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Kutta-Joukowski force expression for viscous flow†

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The Kutta Joukowski (KJ) theorem, relating the lift of an airfoil to circulation, was widely accepted for predicting the lift of viscous high Reynolds number flow without separation. However, this theorem was only proved for inviscid flow and it is thus of academic importance to see whether there is a viscous equivalent of this theorem. For lower Reynolds number flow around objects of small size, it is difficult to measure the lift force directly and it is thus convenient to measure the velocity flow field solely and then, if possible, relate the lift to the circulation in a similar way as for the inviscid KJ theorem. The purpose of this paper is to discuss the relevant conditions under which a viscous equivalent of the KJ theorem exists that reduces to the inviscid KJ theorem for high Reynolds number viscous flow and remains correct for low Reynolds number steady flow. It has been shown that if the lift is expressed as a linear function of the circulation as in the classical KJ theorem, then the freestream velocity must be corrected by a component called mean deficit velocity resulting from the wake. This correction is small only when the Reynolds number is relatively large. Moreover, the circulation, defined along a loop containing the boundary layer and a part of the wake, is generally smaller than that based on inviscid flow assumption. For unsteady viscous flow, there is an inevitable additional correction due to unsteadiness.

lift force, Kutta Joukowski expression, viscous flow

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1 Introduction

For a two dimensional incompressible flow around an airfoil with a sharp trailing edge at incidence, the Kutta-Joukowski (KJ) hypothesis holds good for at least steady unseparated flow. In this hypothesis, the viscosity is explicitly ignored but implicitly incorporated in the Kutta condition (see for instance ref. [1]). The Kutta condition imposes a circulation or a bound vortex attached to the airfoil and creates a starting vortex, of opposite sign, which moves in the downstream direction. Consider an incompressible two-dimensional flow around an airfoil with a velocity field $\boldsymbol{v} = (u, v)$ at constant density ρ, in an unbounded domain R_f. The freestream velocity V_∞ is assumed to be horizontal. The circulation of the bound vortex is defined as $\Gamma_b = \int_{\partial A} (u\mathrm{d}x + v\mathrm{d}y)$ for the closed curve ∂A along the airfoil, with a counter-clockwise path, so that a clockwise circulation has a negative sign. The classical (here called inviscid) KJ expression for inviscid flow expresses the lift (F) and drag (D) per unit span as:

$$F = -\rho V_\infty \Gamma_b, \quad D = 0 \ \text{(KJ theory)}. \tag{1}$$

Though obtained under the strict assumption of inviscid flow, the lift predicted by the Kutta Joukowski expression is accurate even for high Reynolds number viscous flow with small angle of attack, provided the flow is steady and unseparated ([2], p192). The reason being that for high Reynolds number flow at small angle of attack, the viscous effect is confined in a very narrow region near the airfoil such that the pressure field, responsible for the lift force, is essentially the same as for inviscid flow.

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For small Reynolds number problems, as appeared in micro air vehicle flow or insect flow, the viscous boundary layer may be thick enough to have important perturbation on the pressure field. It is thus imperative to investigate whether the inviscid KJ expression would accommodate such cases, or, whether there is an equivalent KJ expression for viscous flow.

Apart from academic importance, it is also significant for practical applications, since for small Reynolds number flow problems, it is difficult to measure the force directly using a force balance. It becomes rational to measure the velocity flow field experimentally and determine the lift force by using integral force formulas that relate the force to the velocity field [3]. Integral approaches used for such applications require the measurement of velocity field and the computation of vorticity in the whole space. It is desirable to have a lift force formula for viscous flow that is simple enough as the inviscid KJ expression, The reason being one just needs to measure the velocity field along a closed loop. Sharma and Deshapande [4] indeed used this idea to determine the lift force. In their work, the lift force generated by a two-dimensional thin flat plate at various angles of attack in low Reynolds number flow was experimentally determined with the application of the KJ expression, where the circulation around the flat plate was obtained from the line integral of velocity measured using LDV (a non-invasive optical technique) along a closed rectangular loop containing the boundary layer and a part of the wake.

Due to the above mentioned importance of applications in small Reynolds number flow, it is important to know whether there exists a viscous equivalent of the inviscid KJ theorem, meeting the following requirements:

(1) it should be in a form similar to the inviscid KJ theorem, for purpose of simplicity;

(2) it is practically applicable to lower Reynolds number flow.

