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Interval design point method for calculating the reliability of structural systems

NI Zao[1*] & QIU ZhiPing[2]

[1] Shanghai Aircraft Design and Research institute, COMAC, Shanghai 201210, China;
[2] Institute of Solid Mechanics, Beijing University of Aeronautics and Astronautics, Beijing 100191, China

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The evaluation of reliability for structural system is important in engineering practices. In this paper, by combining the design point method, JC method, interval analysis theory, and increment load method, we propose a new interval design point method for the reliability of structural systems in which the distribution parameters of random variables are described as interval variables. The proposed method may provide exact probabilistic interval reliability of structures whose random variables can have either a normal or abnormal distribution form. At last, we show the feasibility of the proposed approach through a typical example.

probability, design point method, interval analysis, probabilistic interval reliability, parameters, uncertainty

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1 Introduction

With the development of technology, the structural systems in engineering practices are increasingly complex, and the influence of uncertainties can be more and stronger. Therefore, the analyzing and decision-making must consider the existing uncertainties. Reliability is highly interrelated with uncertainties. Over the past decades, probability theory has enabled reliability estimate of all kinds of industry systems. Many practical numerical techniques, such as the Monte Carlo simulation [1], response surface approximation [2], central point method [3] design point method [4], and JC method [5] have been developed to achieve this goal. Among these methods, the design point method is one of the most efficient and exact computational methods and is widely accepted by engineers.

However, recent research indicates that the probabilistic reliability is very sensitive to small variations of the distribution parameters in probabilistic models. It means that even small perturbations of distribution parameters will lead to the imprecise results of probabilistic predictions. Therefore, the uncertainties in the distribution parameters of probabilistic models have been already considered in a general manner and it is rational to describe the uncertainties by interval variables [6].

Gurov and Utkin [7] pointed out that some structural information can be described in probabilistic theory and the others can be represented by interval estimation, based on the variety of information sources. Consequently, it is necessary to develop rigorous mathematical methods of combining all the information from different sources for obtaining exact estimates of the structural reliabilities [8]. Elishakoff was the pioneer in researching hybrid reliability theories, who combined probabilistic and non-probabilistic information to calculate the structural interval reliability for

*Corresponding author (email: nizao@comac.cc)

the first time [9]. Then, he utilized the combined method to solve many practical issues such as the reliability of space shuttle systems [10–12]. Wu et al. [13] and Elishakoff et al. [14,15] compared probabilistic and non-probabilistic approaches and discussed the compatibility of the two methods.

Recently, Qiu et al. [16,17] combine non-probabilistic interval analysis and probabilistic methodology in an effort to determine the bounds of both the component's and the system's structural reliability. However, the probabilistic model utilized by them is the central point method. The main drawbacks of the central point method can be generalized as follows. For one thing, it is irrational to expand the structural state function at mean value. Also, with the central point method, one may obtain different results from the structural state functions which have the same mechanical meaning and different mathematical expressions. Moreover, the probability distribution forms of random variables are ignored. These inherent deficiencies can lead to the inaccuracy and unreasonable interval evaluation of structural reliability.

In this paper, we first propose a new interval design point method for the reliability of structural components. Then, according to the increment load method [18], we present the enumerating method for the significant failure modes of structural systems when parameters of random variables are interval variables. The present criterion to enumerate the failure modes of structural systems may ensure the worst failure mode included in a few enumerated significant failure modes. Furthermore, the interval design point method for the reliability of structural systems is presented.

2 Conventional probabilistic reliability theory

Assuming that $\boldsymbol{X}=(X_1,X_2,\ldots,X_m)^{\mathrm{T}}$ is the m-dimensional random variable vector denoting the various factors that affect the structural functioning, then

$$Z=g_X(\boldsymbol{X})=g(X_1,X_2,\ldots,X_m) \tag{1}$$

is the state function of structures, and $Z=g_X(\boldsymbol{X})=0$ is the limited state equation of random variable space. The structural reliability should be expressed as:

$$\begin{aligned}P_s&=\int_{Z>0}f_X(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\&=\iint_{Z>0}\cdots\int f_X(x_1,x_2,\ldots,x_m)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n\end{aligned} \tag{2}$$

where $f_X(x_1,x_2,\ldots,x_m)$ is the joint distributional density function of the basic random variables $X_1,X_2,\ldots,X_m$.

It can be seen that calculating eq. (2) is a complex task for multiple integrals. In order to find an effective and accurate method of calculation, engineers introduce the structural reliability index β:

$$\beta=\frac{\mu_Z}{\sigma_Z}, \tag{3}$$

where $\mu_Z=E(Z)$ and $\sigma_Z=\sqrt{Var(Z)}$ are the mean value and standard variance of the state variable Z, respectively.

If $Z=g_X(\boldsymbol{X})$ is a linear function of the vector $\boldsymbol{X}$ which can be expressed as $Z=g_X(\boldsymbol{X})=\sum_{i=1}^{m}a_iX_i$, μ_Z and σ_Z can be written, respectively, as:

$$\mu_Z=\sum_{i=1}^{m}a_i\mu_{X_i} \tag{4}$$

and

$$\sigma_Z=\sqrt{\sum_{i=1}^{m}a_i^2\sigma_{X_i}^2}. \tag{5}$$

On the contrary, if $Z=g(\boldsymbol{X})$ is a non-linear function, μ_{Z_L} and σ_{Z_L} of the linearized Z can be written, respectively, as:

$$\mu_{Z_L}=g_X(\boldsymbol{x}^*)+\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}^*)}{\partial X_i}(\mu_{X_i}-x_i^*) \tag{6}$$

and

$$\sigma_{Z_L}=\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}^*)}{\partial X_i}\right]^2\sigma_{X_i}^2}, \tag{7}$$

where $a_i\ (i=1,2,\ldots,m)$ are the constant coefficients and $\boldsymbol{x}^*$ is the design point on the failure surface.

If each basic random variable $X_i\ (i=1,2,\ldots,m)$ submits normal distribution-that is to say, the state variable Z is a normal distribution variable-the expression for the structural reliability P_s can be rewritten as:

$$P_s=\Phi(\beta)=\Phi\left(\frac{\mu_Z}{\sigma_Z}\right)=\Phi\left(\frac{\mu_{Z_L}}{\sigma_{Z_L}}\right), \tag{8}$$

where $\Phi(\cdot)$ represents the standard normal distribution function and the condition $\beta>0$ which is equivalent to $\mu_Z>0$ should be satisfied, since $P_s\leqslant 0.5$ has no meaning when $\beta\leqslant 0$.

