty}\eta^n\xi_\ell^{(n)}(z), \tag{31}$$

and we obtain the recursive equation from eq. (29) as follows:

$$L^{(0)}[\xi_\ell^{(n)}] = W_\ell^{(n)}, \tag{32}$$

where

$$W_\ell^{(0)} = 0, \tag{33}$$

$$W_\ell^{(n)} = \sum_{i=1}^{n} L^{(i)}[\xi_\ell^{(n-i)}], \quad n = 1,2,3. \tag{34}$$

The solution of $\xi_\ell^{(0)}$ can be expressed as a linear combination of the spherical Bessel functions j_ℓ and n_ℓ, i.e., $\xi_\ell^{(0)} = \alpha^{(0)}j_\ell + \beta^{(0)}n_\ell$. Because n_ℓ does not match with the horizon solution at the leading order of η, we should take $\beta^{(0)} = 0$. Furthermore, because the constant $\alpha^{(0)}$ represents the overall normalization of the solution, which can be chosen arbitrarily, we set $\alpha^{(0)} = 1$. That is, for the zeroth-order solution, we have $f_\ell^{(0)} = j_\ell$ and $g_\ell^{(0)} = 0$. Then, one can immediately write the integral expression for $\xi_\ell^{(n)}$ (where $n > 0$). Noting that $j_\ell n_\ell{}' - n_\ell j_\ell{}' = 1/z^2$, we derive the expression of $\xi_\ell^{(\beta)}$ for $\beta \geq 1$ as follows:

$$\xi_\ell^{(\beta)} = n_\ell\int^z {\rm d}z\left(z^2 j_\ell W_\ell^{(\beta)}\right) - j_\ell\int^z {\rm d}z\left(z^2 n_\ell W_\ell^{(\beta)}\right). \tag{35}$$

In general, $\xi_\ell^{(\beta)}$ can be decomposed into the real and imaginary parts $\xi_\ell^{(\beta)} = f_\ell^{(\beta)} + {\rm i}g_\ell^{(\beta)}$, in which

$$f_\ell^{(\beta)} = n_\ell\int^z {\rm d}z\left(z^2 j_\ell \mathbf{Re}[W_\ell^{(\beta)}]\right) - j_\ell\int^z {\rm d}z\left(z^2 n_\ell \mathbf{Re}[W_\ell^{(\beta)}]\right), \tag{36}$$

$$g_\ell^{(\beta)} = n_\ell\int^z {\rm d}z\left(z^2 j_\ell \mathbf{Im}[W_\ell^{(\beta)}]\right) - j_\ell\int^z {\rm d}z\left(z^2 n_\ell \mathbf{Im}[W_\ell^{(\beta)}]\right), \tag{37}$$

with

$$\mathbf{Re}[W_\ell^{(\beta)}] = \sum_{i=1}^{\beta}\left(\mathbf{Re}[L^{(i)}][f_\ell^{(\beta-i)}] - \mathbf{Im}[L^{(i)}][g_\ell^{(\beta-i)}]\right), \tag{38}$$

$$\mathbf{Im}[W_\ell^{(\beta)}] = \sum_{i=1}^{\beta}\left(\mathbf{Im}[L^{(i)}][f_\ell^{(\beta-i)}] + \mathbf{Re}[L^{(i)}][g_\ell^{(\beta-i)}]\right), \tag{39}$$

where $\mathbf{Re}[x]$ and $\mathbf{Im}[x]$ are the real and imaginary parts of x, respectively. Using the previously presented formula and the method discussed in Supplementary materials, after some tedious calculations, we derive closed analytical formulas of the ingoing-wave S-N-like function for arbitrary ℓ to the first order of η. At the second order of η^2, we can calculate results for any order utilizing eq. (36). However, generalizing these results to encompass all values of ℓ is unattainable. Therefore, for the higher-order results, we only provide results for specific values of ℓ, for $\ell = 2, 3$ to η^2 order, and for $\ell = 2$ to η^3 order. We express the real parts as follows:

$$f_\ell^{(1)} = \frac{(\ell-1)(\ell+3)}{2(\ell+1)(2\ell+1)}j_{\ell+1} - \left(\frac{\ell^2-4}{2\ell(2\ell+1)} + \frac{2\ell-1}{\ell(\ell-1)}\right)j_{\ell-1}$$

$$+R_{\ell,0}j_0 - 2D_\ell^{nj} + \sum_{m=1}^{\ell-2}\left(\frac{1}{m}+\frac{1}{m+1}\right)R_{\ell,m}j_m - \frac{a_2}{3}\Big(\frac{\ell^2+3\ell+4}{2(\ell+1)(2\ell+1)}j_{\ell+1} - \frac{\ell^2-\ell+2}{2\ell(2\ell+1)}j_{\ell-1}\Big), \tag{40}$$

where $R_{m,k}$ is the Lommel polynomial ($R_{m,k} = -R_{k,m}$ for $m < k$), expressed as follows:

$$\begin{aligned} R_{m,k} &= z^2(n_m j_k - j_m n_k) \quad (m > k) \\ &= -\sum_{r=0}^{[\frac{(m-k-1)}{2}]} \frac{(-1)^r(m-k-1-r)!\Gamma\left(m+\frac{1}{2}-r\right)}{r!(m-k-1-2r)!\Gamma\left(k+\frac{3}{2}+r\right)}\left(\frac{2}{z}\right)^{m-k-1-2r}, \end{aligned} \tag{41}$$

B_J is the generalized integral sinusoidal function, and D_ℓ^J is the generalized spherical Bessel function from refs. [38, 59] or see Supplementary materials.