The derivation of an equivalent KJ expression for viscous flow will be presented in sect. 2 and this expression is discussed in sect. 3.

2 Kutta Joukowski expression for viscous flow

The viscous KJ expression is here obtained by using the integral force theory of Wu [5]. In this theory, the lift and drag are expressed as the time variation of the integral of vorticity moment. In the case of two dimensional viscous flow around a fixed airfoil, the lift and drag forces are expressed as the time variation of the integral of vorticity moment

$$\begin{aligned} F &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\infty} x\omega \mathrm{d}x\mathrm{d}y + \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_b} u\mathrm{d}x\mathrm{d}y, \\ D &= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\infty} y\omega \mathrm{d}x\mathrm{d}y + \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_b} v\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2}$$

Here $\omega = \partial v/\partial x - \partial u/\partial y$ is the local vorticity, R_∞ is the entire region including both fluid and solid body, and R_b is the region occupied by the solid body. For a non-accelerating and non-rotating airfoil, with a body fixed frame, we have

$$F = \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{f}} x\omega \mathrm{d}x\mathrm{d}y, D = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{f}} y\omega \mathrm{d}x\mathrm{d}y, \tag{3}$$

where R_f is the region occupied by the fluid only. A useful force formula requires that only the near flow field velocity appears in the formula.

As displayed in Figure 1, let $x = x_\mathrm{d}$ be a cutline downstream of the airfoil. Assume that this cutline is far enough so that the pressure at x_d is the same as at infinity and the shear stress is negligibe. This cutline splits the domain R_f into R_a (containing the airfoil where the flow is steady) and R_d (containing the rest of the flow where the flow may be still unsteady). Due to conservation of total circulation [5]

$$\Gamma_o \triangleq -\int_{R_\mathrm{f}} \omega \mathrm{d}x\mathrm{d}y = 0 \text{ (viscous flow)}, \tag{4}$$

we may rewrite the lift viscous formula in eq. (3) as $F = F_{R_\mathrm{a}} + F_{R_\mathrm{d}}$, with

$$\begin{aligned} F_{R_\mathrm{a}} &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{a}} x\omega \mathrm{d}x\mathrm{d}y - \rho V_\infty \int_{R_\mathrm{a}} \omega \mathrm{d}x\mathrm{d}y, \\ F_{R_\mathrm{d}} &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{d}} x\omega \mathrm{d}x\mathrm{d}y - \rho V_\infty \int_{R_\mathrm{d}} \omega \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{5}$$

Let

$$\Gamma_{\mathrm{b,vis}} \triangleq \int_{R_\mathrm{a}} \omega \mathrm{d}x\mathrm{d}y. \tag{6}$$

Let C_∞ be the contour of the semi-space R_d and $\overline{C}_\infty$ be that part of C_∞ excluding the straight line $x = x_\mathrm{d}$. Due to the condition (4), the flow is uniform on $\overline{C}_\infty$, thus the use of Divergence Theorem yields

$$-\int_{x=x_\mathrm{d}} v\mathrm{d}y = \int_{R_\mathrm{d}} \omega \mathrm{d}x\mathrm{d}y = -\Gamma_{\mathrm{b,vis}}. \tag{7}$$

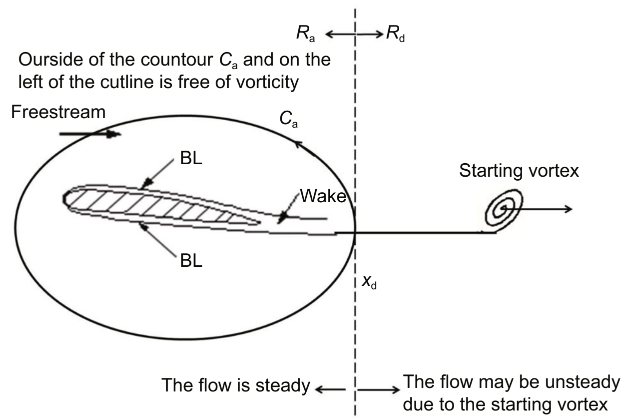

Figure 1 Viscous flow with boundary layer, with wake and starting vortex. There is a cutline $x = x_\mathrm{d}$, downstream of the airfoil, at which some parameters are defined. The countour C_a surrounding the airfoil and boundary layer and tangent to the cutline $x = x_\mathrm{d}$ is used to define the bound circulation $\Gamma_{\mathrm{b,vis}}$.