If several basic random variables are abnormal distribution variables, some approaches such as equivalent normalized approach (i.e. JC method) can be taken to transform the abnormal distribution variables into the normal distribution variables. Then, the state variable Z submitted to normal distribution can be obtained, and eq. (8) computes the structural reliability P_s.

3 Probabilistic interval reliability

3.1 Probabilistic interval reliability with normal distribution variables

Generally, the distributed parameters of random variables have some uncertainties and it is rational to describe them with bounded interval variables. For a normal distribution variable X, its uncertain but bounded distributed parameters μ_X and σ_X can be written, respectively, as:

$$\mu_X \in \mu_X^I = \left[\underline{\mu}_X, \overline{\mu}_X\right] \tag{9}$$

and

$$\sigma_X \in \sigma_X^I = \left[\underline{\sigma}_X, \overline{\sigma}_X\right]. \tag{10}$$

Thus, if distributed parameters of each basic random variable $X_i\left(i=1,2,\ldots,m\right)$ are bounded interval variables, the uncertain but bounded distributed parameters of the state variable Z can be rewritten as follows:

corresponding to eqs. (4) and (5)

$$\begin{aligned}\mu_Z^I &= \sum_{i=1}^{m} a_i \mu_{X_i}^I \\ &= \left[\sum_{i=1}^{m}\min\left\{a_i\underline{\mu}_{X_i}, a_i\overline{\mu}_{X_i}\right\}, \sum_{i=1}^{m}\max\left\{a_i\underline{\mu}_{X_i}, a_i\overline{\mu}_{X_i}\right\}\right] \\ &= \left[\underline{\mu}_Z, \overline{\mu}_Z\right]\end{aligned} \tag{11}$$

and

$$\begin{aligned}\sigma_Z^I &= \sqrt{\sum_{i=1}^{m} a_i^2\left(\sigma_{X_i}^I\right)^2} \\ &= \left[\sqrt{\sum_{i=1}^{m} a_i^2\left(\underline{\sigma}_{X_i}\right)^2}, \sqrt{\sum_{i=1}^{m} a_i^2\left(\overline{\sigma}_{X_i}\right)^2}\right] \\ &= \left[\underline{\sigma}_Z, \overline{\sigma}_Z\right]\end{aligned} \tag{12}$$

and corresponding to eqs. (6) and (7):

$$\begin{aligned}\mu_{Z_L}^I &= g_X\left(\boldsymbol{x}^*\right) + \sum_{i=1}^{m}\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(\mu_{X_i}^I - x_i^*\right) \\ &= g_X\left(\boldsymbol{x}^*\right) + \left[\sum_{i=1}^{m}\min\left\{\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(\underline{\mu}_{X_i} - x_i^*\right), \right.\right. \\ &\qquad \left.\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(\overline{\mu}_{X_i} - x_i^*\right)\right\}, \\ &\qquad \sum_{i=1}^{m}\max\left\{\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(\underline{\mu}_{X_i} - x_i^*\right), \right. \\ &\qquad \left.\left.\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(\overline{\mu}_{X_i} - x_i^*\right)\right\}\right] \\ &= \left[\underline{\mu}_{Z_L}, \overline{\mu}_{Z_L}\right].\end{aligned} \tag{13}$$

and

$$\begin{aligned}\sigma_{Z_L}^I &= \sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\right]^2\left(\sigma_{X_i}^I\right)^2} \\ &= \left[\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\right]^2\left(\underline{\sigma}_{X_i}\right)^2}, \sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\right]^2\left(\overline{\sigma}_{X_i}\right)^2}\right] \\ &= \left[\underline{\sigma}_{Z_L}, \overline{\sigma}_{Z_L}\right],\end{aligned} \tag{14}$$

where the definition of equal intervals in interval analysis [19,20] is utilized.

In eqs. (11) and (12), $\underline{\mu}_Z$, $\overline{\mu}_Z$ and $\underline{\sigma}_Z$, $\overline{\sigma}_Z$ are, respectively, the lower and upper bounds of the mean values and standard variances of the state variable Z. In eqs. (13) and (14), $\underline{\mu}_{Z_L}$, $\overline{\mu}_{Z_L}$ and $\underline{\sigma}_{Z_L}$, $\overline{\sigma}_{Z_L}$ are, respectively, the lower and upper bounds of the mean values and standard variances of the linearized state variable Z_L. The structural reliability index of the structure with uncertain but bounded distributed parameters will become a set as follows:

$$\Gamma = \left\{\beta:\ \beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_{Z_L}}{\sigma_{Z_L}};\quad \begin{array}{c}\underline{\mu}_Z \leqslant \mu_Z \leqslant \overline{\mu}_Z, \underline{\sigma}_Z \leqslant \sigma_Z \leqslant \overline{\sigma}_Z, \\ \underline{\mu}_{Z_L} \leqslant \mu_{Z_L} \leqslant \overline{\mu}_{Z_L}, \underline{\sigma}_{Z_L} \leqslant \sigma_{Z_L} \leqslant \overline{\sigma}_{Z_L}\end{array}\right\}, \tag{15}$$

from eq. (15), we can get the reliability index interval:

$$\beta^I = [\underline{\beta}, \overline{\beta}]. \tag{16}$$

Under the condition that the mean value μ_Z, μ_{Z_L} and standard variances σ_Z, σ_{Z_L} of Z and Z_L are permanent greater than zero, clearly, the minimum value (or the lower bound) $\underline{\beta}$ and the maximum value (or the upper bound) $\overline{\beta}$ can be expressed, respectively, as:

$$\underline{\beta} = \frac{\underline{\mu}_Z}{\overline{\sigma}_Z} = \frac{\underline{\mu}_{Z_L}}{\overline{\sigma}_{Z_L}} \tag{17}$$

and

$$\overline{\beta} = \frac{\overline{\mu}_Z}{\underline{\sigma}_Z} = \frac{\overline{\mu}_{Z_L}}{\underline{\sigma}_{Z_L}}. \tag{18}$$

According to eq. (8) and the monotonicity of $\Phi(\cdot)$, the lower bound and the upper bound of the structural reliability can be calculated, respectively, as:

$$\underline{P}_s = \Phi\left(\underline{\beta}\right) \tag{19}$$

and

$$\overline{P}_s = \Phi\left(\overline{\beta}\right). \tag{20}$$

Thus, the probabilistic interval reliability with normal distribution variables is written as:

$$P_s^I = \left[\underline{P}_s, \overline{P}_s\right]. \tag{21}$$

3.2 Probabilistic interval reliability with abnormal distribution variables

Based on the equivalent normalized approach, we will present the interval equivalent normalized approach in this section.