$$\begin{aligned} f_2^{(2)} =& \left(-\frac{193a_2^2}{1890z}+\frac{45a_2}{14z}-\frac{257a_3}{1008z}-\frac{113}{420z}\right)j_1 \\ &+\left(-\frac{17a_2^2}{1890z}-\frac{10a_2}{63z}-\frac{59a_3}{336z}+\frac{1}{7z}\right)j_3 \\ &+\left(-\frac{16a_2^2}{945}+\frac{a_2}{21}-\frac{5a_3}{126}-\frac{107}{210}\right)j_2\ln z \\ &+\left(\frac{32a_2^2}{315z}-\frac{55a_2}{21z}+\frac{5a_3}{21z}\right)n_0 \\ &+\left(\frac{32a_2^2}{945}-\frac{2a_2}{21}+\frac{5a_3}{63}+\frac{107}{105}\right)D_{-3}^{nj} \\ &+\left(\frac{10}{3}-\frac{2a_2}{15}\right)D_1^{nj}+\frac{14a_2}{45}D_3^{nj}-\frac{389j_0}{70z^2} \\ &-\frac{1}{2}j_2(\ln z)^2+\frac{6D_0^{nj}}{z}-\frac{5D_2^{nj}}{3z}+4D_2^{nnj}, \end{aligned} \tag{42}$$

$$\begin{aligned} f_3^{(2)} =& \left(\frac{20a_2^2}{81z}-\frac{635a_2}{72z}+\frac{5a_3}{9z}-\frac{445}{14z^3}-\frac{1031}{588z}\right)j_0 \\ &+\left(-\frac{197a_2^2}{1134z}+\frac{1093a_2}{168z}-\frac{415a_3}{1008z}+\frac{323}{49z}\right)j_2 \\ &+\left(\frac{2a_2^2}{189z}-\frac{199a_2}{840z}-\frac{65a_3}{504z}+\frac{1}{4z}\right)j_4 \\ &+j_3\ln z\left(-\frac{5a_2^2}{378}+\frac{a_2}{42}-\frac{5a_3}{168}-\frac{13}{42}\right)+4D_3^{nnj} \\ &+\left(\frac{25a_2^2}{63z}-\frac{405a_2}{28z}+\frac{25a_3}{28z}-\frac{65}{6z}\right)n_1 \\ &+\left(-\frac{5a_2^2}{189}+\frac{a_2}{21}-\frac{5a_3}{84}-\frac{13}{21}\right)D_{-4}^{nj}+\left(\frac{13}{3}-\frac{8a_2}{63}\right)D_2^{nj} \\ &+\frac{11a_2}{42}D_4^{nj}-\frac{5065j_1}{294z^2}-\frac{1}{2}j_3(\ln z)^2+\frac{65n_0}{6z^2} \\ &+\frac{30D_0^{nj}}{z^2}+\frac{9D_1^{nj}}{z}-\frac{3D_3^{nj}}{z}. \end{aligned} \tag{43}$$

$$\begin{aligned} f_2^{(3)} =& \left(\frac{9}{4}-\frac{a_2}{30}\right)j_1(\ln z)^2+\left(\frac{7a_2}{90}-\frac{1}{12}\right)j_3(\ln z)^2 \\ &+D_2^{nj}(\ln z)^2+\Bigg(-\frac{16a_2^3}{14175}+\frac{5a_2^2}{63}-\frac{a_3a_2}{378} \\ &-\frac{887a_2}{3150}+\frac{5a_3}{28}+\frac{349}{140}\Bigg)j_1\ln z \\ &+\frac{c_1^2+(c_2-1)(c_1+c_2)}{z}j_2\ln z+\left(\frac{2}{3}-\frac{28a_2}{45}\right)D_3^{nnj} \\ &+\left(\frac{16a_2^3}{6075}-\frac{11a_2^2}{315}+\frac{a_3a_2}{162}+\frac{1543a_2}{18900}-\frac{10a_3}{189}+\frac{29}{252}\right) \\ &\times j_3\ln z+\left(-\frac{16a_2^2}{315}+\frac{a_2}{7}-\frac{5a_3}{42}-\frac{107}{70}\right)n_0\ln z \\ &+\left(\frac{32a_2^2}{945}-\frac{2a_2}{21}+\frac{5a_3}{63}+\frac{107}{105}\right)D_2^{nj}\ln z+\Bigg(-\frac{2381a_2^3}{66150} \\ &+\frac{8207a_2^2}{132300}-\frac{185a_3a_2}{2352}-\frac{83821a_2}{44100}+\frac{21}{100}-\frac{187a_3}{168}\Bigg)j_1 \\ &+\Bigg(\frac{97a_2^3}{36450}+\frac{5339a_2^2}{170100}+\frac{35a_3a_2}{3888}+\frac{9053a_2}{25200}+\frac{4609a_3}{18144} \\ &-\frac{457}{1050}\Bigg)j_3+\Bigg(-\frac{11a_2^3}{142884}+\frac{139a_2^2}{476280}-\frac{43a_3a_2}{95256} \\ &-\frac{277a_2}{105840}+\frac{1}{504}-\frac{109a_3}{45360}\Bigg)j_5+\Bigg(\frac{193a_2^3}{10206}-\frac{110a_2^2}{1701} \\ &+\frac{1033a_3a_2}{27216}+\frac{6911a_2}{5670}+\frac{48353a_3}{90720}-\frac{3(\ln z)^2}{2} \\ &-\frac{2539}{3780}\Bigg)n_0+\Bigg(-\frac{32a_2^3}{1215}+\frac{65a_2^2}{1134}-\frac{5a_3a_2}{81}-\frac{1574a_2}{945} \\ &+\frac{197}{126}-\frac{2455a_3}{3024}\Bigg)n_2+\Bigg(\frac{32a_2^3}{6075}-\frac{58a_2^2}{2835}+\frac{a_3a_2}{81} \\ &+\frac{824a_2}{4725}-\frac{5a_3}{378}-\frac{107}{630}\Bigg)D_{-4}^{nj}+\Bigg(-\frac{32a_2^3}{14175}-\frac{2a_2^2}{45} \\ &-\frac{a_3a_2}{189}+\frac{7468a_2}{1575}-\frac{5a_3}{42}-\frac{457}{70}\Bigg)D_{-2}^{nj} \\ &+\left(-\frac{19a_2^2}{567}+\frac{59a_2}{21}-\frac{2629}{630}-\frac{103a_3}{1512}\right)D_0^{nj} \\ &+\left(\frac{1402a_2^2}{19845}-\frac{925a_2}{441}+\frac{1165a_3}{5292}+\frac{16949}{4410}\right)D_2^{nj} \\ &+\left(\frac{17a_2^2}{6615}+\frac{20a_2}{441}+\frac{59a_3}{1176}-\frac{2}{49}\right)D_4^{nj} \\ &+\left(-\frac{64a_2^2}{945}+\frac{4a_2}{21}-\frac{10a_3}{63}-\frac{214}{105}\right)D_2^{njj} \\ &+\left(-\frac{64a_2^2}{945}+\frac{4a_2}{21}-\frac{10a_3}{63}-\frac{214}{105}\right)D_{-3}^{nnj} \\ &-12D_{-1}^{nnj}+\left(\frac{4a_2}{15}-18\right)D_1^{nnj}-8D_2^{nnnj}. \end{aligned} \tag{44}$$