With $\Gamma_{\mathrm{b,vis}}$ defined by eq. (6) and with the use of eq. (7), we may rewrite F_{R_a} and F_{R_d} as:

$$\begin{aligned} F_{R_\mathrm{a}} &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{a}} x\omega \mathrm{d}x\mathrm{d}y - \rho V_\infty \Gamma_{\mathrm{b,vis}}, \\ F_{R_\mathrm{d}} &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{d}} x\omega \mathrm{d}x\mathrm{d}y + \rho\int_{x=x_\mathrm{d}} V_\infty v\mathrm{d}y. \end{aligned} \tag{8}$$

Eventually only the first term on the right hand side of the expression of F_{R_d} depends on the flow field downstream of $x = x_\mathrm{d}$. Now consider the component F_{R_d}. With the definition $\omega = \partial v/\partial x - \partial u/\partial y$, we have the identity

$$x\omega = \nabla\cdot(xv, -xu) - v$$

and thus

$$\begin{aligned} \int_{R_\mathrm{d}} x\omega\mathrm{d}x\mathrm{d}y &= \int_{R_\mathrm{d}} \left(\nabla\cdot(xv,-xu) - v\right)\mathrm{d}x\mathrm{d}y \\ &= \int_{R_\mathrm{d}} \nabla\cdot(xv,-xu)\,\mathrm{d}x\mathrm{d}y - \int_{R_\mathrm{d}} v\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Apply the Divergence Theorem to R_d we have

$$\int_{R_\mathrm{d}} \nabla\cdot(xv,-xu)\,\mathrm{d}x\mathrm{d}y = -\oint_{C_\infty} xu\mathrm{d}x + xv\mathrm{d}y.$$

Hence

$$\int_{R_\mathrm{d}} \nabla\cdot(xv,-xu)\,\mathrm{d}x\mathrm{d}y = -\int_{x=x_\mathrm{d}} xv\mathrm{d}y,$$

and thus

$$\int_{R_\mathrm{d}} \left(\nabla\cdot(xv,-xu) - v\right)\mathrm{d}x\mathrm{d}y = -\int_{x=x_\mathrm{d}} xv\mathrm{d}y - \int_{R_\mathrm{d}} v\mathrm{d}x\mathrm{d}y.$$

The integral $\int_{x=x_\mathrm{d}}\cdot\mathrm{d}y$ is performed for $-\infty < y < \infty$. Now use the integral form of the y momentum equation of the Navier-Stokes equations to relate the momentum change downstream of $x = x_\mathrm{d}$ to the momentum flux ρuv across the downstream boundary $x = x_\mathrm{d}$

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{d}} v\mathrm{d}x\mathrm{d}y = \int_{x=x_\mathrm{d}} \rho uv\mathrm{d}y.$$

Here, due to the assumption that x_d is relatively far from the airfoil, the pressure and shear stress contribution on $x = x_\mathrm{d}$ is neglected. Hence,

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{d}} x\omega\mathrm{d}x\mathrm{d}y = -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{x=x_\mathrm{d}} xv\mathrm{d}y - \int_{x=x_\mathrm{d}} \rho uv\mathrm{d}y. \tag{9}$$

Inserting eq. (9) into the second formula in eq. (8), we obtain

$$\begin{aligned} F_{R_\mathrm{d}} &= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{x=x_\mathrm{d}} xv\mathrm{d}y - \rho\int_{x=x_\mathrm{d}} (u - V_\infty)\,v\mathrm{d}y \\ &= -\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{x=x_\mathrm{d}} xv\mathrm{d}y - \rho u_p\int_{x=x_\mathrm{d}} v\mathrm{d}y, \end{aligned}$$

where

$$u_p = \int_{x=x_\mathrm{d}} (u - V_\infty)\,v\mathrm{d}y \Big/ \int_{x=x_\mathrm{d}} v\mathrm{d}y \tag{10}$$

is the *mean deficit velocity* u_p at $x = x_\mathrm{d}$, caused by wake arising from the viscous boundary layer.