Assume that the component X_i of the vector $\boldsymbol{X} = \left(X_1, X_2, \ldots, X_m\right)^{\mathrm{T}}$ is the abnormal distribution variable, and $b_1, b_2, \ldots, b_n$ are its uncertain but bounded distributed parameters, which can be written as:

$$b_i \in b_i^I = \left[\underline{b}_i, \overline{b}_i\right], \ (i = 1, 2, \ldots, n). \tag{22}$$

The distribution function $F(x_i)$ and distributed density function $f(x_i)$ of X_i can be expressed, respectively, as:

$$F\left(x_i, b_1, b_2, \ldots, b_n\right), \ (b_i \in b_i^I \ , \ i = 1, 2, \ldots, n) \tag{23}$$

and

$$f\left(x_i, b_1, b_2, \ldots, b_n\right), (b_i \in b_i^I). \tag{24}$$

From eqs. (23) and (24), the lower and upper bounds of $F(x_i)$ and $f(x_i)$ can be obtained as follows:

$$\underline{F}\left(x_i\right) = \min_{b_i \in b_i^I \ (i=1,2,\ldots,n)} F\left(x_i, b_1, b_2, \ldots, b_n\right), \tag{25}$$

$$\overline{F}\left(x_i\right) = \max_{b_i \in b_i^I \ (i=1,2,\ldots,n)} F\left(x_i, b_1, b_2, \ldots, b_n\right) \tag{26}$$

and

$$\underline{f}\left(x_i\right) = \min_{b_i \in b_i^I \ (i=1,2,\ldots,n)} f\left(x_i, b_1, b_2, \ldots, b_n\right), \tag{27}$$

$$\overline{f}\left(x_i\right) = \max_{b_i \in b_i^I \ (i=1,2,\ldots,n)} f\left(x_i, b_1, b_2, \ldots, b_n\right). \tag{28}$$

According to the JC method, the abnormal distribution variable X_i can be transformed into the equivalent normalized variable $\tilde{X}_i$. The upper and lower bounds of standard variance $\sigma_{\tilde{X}_i}$ and equivalent mean value $\mu_{\tilde{X}_i}$ of $\tilde{X}_i$ can respectively be obtained by

$$\underline{\sigma}_{\tilde{X}_i} = \frac{\varphi\left(\Phi^{-1}\left[\overline{F}_{X_i}\left(x_i^*\right)\right]\right)}{\overline{f}\left(x_i^*\right)}, \tag{29}$$

$$\overline{\sigma}_{\tilde{X}_i} = \frac{\varphi\left(\Phi^{-1}\left[\underline{F}_{X_i}\left(x_i^*\right)\right]\right)}{\underline{f}\left(x_i^*\right)} \tag{30}$$

and

$$\underline{\mu}_{\tilde{X}_i} = x_i^* - \Phi^{-1}\left[\overline{F}_{X_i}\left(x_i^*\right)\right]\overline{\sigma}_{\tilde{X}_i}, \tag{31}$$

$$\overline{\mu}_{\tilde{X}_i} = x_i^* - \Phi^{-1}\left[\underline{F}_{X_i}\left(x_i^*\right)\right]\underline{\sigma}_{\tilde{X}_i}, \tag{32}$$

where x_i^* is the design point of X_i, $\Phi^{-1}(\cdot)$ is the inverse function of $\Phi(\cdot)$, and $\varphi(\cdot)$ is the standard normal density function. The monotonicity of $\Phi^{-1}(\cdot)$ & $\varphi(\cdot)$ and the non-negativity of $F(\cdot)$, $f(\cdot)$ and $\varphi(\cdot)$ are utilized to obtain eqs. (29)–(32).

Thus, the interval mean value and standard variance of $\tilde{X}_i$, which is the equivalent normalized variable of the abnormal distribution variable X_i, are expressed as:

$$\mu_{\tilde{X}_i}^I = \left[\underline{\mu}_{\tilde{X}_i}, \overline{\mu}_{\tilde{X}_i}\right] \tag{33}$$

and

$$\sigma_{\tilde{X}_i}^I = \left[\underline{\sigma}_{\tilde{X}_i}, \overline{\sigma}_{\tilde{X}_i}\right]. \tag{34}$$

Replacing eqs. (9) and (10) by eqs. (33) and (34), respectively, and using eqs. (11)–(21), the probabilistic interval reliability with abnormal distribution variables can be obtained:

$$P_s'^I = \left[\underline{P}_s', \overline{P}_s'\right]. \tag{35}$$

4 Interval design point method

In sect. 3, the design point is used to calculate the probabilistic interval reliability of a structure, but determining how to obtain the design point under the condition that the distributed parameters are uncertain becomes the chief problem. In this section, we will present the interval design point method for both normal and abnormal variables needed to solve it.

4.1 Interval design point method for normal distribution variables

Utilizing eqs. (13), (14), (16) and (18), we can obtain

$$\underline{\beta} = \frac{\underline{\mu}_{Z_L}}{\overline{\sigma}_{Z_L}} = \frac{g_X\left(\boldsymbol{x}'\right) + \sum_{i=1}^{m} \min\left\{\frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\left(\underline{\mu}_{X_i} - x_i'\right), \frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\left(\overline{\mu}_{X_i} - x_i'\right)\right\}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\right]^2\left(\overline{\sigma}_{X_i}\right)^2}} \tag{36}$$

and

$$\overline{\beta}=\frac{\overline{\mu}_{Z_L}}{\underline{\sigma}_{Z_L}}=\frac{g_X\left(\boldsymbol{x}''\right)+\sum_{i=1}^{m}\max\left\{\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\left(\underline{\mu}_{X_i}-x_i''\right),\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\left(\overline{\mu}_{X_i}-x_i''\right)\right\}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\right]^2\left(\underline{\sigma}_{X_i}\right)^2}},\tag{37}$$

where $\boldsymbol{x}'=\left(x_1',x_2',\ldots,x_m'\right)^{\mathrm{T}}$ and $\boldsymbol{x}''=\left(x_1'',x_2'',\ldots,x_m''\right)^{\mathrm{T}}$ are two different design points.