The corresponding imaginary parts are expressed as follows:

$$g_\ell^{(1)} = j_\ell \ln z, \tag{45}$$

$$g_\ell^{(2)} = f_\ell^{(1)} \ln z - \frac{\mathbb{T}_1}{z} j_\ell + \varsigma_\ell^{(2)}, \tag{46}$$

$$g_\ell^{(3)} = f_\ell^{(2)} \ln z - \frac{\mathbb{T}_1 f_\ell^{(1)}}{z} - \frac{\mathbb{T}_2 j_\ell}{2z^2} + \frac{1}{3} j_\ell (\ln z)^3 + \varsigma_\ell^{(3)}, \tag{47}$$

with

$$\begin{aligned}\mathbb{T}_1 &= 1 + c_1^2 - (c_1 + c_2)(1 - c_2),\\ \mathbb{T}_2 &= c_1 c_2(-c_1 - c_2 + 3) + 2(c_1 - 1)c_1 + 2(c_2 - 1)c_2 + 1,\end{aligned} \tag{48}$$

where $\varsigma_\ell^{(n)}$ for $\ell = 2$ can be expressed as follows:

$$\varsigma_2^{(2)} = \left(\frac{2a_2^2}{81} + \frac{5a_3}{108} + \frac{a_2}{180}\right) j_3 + \frac{a_2}{30} j_1, \tag{49}$$

$$\begin{aligned}\varsigma_2^{(3)} =& \left(-\frac{4a_2^2}{81} - \frac{a_2}{90} - \frac{5a_3}{54}\right) D_3^{nj}\\ &+ \left(\frac{176a_2^3}{2835z} - \frac{1469a_2^2}{3780z} + \frac{40a_2a_3}{189z} - \frac{5a_2}{24z} - \frac{181a_3}{252z}\right) j_1 - \frac{a_2}{15} D_1^{nj}\\ &+ \left(\frac{5a_2^3}{1701z} + \frac{83a_2^2}{3780z} + \frac{20a_2a_3}{567z} + \frac{a_2}{144z} + \frac{13a_3}{252z}\right) j_3\\ &+ \left(\frac{22a_2^3}{2835} + \frac{a_2^2}{105} + \frac{5a_2a_3}{189} + \frac{a_3}{56}\right) j_2 \ln z\\ &+ \left(-\frac{44a_2^3}{945z} + \frac{296a_2^2}{945z} - \frac{10a_2a_3}{63z} + \frac{a_2}{12z} + \frac{37a_3}{63z}\right) n_0\\ &+ \left(-\frac{44a_2^3}{2835} - \frac{2a_2^2}{105} - \frac{10a_2a_3}{189} - \frac{a_3}{28}\right) D_{-3}^{nj}.\end{aligned} \tag{50}$$

Inserting these expressions into eq. (27) and expanding the result with respect to η, we find that

$$X_{\ell m\omega}^{\rm in} = X_\ell^{(0)} + \eta X_\ell^{(1)} + \eta^2 X_\ell^{(2)} + \eta^3 X_\ell^{(3)}, \tag{51}$$

where

$$\begin{aligned}X_\ell^{(0)} =& z j_\ell,\\ X_\ell^{(1)} =& z f_\ell^{(1)},\\ X_\ell^{(2)} =& z\left[f_\ell^{(2)} + \frac{1}{2} j_\ell (\ln z)^2 + {\rm i}\varsigma_\ell^{(2)}\right],\\ X_\ell^{(3)} =& z\left[f_\ell^{(3)} + \frac{1}{2} f_\ell^{(1)} (\ln z)^2 - \frac{\mathbb{T}_1}{z} j_\ell \ln z + \varsigma_\ell^{(2)} \ln z + {\rm i}\varsigma_\ell^{(3)}\right].\end{aligned} \tag{52}$$