Summation of this with F_{R_a} defined in eq. (8) and using eq. (7), we have

$$F = -\rho\left(V_\infty + u_p\right)\Gamma_{\mathrm{b,vis}} + F_u,$$

where

$$F_u = \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_\mathrm{a}} x\omega\mathrm{d}x\mathrm{d}y - \rho\frac{\mathrm{d}}{\mathrm{d}t}\int_{x=x_\mathrm{d}} xv\mathrm{d}y \tag{11}$$

is due to unsteadiness. Since this paper exclusively considers steady flow, $F_u = 0$ and thus

$$F = -\rho\left(V_\infty + u_p\right)\Gamma_{\mathrm{b,vis}}. \tag{12}$$

With the decomposition of $y\omega = \nabla\cdot(yv, -yu) + u$ and repetition of the same analysis for the second formula in eq. (15), we obtain the drag formula:

$$D = \rho V_\infty \int_{x=x_c} (V_\infty - u)\mathrm{d}y. \tag{13}$$

In summary, if, downstream of the body, we choose a cut-line $x = x_\mathrm{d}$ where the pressure is close to that at infinity and the shear stress is negligible, then we have a viscous equivalent of the Kutta Joukowski formula:

$$\begin{aligned} &F = -\rho\left(V_\infty + u_p\right)\Gamma_{\mathrm{b,vis}}, \\ &D = \rho V_\infty \int_{x=x_c} (V_\infty - u)\mathrm{d}y, \\ &\Gamma_{\mathrm{b,vis}} \triangleq \int_{R_\mathrm{a}} \omega\mathrm{d}x\mathrm{d}y, \qquad \text{(Viscous KJ expression)} \\ &u_p = \int_{x=x_\mathrm{d}} (u - V_\infty)\,v\mathrm{d}y \Big/ \int_{x=x_\mathrm{d}} v\mathrm{d}y. \end{aligned} \tag{14}$$

The viscous bound circulation $\Gamma_{\mathrm{b,vis}}$, that will be defined along a loop containing the boundary layer and a part of the wake, can not be replaced by the circulation along the surface of the airfoil, since for viscous flow the velocity vanishes on the body.

3 Discussions

Compared to the inviscid KJ expression (1), the freestream velocity has a correction due to the mean deficit velocity u_p in the viscous case. From the drag formula in eq. (2), we see that the physical existence of drag for viscous flow implies that $u_p < 0$, so that there is a lift decrease due to drag. Now we will consider three problems: (1) the connection of the force formula (2) to the inviscid KJ theorem; (2) whether the viscous expression (2) gives the same force for high Reynolds number unseparated flow; (3) restriction of eq. (2) in low Reynolds number flow.

3.1 Connection to the inviscid theorem

Several authors have discussed the connection between integral force approaches and the inviscid KJ expression [5–7], however, they are focused on the reduction of their formulas to the inviscid KJ expression in cases where the flow is steady and inviscid. Wu (1981) [5] attempted to obtain a viscous equivalent of the KJ expression. For this purpose, the vortical region is decomposed into four parts: (1) near vortical region of the airfoil (including majority of the boundary layer), (2) near wake region (part of the wake starting from the trailing edge and translating at a velocity different from the freestream velocity), (3) far wake region (the rest of the wake translating at a velocity at V_∞) and (4) starting vortex region. Let x_c be the cutline separating regions (2) and (3). Wu (1981) [5] showed that for steady airfoil viscous flow, the forces can be expressed as:

$$F = -\rho V_\infty (\Gamma_a + \Gamma_s)\,,\; D = \rho V_\infty \int_{-\infty}^{+\infty} (V_\infty - u(x_c, y))\mathrm{d}y. \quad (15)$$

Here Γ_a is the circulation for a contour enclosing region (3) (far wake region) and Γ_s is the circulation of the starting vortex. By definition, $\Gamma_b = \Gamma_a + \Gamma_s$ where Γ_b may be regarded as the circulation integrated over a contour enclosing regions (1) and (2); the first formula in eq. (15) has exactly the same form as the inviscid KJ expression. However, there is an inconsistency for eq. (15). Wu's formula (15) requires a cutline $x = x_c$ at which $u(x_c, y) = V_\infty$. This means, by the drag formula, that $D = 0$. Hence, physically, due to the fact we necessarily have a drag for viscous flow, we can never have a cutline with $u(x_c, y) = V_\infty$. Compared to Wu's formula (15), the present formula (2) does not require the constraint $u = V_\infty$ at the cutline, thus eliminates the inconsistency (of whether $D = 0$ or there is no cutline at which $u = V_\infty$).