From eq. (6), we know that the linearized limited state equation is expressed as:

$$Z_L=g_X\left(\boldsymbol{x}^*\right)+\sum_{i=1}^{m}\frac{\partial g_X\left(\boldsymbol{x}^*\right)}{\partial X_i}\left(X_i-x_i^*\right)=0\,.\tag{38}$$

The standardized forms of eq. (38) corresponding to eqs. (36) and (37) are, respectively,

$$-\frac{g_X\left(\boldsymbol{x}'\right)+\sum_{i=1}^{m}\min\left\{\frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\left(\underline{\mu}_{X_i}-x_i'\right),\frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\left(\overline{\mu}_{X_i}-x_i'\right)\right\}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}'\right)}{\partial X_i}\right]^2\left(\overline{\sigma}_{X_i}\right)^2}}-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\overline{\sigma}_{X_i}Y_i'}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\right]^2\overline{\sigma}_{X_i}^2}}=0\tag{39}$$

and

$$-\frac{g_X\left(\boldsymbol{x}''\right)+\sum_{i=1}^{m}\max\left\{\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\left(\underline{\mu}_{X_i}-x_i''\right),\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\left(\overline{\mu}_{X_i}-x_i''\right)\right\}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X\left(\boldsymbol{x}''\right)}{\partial X_i}\right]^2\left(\underline{\sigma}_{X_i}\right)^2}}-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\underline{\sigma}_{X_i}Y_i''}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\right]^2\underline{\sigma}_{X_i}^2}}=0\,,\tag{40}$$

where Y_i' and Y_i'' are standardized variables of X_i corresponding to eqs. (36) and (37), respectively.

Substituting eqs. (36) and (37) into eqs. (39) and (40), respectively, eqs. (39) and (40) can be rewritten as:

$$-\underline{\beta}-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\overline{\sigma}_{X_i}Y_i'}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\right]^2\overline{\sigma}_{X_i}^2}}=0\tag{41}$$

and

$$-\overline{\beta}-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\underline{\sigma}_{X_i}Y_i''}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\right]^2\underline{\sigma}_{X_i}^2}}=0\,.\tag{42}$$

The sensitivity factors of Y_i' and Y_i'' can respectively be defined as:

$$\alpha_{Y_i'}=\cos\theta_{Y_i'}=-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\overline{\sigma}_{X_i}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\right]^2\overline{\sigma}_{X_i}^2}}\tag{43}$$

and

$$\alpha_{Y_i''}=\cos\theta_{Y_i''}=-\frac{\sum_{i=1}^{m}\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\underline{\sigma}_{X_i}}{\sqrt{\sum_{i=1}^{m}\left[\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\right]^2\underline{\sigma}_{X_i}^2}}\,.\tag{44}$$

Thus, eqs. (41) and (42) can be transformed into

$$\sum_{i=1}^{m}\cos\theta_{Y_i'}Y_i'-\underline{\beta}=0\tag{45}$$

and

$$\sum_{i=1}^{m}\cos\theta_{Y_i''}Y_i''-\overline{\beta}=0\,.\tag{46}$$

From eqs. (45) and (46), we can get the relation between the reliability indexes $\underline{\beta}$ & $\overline{\beta}$ and the coordinates of the design points in $\boldsymbol{Y}'$ and $\boldsymbol{Y}''$ space, respectively. They are

$$y_i'=\underline{\beta}\cos\theta_{Y_i'},\ \ (i=1,2,\ldots,m)\tag{47}$$

and

$$y_i''=\overline{\beta}\cos\theta_{Y_i''},\ \ (i=1,2,\ldots,m)\,.\tag{48}$$

According to the upward equations, the corresponding coordinates of the design points in $\boldsymbol{X}'$ and $\boldsymbol{X}''$ space are

$$x_i' = \mu_{X_i}' + \underline{\beta}\overline{\sigma}_{X_i} \cos\theta_{Y_i'}, \ (i = 1,2,\ldots,m) \tag{49}$$

and

$$x_i'' = \mu_{X_i}'' + \overline{\beta}\underline{\sigma}_{X_i} \cos\theta_{Y_i''}, \ (i = 1,2,\ldots,m)\,, \tag{50}$$

where $\mu_{X_i}' \in S$ ($S = \{\mu_{X_i} \mid \mu_{X_i} = \underline{\mu}_{X_i}$ or $\mu_{X_i} = \overline{\mu}_{X_i}\}$) and

$$\frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\left(\mu_{X_i}' - x_i'\right) = \min_{\mu_{X_i}\in S} \frac{\partial g_X(\boldsymbol{x}')}{\partial X_i}\left(\mu_{X_i} - x_i'\right), \tag{51}$$

$\mu_{X_i}'' \in S$ and

$$\frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\left(\mu_{X_i}'' - x_i''\right) = \max_{\mu_{X_i}\in S} \frac{\partial g_X(\boldsymbol{x}'')}{\partial X_i}\left(\mu_{X_i} - x_i''\right). \tag{52}$$

Combining eqs. (36), (43) and (49), the lower bound reliability index $\underline{\beta}$ and the design point coordinate $\boldsymbol{x}' = \left(x_1', x_2', \ldots, x_m'\right)^{\mathrm{T}}$ can be obtained. Here, the iterative method is utilized; the iterative steps are written as follows:

(1) Assume an initial design point $\boldsymbol{x}'^{(0)}$; usually assume $\boldsymbol{x}'^{(0)} = \boldsymbol{\mu}_X^c = (\underline{\boldsymbol{\mu}}_X + \overline{\boldsymbol{\mu}}_X)/2$;

(2) Utilizing eq. (43), compute $\cos\theta_{Y_i'}$;

(3) Utilizing eq. (36), compute $\underline{\beta}$;

(4) Utilizing eq. (49), compute a new design point $\boldsymbol{x}'^{(1)}$;

(5) Repeat steps (2)–(4) until $\left|\boldsymbol{x}'^{(n)} - \boldsymbol{x}'^{(n-1)}\right| < \varepsilon$.

Combining eqs. (37), (44), and (50), the upper bound reliability index $\overline{\beta}$ and the design point coordinate $\boldsymbol{x}'' = \left(x_1'', x_2'', \ldots, x_m''\right)^{\mathrm{T}}$ can be obtained. The corresponding iterative steps are written as follows:

1) Assume an initial design point $\boldsymbol{x}''^{(0)}$, usually assume $\boldsymbol{x}''^{(0)} = \boldsymbol{\mu}_X^c = (\underline{\boldsymbol{\mu}}_X + \overline{\boldsymbol{\mu}}_X)/2$;

2) Utilizing eq. (37), compute $\cos\theta_{Y_i''}$;

3) Utilizing eq. (44), compute $\overline{\beta}$;

4) Utilizing eq. (50), compute new design point $\boldsymbol{x}''^{(1)}$;

5) Repeat steps 1) to 4) until $\left|\boldsymbol{x}''^{(n)} - \boldsymbol{x}''^{(n-1)}\right| < \varepsilon$.