3.2.3 Coefficient of amplitude $A_\ell^{\rm in}$

Noting ${\rm e}^{-{\rm i}\eta(-b_1 \ln(z-c_1\eta)+b_2\ln(z-c_2\eta)+b_h\ln(z-c_h\eta))} = {\rm e}^{-{\rm i}z^*}{\rm e}^{{\rm i}z} \xrightarrow{z\to\infty} 1$, taking the expressions of the spherical Hankel functions of the first and second kinds $h_\ell^{(1)}$ and $h_\ell^{(2)}$ as:

$$h_\ell^{(1)} = j_\ell + {\rm i}n_\ell \xrightarrow{z\to\infty} (-{\rm i})^{\ell+1}\frac{{\rm e}^{{\rm i}z}}{z}, \tag{53}$$

$$h_\ell^{(2)} = j_\ell - {\rm i}n_\ell \xrightarrow{z\to\infty} {\rm i}^{\ell+1}\frac{{\rm e}^{-{\rm i}z}}{z}, \tag{54}$$

and using the asymptotic behavior of B_J and D_ℓ^J in ref. [59], we obtain the following expression:

$$\begin{aligned}A_\ell^{\rm in} =& \frac{1}{2}{\rm i}^{\ell+1}{\rm e}^{-{\rm i}\eta(\ln 2\eta+\mathbf{elg})}{\rm e}^{{\rm i}\left[\eta p_\ell^{(0)} - \pi\eta^2 p_\ell^{(1)} + \eta^3\left(p_\ell^{(2)} - \pi^2 p_\ell^{(3)} + p_\ell^{(4)}\mathbf{RiZ}(3)\right)\right]}\\ &\times\left\{1 - \frac{\pi}{2}\eta + \eta^2\left[2(\mathbf{elg} + \ln 2)p_\ell^{(1)} + q_\ell^{(1)} + \frac{5\pi^2}{24}\right]\right.\\ &\left. + \eta^3\left[\pi q_\ell^{(2)} + \pi^3 q_\ell^{(4)} + \pi(\mathbf{elg} + \ln 2)q_\ell^{(3)}\right]\right\},\end{aligned} \tag{55}$$

where **elg** is the EulerGamma constant ($\mathbf{elg} = 0.57721\cdots$), $\mathbf{RiZ}(n)$ is the Riemann zeta function ($\mathbf{RiZ}(3) = 1.202\cdots$), and the coefficients of $A_2^{\rm in}$ are

$$\begin{aligned}p_2^{(0)} &= \frac{15 - 2a_2}{9}, \quad p_2^{(1)} = \frac{32a_2^2 - 90a_2 + 75a_3 + 963}{3780},\\ p_2^{(2)} &= \frac{-292a_2^3 + 1962a_2^2 - 765a_2a_3 - 1782a_2 + 3564a_3 + 1566}{34992},\\ p_2^{(3)} &= \frac{32a_2^2 - 90a_2 + 75a_3 + 963}{11340}, \quad p_2^{(4)} = \frac{1}{3},\\ q_2^{(1)} &= \frac{-37a_2 - 5a_3 + 150}{108}, \quad q_2^{(4)} = -\frac{1}{16},\\ q_2^{(2)} &= \frac{-176a_2^3 - 216a_2^2 - 600a_2a_3 + 7770a_2 + 645a_3 - 31500}{45360},\end{aligned}$$

and

$$\begin{aligned}q_2^{(3)} =& \frac{-32a_2^2 + 90a_2 - 75a_3 - 963}{3780}\\ &- {\rm i}\frac{176a_2^3 + 216a_2^2 + 600a_2a_3 + 405a_3}{22680\pi}.\end{aligned} \tag{56}$$

3.3 Quasi-circular orbit on the equatorial plane around an EOB

In this section, we consider a quasi-circular orbit. In this case, we assume that the orbit lies on the equatorial plane ($\theta = \pi/2$) without loss of generality. By setting $V_r(r_0) = \partial V_r/\partial r(r_0) = 0$, the effective energy E and effective angular momentum L are given by

$$E/\mu = \frac{\sqrt{2}[a_3(GM)^3 + r_0(a_2(GM)^2 + r_0(r_0 - 2GM))]}{r_0\sqrt{r_0}\sqrt{5a_3(GM)^3 + 2r_0(2a_2(GM)^2 + r_0(r_0 - 3GM))}}, \tag{57}$$

$$L/\mu = \frac{r_0\sqrt{-3a_3(GM)^3 + 2GMr_0(r_0 - a_2GM)}}{\sqrt{5a_3(GM)^3 + 2r_0(2a_2(GM)^2 + r_0(r_0 - 3GM))}}, \tag{58}$$

where r_0 is the orbital radius. By defining

$${}_0b_{\ell m} = \frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{2\sqrt{2\pi}\Delta_r}$$

$$\times\frac{15a_3(GM)^2+8a_2GMr_0-6r_0^2}{5a_3(GM)^2+4a_2GMr_0-6r_0^2}{}_0Y_{\ell m}\left(\frac{\pi}{2},0\right)\times r_0^2E,$$

$$_{-1}b_{\ell m}=\frac{2\sqrt{(\ell-1)(\ell+2)}}{\sqrt{2\pi}r_0}\times\frac{5a_3(GM)^2+3a_2GMr_0-3r_0^2}{\left(5a_3(GM)^2+4a_2GMr_0-6r_0^2\right)^3}{}_{-1}Y_{\ell m}\left(\frac{\pi}{2},0\right)\cdot\mathcal{P}_r\cdot L,$$

$$_{-2}b_{\ell m}=\frac{\Delta_r}{\sqrt{2\pi}r^4E}{}_{-1}Y_{\ell m}(\frac{\pi}{2},0)L^2,$$

$$\mathcal{P}_r=75a_3^2(GM)^4+10a_3(GM)^2r_0(11a_2GM-18r_0)+4r_0^2(8a_2^2(GM)^2-21a_2GMr_0+9r_0^2),\tag{59}$$

$$\mathcal{B}_r=r_0\Delta_r\left(8a_2r_0+30a_3GM\right)+6a_3r_0^2(GM)^2-8a_2a_3r_0(GM)^3-15a_3^2(GM)^4,$$