Let C_a be a contour surrounding the near vortical region of the airfoil and lying inside R_a in such a way that in that part of R_a outside of C_a, the flow is vortex free, then we have

$$\int_{R_a} \omega \mathrm{d}x\mathrm{d}y = \int_{\text{inside } C_a} \omega \mathrm{d}x\mathrm{d}y.$$

By using the Divergence Theorem, we have

$$\int_{\text{inside } C_a} \omega \mathrm{d}x\mathrm{d}y = \oint_{C_a} (u\mathrm{d}x - v\mathrm{d}y) - \oint_{body} (u\mathrm{d}x - v\mathrm{d}y)\,.$$

On the body surface for viscous flow, the velocity vanishes. Thus

$$\Gamma_{\mathrm{b,vis}} = \oint_{C_a} u\mathrm{d}x - v\mathrm{d}y. \quad (16)$$

With this definition for $\Gamma_{\mathrm{b,vis}}$, the viscous KJ expression (2) is similar to the inviscid one, except that we have a correction u_p to the free-stream velocity and have $D > 0$.

3.2 High Reynolds number flow case

Now, consider the second question, for an airfoil of chord length c_A. The airfoil is assumed to be thin enough and at a Reynolds number Re large enough, for the wake to be narrow in the vertical direction. The center of the airfoil is at $x = 0$. Put $x_d = \frac{1}{2}c_A$ (just at the trailing edge), in this case, $u(x_d, y) \approx V_\infty$ except at the narrow region inside the wake. Inside the wake, v is approximately constant and $\tau_y \approx 0$, so that

$$u_p \approx \int_{x=x_d} (u - V_\infty)\mathrm{d}y \frac{v(x_d, 0)}{\int_{x=x_d} v\mathrm{d}y} = -\frac{v(x_d, 0)}{\rho V_\infty \int_{x=x_d} v\mathrm{d}y} D.$$

Since the flow is assumed steady near the airfoil, the starting vortex has no influence on $v(x_d, 0)$. The velocity $v(x_d, 0)$ is essentially induced by the bound vortex, regarded as a point vortex

$$v(x_d, 0) \approx \frac{\Gamma_{\mathrm{b,vis}}}{2\pi x_d}.$$

By eqs. (4), (6) and (7), we have $\int_{x=x_d} v\mathrm{d}y = \Gamma_{\mathrm{b,vis}}$, hence

$$u_p \approx -\frac{1}{2\pi x_d \rho V_\infty} D = -\frac{1}{\pi c_A \rho V_\infty} D = -\frac{c_d}{2\pi} V_\infty,$$

where $c_d = D/\frac{1}{2}c_A \rho V_\infty^2$ is the drag coefficient. Putting this into eq. (14) we obtain

$$F \approx -\left(1 - \frac{c_d}{2\pi}\right)\rho V_\infty \Gamma_{\mathrm{b,vis}}.$$

For thin airfoil with angle of attack small enough, we may estimate c_d by a flat plate, as:

$$c_d \approx \begin{cases} \dfrac{1.328}{Re^{\frac{1}{2}}} \text{ (laminar)}, \\ \dfrac{0.148}{Re^{\frac{1}{5}}} \text{ (turbulent)}. \end{cases}$$

For typical aeronautical applications, Re is of the order of 10^6 and higher, such that the correction factor $\frac{c_d}{2\pi}$ is of the order of 0.1% or less. For insects and micro air vehicles with Re is of the order of 10^3, such that the correction is of the order of 1%. For these types of flow, the viscous correction to the inviscid KJ expression is small or rather negligible.

3.3 Lower Reynolds number flow case

The viscous KJ expression (2) has been obtained for steady flow. For unsteady flow, the unsteady correction (11) may be important. Lower Reynolds number flow is generally unsteady and involves free vortices and vortex shedding. Even for unsteady inviscid flow with free vortices, the KJ expression involves contributions by free vortex induction, vortex motion and vortex producton [8,9]. For general viscous flow, if unsteadiness is important, one should use the more complex force formulas as reviewed by Noca et al. [3] and Wu et al. [10] to determine the forces.