Substitute $\boldsymbol{x}'^{(n)}$ and $\boldsymbol{x}''^{(n)}$ into $g_X(\cdot)$, if $g_X(\boldsymbol{x}'^{(n)}) = g_X(\boldsymbol{x}''^{(n)}) = 0$, $\boldsymbol{x}'^{(n)} = \boldsymbol{x}'$ and $\boldsymbol{x}''^{(n)} = \boldsymbol{x}''$ are just the design points we want; if $g_X(\boldsymbol{x}'^{(n)}) \neq 0$ or $g_X(\boldsymbol{x}''^{(n)}) \neq 0$, we can assume a new initial design point and repeat the corresponding steps (1)–(5) or 1)–5).

4.2 Interval design point method for abnormal distribution variables

For abnormal distribution variables, the process of interval equivalent normalized stated in sect. 3.2 should be utilized.

Replacing the variables $\underline{\mu}_{X_i}$, $\overline{\mu}_{X_i}$, $\underline{\sigma}_{X_i}$ and $\overline{\sigma}_{X_i}$ in eqs. (36)–(52) with $\underline{\mu}_{\tilde{X}_i}$, $\overline{\mu}_{\tilde{X}_i}$, $\underline{\sigma}_{\tilde{X}_i}$ and $\overline{\sigma}_{\tilde{X}_i}$ in eqs. (29)–(32), respectively, new equations will be generated. Utilizing them and executing the similar iterative steps of (1)–(5) and 1)–5) in sect. 4.1, the design points for abnormal distribution variables can be obtained.

5 Probabilistic interval reliability of structural systems

5.1 Enumerating significant failure modes of structural systems

Based on the incremental loads approach, in this section, the enumerating method is presented for the significant failure modes of structural systems when parameters of random variables are interval variables. First, we should determine the most serious and less serious critical members under the first incremental loading.

Assumed that the mean value and standard variance of the *i*th member's strength R_i are expressed as eqs. (9) and (10), respectively. The expression for the load effect of the *i*th member F_{i1} and the first incremental loading S_1 is

$$F_{i1} = a_{i1}S_1 \tag{53}$$

where a_{i1} is the load utilization of the *i*th member. In order to ensure the worst failure mode included in a few enumerated significant failure modes, we will consider two kinds of load effect: F_{i1}^c and F_{i1}^l. If we consider that any member's load effects reach their full strength, then

$$F_{i1cr}^c = \mu_{R_i}^c, \ (i = 1,2,\ldots,n)\,, \tag{54}$$

$$F_{i1cr}^l = (\underline{\mu}_{R_i} - k\overline{\sigma}_{R_i}), \ (i = 1,2,\ldots,n)\,, \tag{55}$$

where n is the total number of the members of the entire structure and subscript *cr* means that the *i*th member has reached its critical case under the first incremental loading; $\mu_{R_i}^c = (\underline{\mu}_{R_i} + \overline{\mu}_{R_i})/2$ and k is a constant ($k = 3$ here according to "3σ" theorem in probability theory [21]).

Substituting eqs. (54) and (55) into eq. (53), respectively, one obtains

$$F_{i1cr}^c = \mu_{R_i}^c = a_{i1}S_{i1cr}^c\,, \tag{56}$$

$$F_{i1cr}^l = (\underline{\mu}_{R_i} - k\overline{\sigma}_{R_i}) = a_{i1}S_{i1cr}^l\,. \tag{57}$$

Then

$$S_{i1cr}^c = \mu_{R_i}^c / a_{i1}\,, \tag{58}$$

$$S_{i1cr}^l = (\underline{\mu}_{R_i} - k\overline{\sigma}_{R_i}) / a_{i1}\,. \tag{59}$$

The symbols S_{1cr}^c and S_{1cr}^l are defined as:

$$S_{1cr}^c = \min_{1\leqslant i\leqslant n}(S_{i1cr}^c), \tag{60}$$

$$S_{1cr}^l = \min_{1\leqslant i\leqslant n}(S_{i1cr}^l). \tag{61}$$

Let

$$S_{i1} = \max\left\{\frac{S_{1cr}^c}{S_{i1cr}^c}, \frac{S_{1cr}^l}{S_{i1cr}^l}\right\}, \ (i=1,2,\ldots,n). \tag{62}$$

Then, the first group of serious critical members should be determined by using the following inequality.

$$C_1 \leqslant S_{i1} \leqslant 1, \tag{63}$$

eq. (63) means that we take S_{i1}=1 as the most serious critical case and $C_1 \leqslant S_{i1} < 1$ as less serious critical cases. We take the value of C_1 as 0.8~0.9.

The approach to determining the j-th ($j\geqslant 2$) group of serious critical members under the j-th ($j\geqslant 2$) incremental loading is explained as follows:

$$S_{ijcr} = (\mu_{R_i}^c - \sum_{k=1}^{j-1} a_{ik} S_k) / a_{ij}. \tag{64}$$

Then

$$\begin{aligned} &S_{jcr} = \min_i (S_{ijcr}), \\ &(i=1,2,\ldots,n, \ \text{ except the } j-1 \text{ critical members}). \end{aligned} \tag{65}$$

Let

$$S_{ij} = S_{jcr} / S_{ijcr}. \tag{66}$$

Therefore, the j-th group of serious critical members should be determined by using the following inequality.

$$C_j \leqslant S_{ij} \leqslant 1. \tag{67}$$

In general, the value of the constant C_j ($j\geqslant 2$) may be taken as 0.8~0.9.

After enumerating the significant failure modes, the limited state equation of each failure mode can be determined by increment load procedure. The relation between the incremental load S_i $(i=1,2,\ldots,n)$ and the strength R_i $(i=1,2,\ldots,n)$ is

$$\begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{Bmatrix} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{Bmatrix}. \tag{68}$$

Therefore, the strength of structural system R_s and the corresponding limited state equation are, respectively

$$R_s = \sum_j S_j = \sum_j d_j R_j, \tag{69}$$

$$\sum_j d_j R_j - P = 0, \tag{70}$$

where d_i is the coefficient determined by the load utilization ratio a_{ij} $(i,j=1,2,\ldots,n)$, and P is the generalized load used on the structural system.