we obtain

$$\mathbf{A}_0=\frac{1}{2r_0^2}\cdot\Bigg\{2{}_0b_{\ell m}+4\mathrm{i}{}_{-1}b_{\ell m}\Bigg[1+\frac{\mathrm{i}}{2}\omega\frac{r_0^3}{\Delta_r}\Bigg(1+\frac{2GMr_0}{\mathcal{P}_r}\times(6a_2r_0^2+30a_3GMr_0-5a_2a_3(GM)^2)\Bigg)\Bigg]-2\mathrm{i}\frac{{}_{-2}b_{\ell m}\omega r_0^3}{\Delta_r^2}\Bigg[r_0^2-GMr_0+\frac{GM\times\mathcal{B}_r}{15a_3(GM)^2+8a_2GMr_0-6r_0^2}+\frac{1}{2}\mathrm{i}\omega r_0^3+6\mathrm{i}\frac{GM\Delta_r^2\cdot\left(2a_2r_0^2+15a_3GMr_0-5a_2a_3(GM)^2\right)}{r_0^2\omega(15a_3(GM)^2+8a_2GMr_0-6r_0^2)^2}\Bigg]\Bigg\},\tag{60}$$

$$\mathbf{A}_1=\mathrm{i}\frac{{}_{-1}b_{\ell m}}{r_0}\left(1+\frac{2GMr_0\left(6a_2r_0^2+30a_3GMr_0-5a_2a_3(GM)^2\right)}{\mathcal{P}_r}\right)-\frac{{}_{-2}b_{\ell m}}{r_0}\left(1+\frac{GM\cdot(15a_3GM+4a_2r_0)}{15a_3(GM)^2+8a_2GMr_0-6r_0^2}+\mathrm{i}\frac{r_0^3\omega}{\Delta_r}\right),\tag{61}$$

$$\mathbf{A}_2=-\frac{{}_{-2}b_{\ell m}}{2}.\tag{62}$$

4 Energy radiation rate and gravitational waveforms

Inserting the aforementioned result of $\mathbf{A}_i(i=1,2,3)$, $B_\ell^{\rm in}$, and $R_\ell^{\rm in}$ into eq. (17), we can obtain the following expression:

$$Z_{\ell m\omega_0}=Z^{(N,\epsilon)}_{\ell m\omega_0}\tilde{Z}_{\ell m\omega_0},\tag{63}$$

$$Z^{(N,\epsilon)}_{\ell m\omega_0}=(-1)^{\ell+\epsilon+1}m^2\frac{\nu}{2M}n^{(\epsilon)}_{\ell m}\nu^{\ell+\epsilon+6}{}_0Y_{\ell-\epsilon,-m}(\pi/2,\varphi),\tag{64}$$

$$n^{(\epsilon)}_{\ell m}=\begin{cases}(\mathrm{i}m)^\ell\dfrac{8\pi}{(2\ell+1)!!}\sqrt{\dfrac{(\ell+1)(\ell+2)}{\ell(\ell-1)}}, & \epsilon=0,\\ -(\mathrm{i}m)^\ell\dfrac{16\pi}{(2\ell+1)!!}\sqrt{\dfrac{(2\ell+1)(\ell+2)(\ell^2-m^2)}{(2\ell-1)(\ell+1)\ell(\ell-1)}}, & \epsilon=1,\end{cases}\tag{65}$$

where we define $\nu=(GM\Omega)^{1/3}$, $\omega_0=m\Omega$, $\epsilon=1$ when $\ell+m=1$, and $\epsilon=0$ when $\ell+m=0$. We can divide the higher-order term $\tilde{Z}_{\ell m\omega_0}$ into two parts: the $\tilde{Z}^{(S)}_{\ell m\omega_0}$ is computed in the Schwarzschild case [62] and the $\tilde{Z}^{(R)}_{\ell m\omega_0}$ is the 2PM and 3PM perturbation terms:

$$\tilde{Z}_{\ell m\omega_0}=\tilde{Z}^{(S)}_{\ell m\omega_0}+\tilde{Z}^{(R)}_{\ell m\omega_0}.\tag{66}$$

The explicit expression of $\tilde{Z}^{(R)}_{\ell m\omega_0}$ is presented in Supplementary materials. In the test particle limit, i.e., $\nu\to0$, we note that $\tilde{Z}^{(R)}_{\ell m\omega_0}$ vanishes completely because a_2 and a_3 approach 0. That is, our results revert to the Schwarzschild case in the test particle limit.

In eq. (18), utilizing the symmetry of the spin-weighted spherical harmonics, ${}_sY_{\ell,-m}(\frac{\pi}{2},0)=(-1)^{s+\ell}{}_sY_{\ell m}(\frac{\pi}{2},0)$, we know that $Z_{\ell(-m)\omega}=(-1)^\ell Z^*_{\ell m\omega}$, where $Z^*_{\ell m\omega}$ is the complex conjugate of $Z_{\ell m\omega}$. In terms of the amplitude $Z_{\ell m\omega}$, we find from eq. (18) that the gravitational waveform [14, 27, 58] at infinity is given by

$$h_+-\mathrm{i}h_\times=\sum_{\ell m}h_{\ell m}\frac{{}_{-2}Y_{\ell m}}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\omega_0(r^*-t)+\mathrm{i}m\varphi},\tag{67}$$

with

$$h_{\ell m}=-\frac{2}{\mathcal{R}\omega_0^2}Z_{\ell m\omega_0}=h^{(S)}_{\ell m}+h^{(R)}_{\ell m},\tag{68}$$

where

$$h^{(S)}_{\ell m}=h^{(N,\epsilon)}_{\ell m}\tilde{Z}^{(S)}_{\ell m\omega_0},\quad h^{(R)}_{\ell m}=h^{(N,\epsilon)}_{\ell m}\tilde{Z}^{(R)}_{\ell m\omega_0},\tag{69}$$

$$h^{(N,\epsilon)}_{\ell m}=\frac{GM\nu}{\mathcal{R}}n^{(\epsilon)}_{\ell m}c_{\ell+\epsilon}(\nu)\nu^{\ell+\epsilon}{}_0Y_{\ell-\epsilon,-m}(\pi/2,\varphi).\tag{70}$$