For the particular case of lower Reynolds number steady flow, the viscous KJ expression (2) obtained here not only has academic value but may also be used to pick up forces from the data of the velocity field (for applications where it is difficult to measure the force directly), as done experimentally by Sharma and Deshapande [4]. They directly used the original inviscid KJ theorem for lower Reynolds number flow with wake, probably with negligible unsteady effect. The method to apply (2) is simple. It can be rewritten as:

$$\begin{aligned}
&F = -\rho\left(V_\infty + u_p\right)\Gamma_{\mathrm{b,vis}},\\
&D = \rho V_\infty \int_{-Y}^{Y} (V_\infty - u)|_{x=x_\mathrm{d}}\,\mathrm{d}y,\\
&\Gamma_{\mathrm{b,vis}} = \oint_{C_\mathrm{a}} u\mathrm{d}x - v\mathrm{d}y,\\
&u_p = \Gamma_{\mathrm{b,vis}}^{-1} \int_{-Y}^{Y} \left((u - V_\infty)v + \tau_y\right)\Big|_{x=x_\mathrm{d}}\mathrm{d}y,
\end{aligned} \tag{17}$$

where $Y \to \infty$ but one may set Y to be several times of c_A as $V_\infty - u$ quickly drops to zero away from the wake. Choose a cutline $x = x_\mathrm{d}$ downstream of the airfoil and a loop C_a around the airfoil containing the boundary layers, such that the right boundary of C_a coincides with $x = x_\mathrm{d}$, so that the velocity can be conveniently obtained along the loop C_a and along the line $x = x_\mathrm{d}$, $-Y < y < Y$ from experimental data. The only constraint is that flow should be steady at and upstream of $x = x_\mathrm{d}$. Since u_p is small for moderate to large Reynolds number flow, we may set $u_p = 0$ for such cases.

In summary, for steady viscous flow, the viscous equivalent of the Kutta Joukowski expression is given by eq. (2), where the cutline $x = x_\mathrm{d}$ is sufficiently far away from the airfoil so that the pressure difference $p - p_\infty$ and the shear stress at $x = x_\mathrm{d}$ are negligible. The correction by the mean deficit velocity u_p to the inviscid KJ expression is due to the wake, and is negligible when the Reynolds number is high enough. For unsteady flow, with vortex shedding, the unsteady force component as defined by eq. (11) which may be important and in this case there is no force formula as simple as for the inviscid KJ theorem.

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1 Batchelor F R S. An Introduction to Fluid Dynamics. Cambridge: Cambridge University Press, 1967
2 Anderson J. Fundamentals of Aerodynamics, Mcgraw-Hill Series in Aeronautical and Aerospace Engineering. New York: McGraw-Hill Education, 1984
3 Noca F, Shiels D, Jeon D. A comparison of methods for evaluating time-dependent fluid dynamic forces on bodies, using only velocity fields and their derivatives. J Fluids Struct, 1999, 13: 551–578
4 Sharma S D, Deshpande P J. Kutta-Joukowsky theorem in viscous and unsteady flow. Exp Fluids, 2012, 52: 1581–1591
5 Wu J C. Theory for aerodynamic force and moment in viscous flows. AIAA J, 1981, 19: 432–441
6 Saffman P G. Vortex Dynamics. New York: Cambridge University Press, 1992
7 Howe M S. On the force and moment on a body in an incompressible fluid, with application to rigid bodies and bubbles at high Reynolds numbers. Q J Mech Appl Math, 1995, 48: 401–425
8 Bai C Y, Wu Z N. Generalized Kutta-Joukowski theorem for multi-vortex and multi-airfoil flow (a lumped vortex model). Chin J Aeronaut, 2014, 27: 34–39
9 Bai C Y, Li J, Wu Z N. Generalized Kutta-Joukowski theorem for multi-vortex and multi-airfoil flow with vortex production (general model). Chin J Aeronaut, 2014, in press
10 Wu J C, Lu X Y, Zhuang L X. Integral force acting on a body due to local flow structures. J Fluid Mech, 2007, 576: 265–286