5.2 Computation of the probabilistic interval reliability of structural system

For each significant failure mode, there is a corresponding limited state equation as eq. (70). One can obtain the probabilistic interval reliability of each significant failure mode using the present interval design point method for structural members. Given a structural system which has k significant failure modes, the probabilistic interval reliability of each mode can be written as:

$$(P_s^I)_i = \left[(\underline{P}_s)_i, (\overline{P}_s)_i\right], \ (i=1,2,\ldots,k). \tag{71}$$

Then, the lower and upper bounds of the probabilistic interval reliability of the structural system should be, respectively

$$\underline{P}_s = \min\{(\underline{P}_s)_i, i=1,2,\ldots,k\} \tag{72}$$

and

$$\overline{P}_s = \min\{(\overline{P}_s)_i, i=1,2,\ldots,k\}, \tag{73}$$

since the reliability of structural system is mainly decided by the failure mode which has the minimum reliability.

Thus, the probabilistic interval reliability of the structural system is

$$P_s^I = [\underline{P}_s, \overline{P}_s] = [1-\overline{P}_f, \ 1-\underline{P}_f], \tag{74}$$

where P_f is the failure probability.

6 Numerical examples

Consider a 14-bar 2D truss structure in Figure 1. The elastic module and cross-sectional areas for all members are the same, which are E=70 GPa and $A_i = 0.004$ m^2 $(i=1, 2,\cdots,14)$. The strength of each bar is normally distributed. The load P submits to lognormal distribution. The mean value and standard variance of R_i $(i=1,2,\ldots,14)$ and P are uncertain, changing within the following intervals, respectively,

$$\begin{aligned} &\mu_{R_i}^I = [60(1-\alpha), 60(1+\alpha)] \text{ MPa}, \\ &\sigma_{R_i}^I = [13(1-\alpha), 13(1+\alpha)] \text{ MPa}, \ (i=1,2,\ldots,14), \\ &\mu_P^I = [120(1-\alpha), 120(1+\alpha)] \text{ kN}, \\ &\sigma_P^I = [20(1-\alpha), 20(1+\alpha)] \text{ kN}. \end{aligned} \tag{75}$$

where α is the coefficient in the range of 0–0.1.

Utilizing the present enumerating method, we get 6 significant failure modes of the system. The failure tree is shown in Figure 2, and the corresponding limited state equations are listed in Table 1. The comparison of the interval central point method and the present method on calculating the probability interval reliability index and interval failure probability of each failure path as α varies is showed in Figures 3–8.

In Figures 3–8, the subscript D means the interval design point method and the subscript C represents the interval central point method, Figure (a) shows the probabilistic interval reliability indexes while Figure (b) shows the probabilistic interval failure probabilities. As can be seen from these Figures, the bounds of the reliability index and reliability increase monotonically with increasing α and the bounds obtained using the interval design point and interval central point methods are different. When α=0 which implies all the parameters are deterministic, the failure probabilities of Z_1 to Z_6 obtained by Monte Carlo simulation are 9.4×10^{-4}, 9.96×10^{-4}, 4.03×10^{-3}, 4.21×10^{-3}, 2.72×10^{-3} and 2.05×10^{-3}, which are identical to those obtained by the interval design point method. It is obvious that the result obtained by the interval central point method is incorrect, since the distribution characteristics of each random variable are neglected.

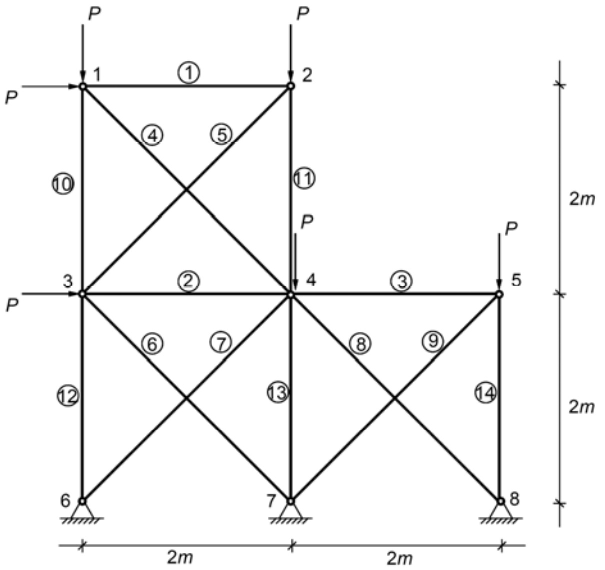

Figure 1 A 14-bar 2D truss structure.

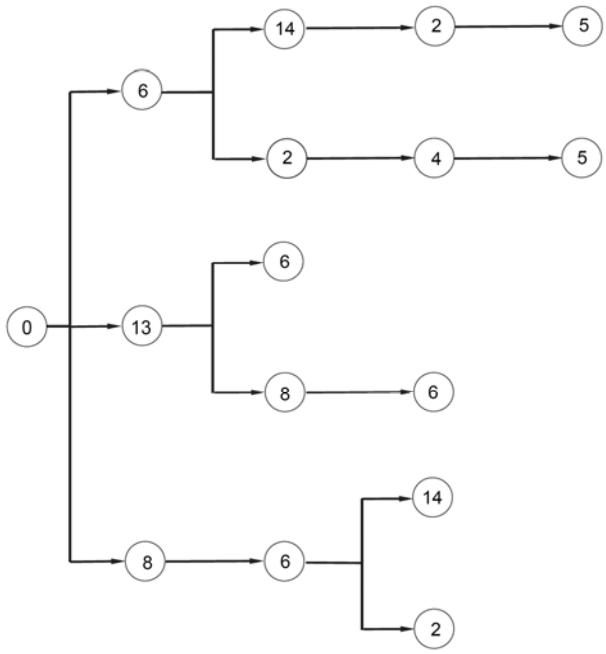

Figure 2 Fault tree of the 14-bar 2D truss structure.

Utilizing the present method for calculating the probabilistic interval reliability of the structural system, we can obtain the probabilistic interval reliability index and reliability of the frame system as shown in Figure 9. When α=0.1, the probabilistic interval reliability of the reliability system calculated by the interval design point method and the interval central point method are [0.9782, 0.9993] and [0.9051, 0.9991], respectively.

Moreover, we take the first failure path as an example to show the iteration process of the interval design point method. The iteration process for both $\underline{\beta}_1$ and $\overline{\beta}_1$ when α=0.05 are listed in Tables 2 and 3.