The energy loss rate along any orbit, in polar coordinates, can be expressed as $\frac{\mathrm{d}E[g^{\rm eff}_{\mu\nu}]}{\mathrm{d}t}=\dot{R}\mathcal{F}_R[g^{\rm eff}_{\mu\nu}]+\dot{\varphi}\mathcal{F}_\varphi[g^{\rm eff}_{\mu\nu}]$. By simply replacing the radial component with zero, an excellent approximation of the radiation reaction forces can be obtained [26]. Thus, from eq. (67), we know that, for given energy ω_n, the energy loss rate [14, 27] for the "plus" and "cross" modes of the gravitational wave is described by the following expression:

$$\frac{\mathrm{d}E[g^{\rm eff}_{\mu\nu}]}{\mathrm{d}t}=\frac{1}{2}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_N\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\Pi_{\ell m},\tag{71}$$

with

$$\Pi_{\ell m}=\mathcal{E}^{(S)}_{\ell m}+\mathcal{E}^{(R)}_{\ell m},\tag{72}$$

$$\mathcal{E}^{(S)}_{\ell m}=\frac{|\,Z^{(N,\epsilon)}_{\ell m\omega_0}\tilde{Z}^{(S)}_{\ell m\omega_0}\,|^2}{2\pi G^2\omega_0^2\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_N},\tag{73}$$

$$\mathcal{E}_{\ell m}^{(R)} = \frac{|Z_{\ell m\omega_0}^{(N,\epsilon)}|^2}{2\pi G^2\omega_0^2\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_N} \times \left\{\tilde{Z}_{\ell m\omega_0}^{(S)}\tilde{Z}_{\ell m\omega_0}^{(R)*} + \tilde{Z}_{\ell m\omega_0}^{(R)}\tilde{Z}_{\ell m\omega_0}^{(S)*} + \tilde{Z}_{\ell m\omega_0}^{(R)}\tilde{Z}_{\ell m\omega_0}^{(R)*}\right\}, \tag{74}$$

where $(\mathrm{d}E/\mathrm{d}t)_N = 32\nu^2 v^{10}/5$ is the Newtonian quadrupole luminosity and the superscript $*$ denotes the complex conjugation of the corresponding expression. In Figure 1, we present the curves of Π_{22} and Π_{33} as the symmetric mass ratio ν takes different values.

Then, we determine the radiation reaction forces for the "plus" and "cross" modes of the gravitational wave as follows:

$$\mathcal{F}_{\varphi}^{\mathrm{circ}}[g_{\mu\nu}^{\mathrm{eff}}] \simeq \frac{1}{\dot{\varphi}}\frac{\mathrm{d}E[g_{\mu\nu}^{\mathrm{eff}}]}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_N \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\mathbf{F}_{\ell m}^{(S)} + \mathbf{F}_{\ell m}^{(R)}\right), \tag{75}$$

where $\mathbf{F}_{\ell m}^{(S)} = \mathcal{E}_{\ell m}^{(S)}/\frac{1}{\dot{\varphi}}$ and $\mathbf{F}_{\ell m}^{(R)} = \mathcal{E}_{\ell m}^{(R)}/\frac{1}{\dot{\varphi}}$.

Eqs. (67), (71), and (75) indicate that all of the gravitational waveforms, the energy radiation rate, and the radiation reaction forces are based on the effective spacetime.

5 Conclusions

In this study, we investigate the waveforms and energy radiation rate of gravitational waves generated by coalescing spinless binary systems up to 3PM approximation in the EOB theory. We focus on the radiation reaction forces in the Hamilton equation, which can be described by the energy radiation rate $\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{4\pi G\omega^2}\int|\Psi_4^B|^2 r^2\mathrm{d}\Omega$ and the "plus" and "cross" modes of gravitational waves, which are related to the null tetrad components of the gravitational perturbed Weyl tensor by $\Psi_4^B = \frac{1}{2}(\ddot{h}_+ - \mathrm{i}\ddot{h}_\times)$. Clearly, to find the energy radiation rate and construct gravitational waveforms, the key step is to seek the solution of Ψ_4^B. Therefore, the main task of this study is to solve the decoupled and separated equations of the null tetrad components of gravitational perturbed Weyl tensor Ψ_4^B in the effective spacetime by employing the Green's function method.

To achieve this goal, noting that the potential function in the radial Teukolsky-like equation is a long-range potential, we first transform it into an S-N-like equation, which has a short-range potential. Then, by expanding the homogeneous S-N-like equation with $\eta = 2GM\omega$, where ω denotes the angular frequency of the wave, we derive closed analytical expressions for the solutions of each order. These solutions are essential for constructing the Green function and asymptotic amplitude. The lowest-order solution is expressed as a linear combination of spherical Bessel functions, which allows us to perform iterative calculations to obtain higher-order solutions. This approach simplifies the problem and enables us to efficiently study the radial Teukolsky-like equation. In the calculation process, we use a low-frequency approximation and considered the conditions of quasi-circular orbits. These conditions are represented by the relationships $z \propto v$, $\eta \propto v^3$ and $\omega = m\Omega$. As a result, the obtained results are accurate to $O(v^{9-2(\ell-2)-\epsilon})$, which means that the accuracy of the results of this study reaches the 4.5PN order [62, 63].

This article also presents a more general integral formula than that given by Sasaki's group [59], which can be extended to higher orders or even arbitrary orders without additional treatments. In Supplementary materials, the general integral formulas, which can theoretically derive the series solution of the homogeneous S-N-like equation to any order, are presented. However, when constructing the general solution of the nonhomogeneous equation using Green's function method, it is necessary to specify the amplitude at infinity, which requires finding the asymptotic behavior of B_J as $z \to \infty$. Although we know what needs to be done at each step, we have not yet been able to implement our ideas

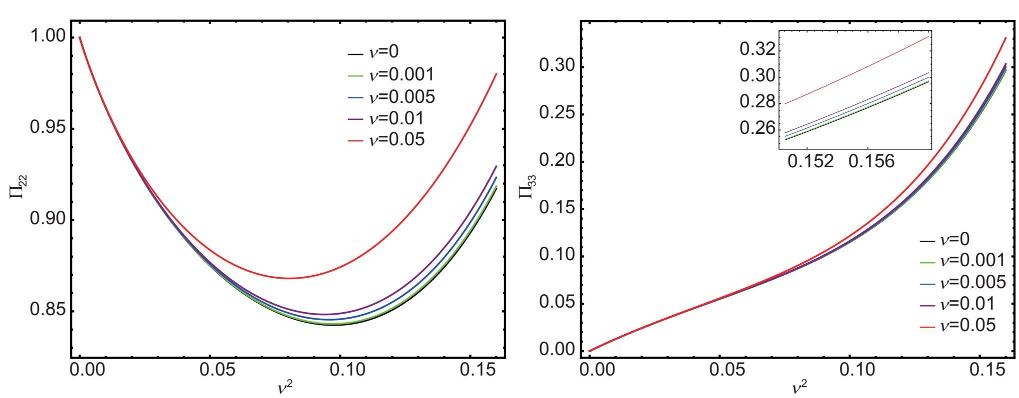

Figure 1 (Color online) $\Pi_{\ell m} = \Pi_{\ell m}(v^2, \nu)$ with respect to v^2 and ν, where the curve of $\nu = 0$ corresponds to the 4.5PN result obtained for the Schwarzschild case. The figure on the left is for $\Pi_{22} = \Pi_{22}(v^2, \nu)$, and the right one is for $\Pi_{33} = \Pi_{33}(v^2, \nu)$.

using a computer, but we can obtain specific results through complex calculations. Therefore, combining the outstanding works of Sasaki et al. with the useful formulas presented in Supplementary materials, we have confidence that this method will yield good results in the future.

From the analysis provided, it is evident that the effective metric degenerates into the Schwarzschild case in the test particle limit ($\nu \to 0$). This limit is characterized by vanishing of the coefficients a_2 and a_3. Therefore, the gravitational waveforms and energy radiation rate calculated in this study were divided into two parts: the Schwarzschild part and the correction part related to PM parameters a_2 and a_3. Handling spinning binary systems will involve additional complexities. However, the results of this paper and the listed mathematical techniques will be valuable for understanding the energy flux and waveforms in spinning binary systems.

This work was supported by the National Natural Science Foundation of China (Grant No. 12035005), and the National Key Research and Development Program of China (Grant No. 2020YFC2201400). We would like to thank professors S. Chen and Q. Pan for useful discussions on the manuscript.

Conflict of interest The authors declare that they have no conflict of interest.

Supporting Information

The supporting information is available online at http://phys.scichina.com and https://link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