7 Conclusions

In this paper, a new interval design point method for the reliability of structural systems is presented. The uncertain parameters are described as interval variables in this ap-

Table 1 The limited state equations of the main failure modes of the 2D truss structure

Sequence number	Failure path	Limited state equation
1	6-14-2-5	$Z_1 = 1.25383R_{14} + 0.772071R_2 + 2.97442R_5 - 0.11376R_6 - P = 0$
2	6-2-4-5	$Z_2 = 1.8336R_2 + 0.546651R_4 + 1.73491R_5 + 0.830773R_6 - P = 0$
3	13-6	$Z_3 = 0.39888R_{13} + 3.32557R_6 - P = 0$
4	13-8-6	$Z_4 = 0.6977R_{13} + 1.96348R_6 + 1.03534R_8 - P = 0$
5	8-6-14	$Z_5 = 4R_{14} - P = 0$
6	8-6-2	$Z_6 = 1.81225R_2 + 1.69342R_6 + 0.716037R_8 - P = 0$

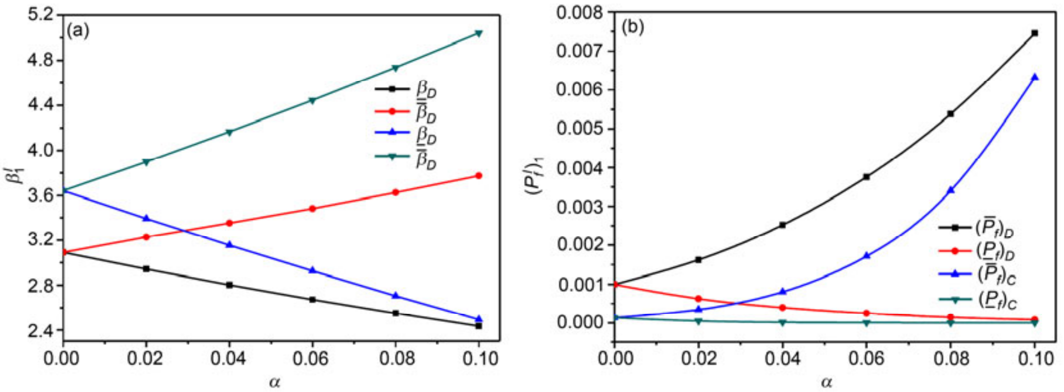

Figure 3 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_1 (b) obtained using the interval design point and interval central point methods as α varies for the 2D truss structure.

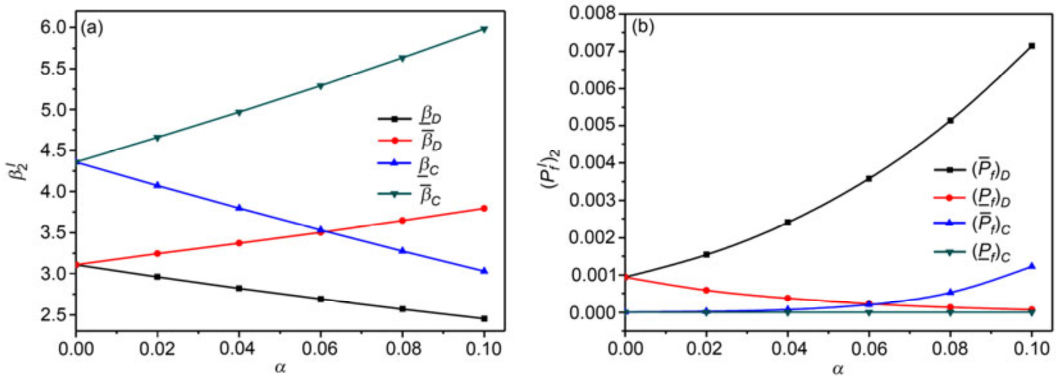

Figure 4 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_2 (b) obtained using the interval design point method and interval central point method as α varies for the 2D truss structure.

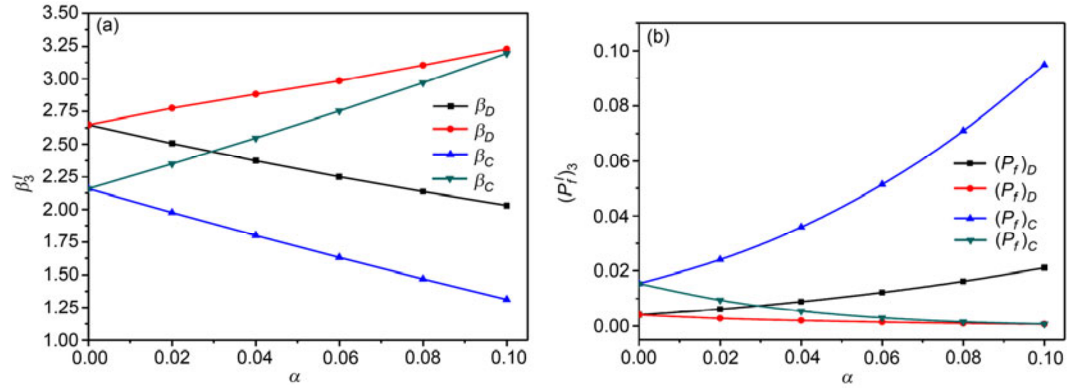

Figure 5 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_3 (b) obtained using the interval design point and interval central point methods as α varies for the 2D truss structure.

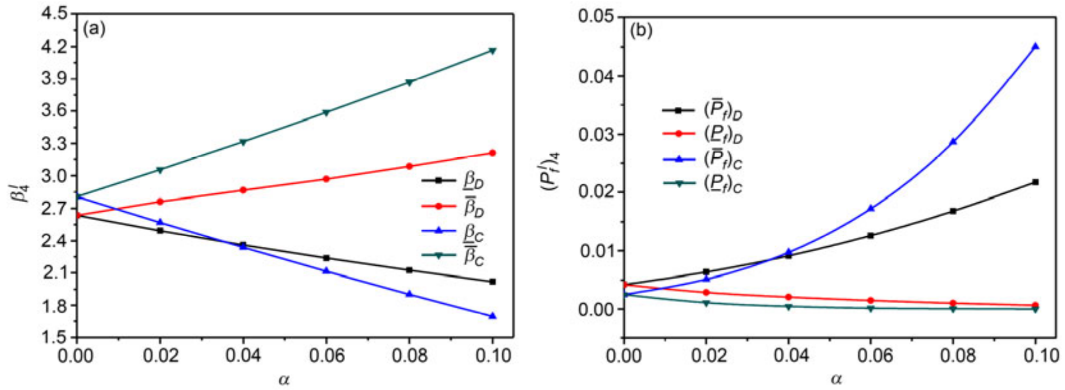

Figure 6 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_4 (b) obtained using the interval design point and interval central point methods as α varies for the 2D truss structure.