1 B. P. Abbott, et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **116**, 241103 (2016), arXiv: 1606.04855.
2 B. P. Abbott, et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **118**, 221101 (2017), arXiv: 1706.01812.
3 B. P. Abbott, et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **119**, 141101 (2017), arXiv: 1709.09660.
4 B. P. Abbott, et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. Lett. **119**, 161101 (2017), arXiv: 1710.05832.
5 R. Abbott, et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. D **102**, 043015 (2020), arXiv: 2004.08342.
6 R. Abbott,et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. X **11**, 021053 (2021), arXiv: 2010.14527.
7 R. Abbott,et al. (LIGO Scientific and Virgo Collaboration), Phys. Rev. D **105**, 122001 (2022), arXiv: 2110.09834.
8 A. H. Nitz, C. D. Capano, S. Kumar, Y. F. Wang, S. Kastha, M. Schäfer, R. Dhurkunde, and M. Cabero, Astrophys. J. **922**, 76 (2021), arXiv: 2105.09151.
9 R. Abbott, et al. (LIGO Scientific, VIRGO, KAGRA Collaboration), arXiv: 2111.03606.
10 R. Abbott, et al. (LIGO Scientific, VIRGO Collaboration), arXiv: 2108.01045.
11 T. Damour, and G. Schafer, Nuov. Cim. B **101**, 127 (1988).
12 A. Buonanno, and T. Damour, Phys. Rev. D **59**, 084006 (1999), arXiv: gr-qc/9811091.
13 T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **62**, 084011 (2000), arXiv: gr-qc/0005034.
14 A. Buonanno, G. B. Cook, and F. Pretorius, Phys. Rev. D **75**, 124018 (2007), arXiv: gr-qc/0610122.
15 T. Damour, Phys. Rev. D **94**, 104015 (2016), arXiv: 1609.00354.
16 D. Bini, and T. Damour, Phys. Rev. D **98**, 044036 (2018), arXiv: 1805.10809.
17 T. Damour, and P. Rettegno, Phys. Rev. D **107**, 064051 (2023), arXiv: 2211.01399.
18 C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Phys. Rev. Lett. **128**, 161104 (2022), arXiv: 2112.11296.
19 C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, Phys. Rev. Lett. **130**, 101401 (2023), arXiv: 2210.05541.
20 G. Kälin, Z. Liu, and R. A. Porto, Phys. Rev. Lett. **125**, 261103 (2020), arXiv: 2007.04977.
21 T. Damour, Int. J. Mod. Phys. A **23**, 1130 (2008), arXiv: 0802.4047.
22 T. Damour, and A. Nagar, Phys. Rev. D **77**, 024043 (2008), arXiv: 0711.2628.
23 T. Damour, and A. Nagar, Fundam. Theor. Phys. **162**, 211 (2011).
24 T. Damour, and A. Nagar, Phys. Rev. D **76**, 064028 (2007), arXiv: 0705.2519.
25 T. Damour, Phys. Rev. D **102**, 024060 (2020), arXiv: 1912.02139.
26 A. Buonanno, and T. Damour, Phys. Rev. D **62**, 064015 (2000), arXiv: gr-qc/0001013.
27 E. Poisson, Phys. Rev. D **47**, 1497 (1993).
28 S. Long, Y. Zou, and J. Jing, Class. Quantum Grav. **40**, 225006 (2023).
29 J. Jing, S. Long, W. Deng, M. Wang, and J. Wang, Sci. China-Phys. Mech. Astron. **65**, 100411 (2022), arXiv: 2208.02420.
30 J. Jing, W. Deng, S. Long, and J. Wang, Sci. China-Phys. Mech. Astron. **66**, 270411 (2023), arXiv: 2305.03225.
31 X. He, M. Sun, J. Jing, and Z. Cao, Eur. Phys. J. C **81**, 97 (2021).
32 J. Jing, W. Deng, S. Long, and J. Wang, Eur. Phys. J. C **83**, 608 (2023), arXiv: 2307.05971.
33 S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
34 J. M. Bardeen, and W. H. Press, J. Math. Phys. **14**, 7 (1973).
35 S. Chandrasekhar, Proc. R. Soc. Lond. A **343**, 289 (1975).
36 S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Springer, Heidelberg, 1983).
37 S. Mano, H. Suzuki, and E. Takasugi, Prog. Theor. Phys. **95**, 1079 (1996), arXiv: gr-qc/9603020.
38 M. Sasaki, and H. Tagoshi, Living Rev. Relativ. **6**, 6 (2003), arXiv: gr-qc/0306120.
39 G. B. Cook, and M. Zalutskiy, Phys. Rev. D **90**, 124021 (2014), arXiv: 1410.7698.
40 P. Fiziev, and D. Staicova, Phys. Rev. D **84**, 127502 (2011), arXiv: 1109.1532.
41 E. W. Leaver, J. Math. Phys. **27**, 1238 (1986).
42 E. W. Leaver, Proc. R. Soc. Lond. A **402**, 285 (1985).
43 E. W. Leaver, Phys. Rev. D **34**, 384 (1986).
44 C. Chen, and J. Jing, J. Cosmol. Astropart. Phys. **11**, 70 (2023), arXiv: 2307.14616.
45 W. B. Han, and Z. Cao, Phys. Rev. D **84**, 044014 (2011), arXiv: 1108.0995.
46 A. Nagar, G. Riemenschneider, and G. Pratten, Phys. Rev. D **96**, 084045 (2017), arXiv: 1703.06814.
47 A. Nagar, J. Healy, C. O. Lousto, S. Bernuzzi, and A. Albertini, Phys. Rev. D **105**, 124061 (2022), arXiv: 2202.05643.
48 A. Bohé, L. Shao, A. Taracchini, A. Buonanno, S. Babak, I. W. Harry, I. Hinder, S. Ossokine, M. Pürrer, V. Raymond, T. Chu, H. Fong, P. Kumar, H. P. Pfeiffer, M. Boyle, D. A. Hemberger, L. E. Kidder, G. Lovelace, M. A. Scheel, and B. Szilágyi, Phys. Rev. D **95**, 044028 (2017), arXiv: 1611.03703.
49 S. Xin, B. Chen, R. K. L. Lo, L. Sun, W. B. Han, X. Zhong, M. Srivastava, S. Ma, Q. Wang, and Y. Chen, Phys. Rev. D **104**, 104005 (2021), arXiv: 2105.12313.
50 C. Zhang, W. B. Han, X. Y. Zhong, and G. Wang, Phys. Rev. D **104**, 024050 (2021), arXiv: 2102.05391.
51 C. Zhang, W. B. Han, and S. C. Yang, Commun. Theor. Phys. **73**, 085401 (2021), arXiv: 2001.06763.

52 R. Cheng, and W.-B. Han, arXiv: 1706.03884.

53 Z. Cao, and W. B. Han, Phys. Rev. D **96**, 044028 (2017), arXiv: 1708.00166.

54 R. Gamba, S. Akçay, S. Bernuzzi, and J. Williams, Phys. Rev. D **106**, 024020 (2022), arXiv: 2111.03675.

55 S. Ossokine, A. Buonanno, S. Marsat, R. Cotesta, S. Babak, T. Dietrich, R. Haas, I. Hinder, H. P. Pfeiffer, M. Pürrer, C. J. Woodford, M. Boyle, L. E. Kidder, M. A. Scheel, and B. Szilágyi, Phys. Rev. D **102**, 044055 (2020), arXiv: 2004.09442.

56 T. Damour, and A. Nagar, Phys. Rev. D **90**, 044018 (2014), arXiv: 1406.6913.

57 N. Afshordi, et al. (LISA Consortium Waveform Working Group), arXiv: 2311.01300.

58 M. Sasaki, Prog. Theor. Phys. **92**, 17 (1994), arXiv: gr-qc/9402042.

59 Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi, and T. Tanaka, Prog. Theor. Phys. Suppl. **128**, 1 (1997), arXiv: gr-qc/9712057.

60 T. Dray, J. Math. Phys. **26**, 1030 (1985).

61 E. Poisson, Phys. Rev. D **55**, 639 (1997), arXiv: gr-qc/9606078.

62 Y. Pan, A. Buonanno, R. Fujita, E. Racine, and H. Tagoshi, Phys. Rev. D **83**, 064003 (2011), arXiv: 1006.0431.

63 T. Damour, B. R. Iyer, and A. Nagar, Phys. Rev. D **79**, 064004 (2009), arXiv: 0811.2069.