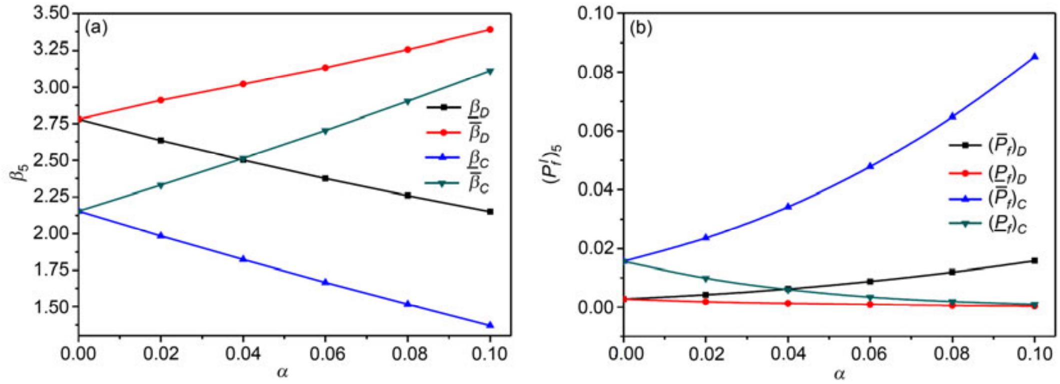

Figure 7 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_5 (b) obtained using the interval design point and interval central point methods as α varies for the 2D truss structure.

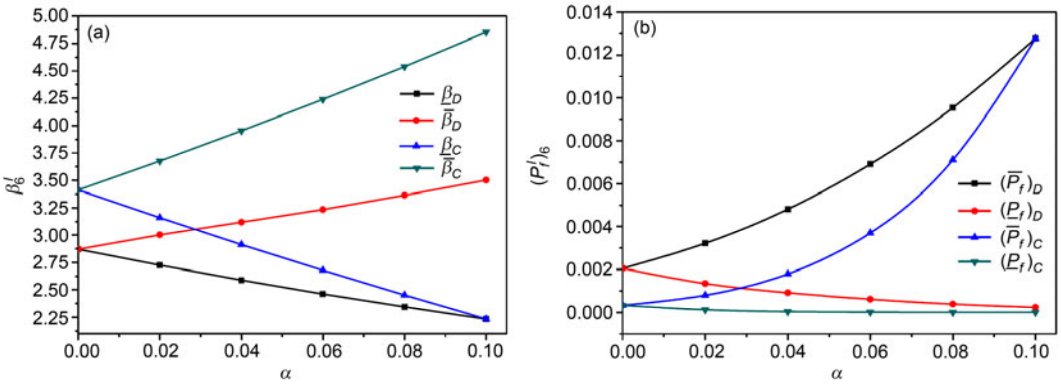

Figure 8 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of Z_6 (b) obtained using the interval design point and interval central point methods as α varies for the 2D truss structure.

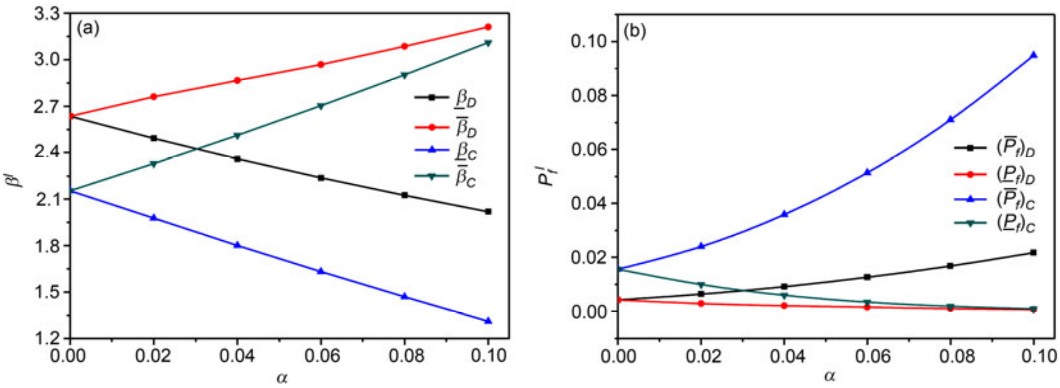

Figure 9 (Color online) The comparison of the probabilistic interval reliability index (a) and failure probability of 2D truss structure (b) obtained using the interval design point and interval central point methods as α varies.

Table 2 The iteration process of the interval design point method for the lower bound reliability index $\underline{\beta}_1$ of Z_1 when α=0.05

Iterations		1	2	3	4	5	6	7
Initial coordinate of the design point	$R'^{(0)}$	60	29.8345	18.2643	20.0873	20.4649	20.5352	20.5480
	$P'^{(0)}$	120	94.2121	99.1748	100.192	100.381	100.415	100.422
Reliability index	$\underline{\beta}$	2.1029	2.9007	2.7706	2.7436	2.7386	2.7377	2.7375
	$\Delta\underline{\beta}$	——	0.7978	0.1301	0.0270	0.0050	0.0009	0.0002
New coordinate of the design point	$R'^{(1)}$	29.8345	18.2643	20.0873	20.4649	20.5352	20.5480	20.5504
	$P'^{(1)}$	94.2121	99.1748	100.192	100.381	100.415	100.422	100.423

Table 3 The iteration process of the interval design point method for the upper bound reliability index $\overline{\beta}_1$ of Z_1 when α=0.05

Iterations		1	2	3	4	5
Initial coordinate of the design point	$R'^{(0)}$	26.5108	22.9062	22.3147	22.2880	22.2864
	$P'^{(0)}$	110.454	108.975	108.908	108.904	108.903
Reliability index	$\underline{\beta}$	3.0663	3.3669	3.4144	3.4165	3.4166
	$\Delta\underline{\beta}$	–	0.3006	0.0475	0.0021	0.0001
New coordinate of the design point	$R'^{(1)}$	22.9062	22.3147	22.2880	22.2864	22.2863
	$P'^{(1)}$	108.975	108.908	108.904	108.903	108.903

proach. The present criterion to enumerate the failure modes of structural systems may ensure the worst failure mode included in a few enumerated significant failure modes.

From the results of numerical examples, we may find that the proposed method can overcome the inherent deficiencies of the interval central point method and may provide exact probabilistic interval reliability of structures whose random variables can have either normal or abnormal distribution form. Moreover, the obtained exact probabilistic interval reliability is expressed in the form of interval range which can provide more information than any single number result on the safety degree of structures.

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