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Nonstationary probability densities of a class of nonlinear system excited by external colored noise

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This paper deals with the approximate nonstationary probability density of a class of nonlinear vibrating system excited by colored noise. First, the stochastic averaging method is adopted to obtain the averaged Itô equation for the amplitude of the system. The corresponding Fokker-Planck-Kolmogorov equation governing the evolutionary probability density function is deduced. Then, the approximate solution of the Fokker-Planck-Kolmogorov equation is derived by applying the Galerkin method. The solution is expressed as a sum of a series of expansion in terms of a set of proper basis functions with time-depended coefficients. Finally, an example is given to illustrate the proposed procedure. The validity of the proposed method is confirmed by Monte Carlo Simulation.

colored noise, nonstationary probability density, Galerkin method, stochastic averaging method

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Nonlinear systems subjected to random excitations are often found in science and engineering. A case in point is the helicopter rotor blade vibration during forward flight in a turbulent atmosphere. The response of such systems has been studied for decades. However, the exact probability density functions (PDF) are obtained only for some linear systems and a few particular first-order nonlinear systems [1–4]. For some specific second order nonlinear systems, exact stationary PDF can be deduced [5–7]. In order to study the response of nonlinear stochastic systems, the stochastic averaging method was introduced by Stratonovitch [8] and verified theoretically by Khasminskii [9]. Then, this method was reproposed by Zhu and Lin [10] based on a theorem proposed by Khasminskii. The stochastic averaging method has been proved to be a powerful strategy to predict response [11–14]. The main advantage of the stochastic averaging method is that it leads to a Markov process, and that a much more tractable mathematical problem of determining the response of the system can be obtained.

The transient response is usually considered in the area of electrical engineering [15]. The nonstationary PDF governing the dynamic evolution of the stochastic nonlinear dynamic system is difficult to obtain. Some approximate strategies have been proposed, such as the cell mapping method [16], path integration [17,18], and the Galerkin method [19]. This paper is limited to the Galerkin method. The Galerkin method, as a variation method, has been adopted to study both the stationary and nonstationary responses of nonlinear systems. Atkinson [19] adopted this method to study the stationary response of several second-order nonlinear systems, and a non-stationary case was studied by Wen [20]. Recently, Spanos [21] further studied the nonstationary response of nonlinear oscillators subjected to additive Gaussian White noise and Jin [22]

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studied the response of such oscillators with time delay.

White noise is an ideal mathematical model. The correlation time is supposed to be zero, which never happens in the real world. Real noise is considered to be colored noise. Some work has been done on the response of the system excited by colored noise [23,24]. To the authors' knowledge, little work has been done on the nonstationary response envelope probability densities of nonlinear oscillators subjected to colored noise.

The present paper deals with the nonstationary response PDF of a class of nonlinear oscillators subjected to colored noise. The results obtained are authenticated by those obtained from Monte Carlo Simulation (MCS) of the original oscillator.

1 Simplification of the model

Consider a nonlinear system driven by colored noise

$$\begin{aligned}&\ddot{X}(t)+\omega_0^2X(t)+2\gamma\omega_0\dot{X}(t)+h(X,\dot{X})\dot{X}(t)=f_k\xi_k(t),\\&k=1,2,\cdots,m,\end{aligned} \tag{1}$$

where ω_0 and γ are the frequency and the ratio of damping of the system, respectively. Suppose δ is a small positive parameter, and $h(X,\dot{X})$ represents lightly nonlinear damping of order δ. f_k represents the amplitude of external excitation $\xi_k(t)$ of the order of $\delta^{1/2}$. $\xi_k(t)$ are wide (or narrow)-band colored noise with zero mean and correlation functions $R_{kr}(\tau)$ or spectral functions $S_{kr}(\omega)$.

The response of system (1) can be assumed of the following form [13]:

$$\begin{aligned}&X(t)=A\cos\Phi(t),\\&\dot{X}(t)=-A\omega_0\sin\Phi(t),\\&\Phi(t)=\omega_0t+\Theta,\end{aligned} \tag{2}$$

where A, Φ and Θ are all stochastic processes. Regarding eq. (2) as generalized van der Pol transformation from X, $\dot{X}$ to A, Θ, one can obtain the following equations:

$$\begin{aligned}&\frac{\mathrm{d}A}{\mathrm{d}t}=s_1(A,\Theta)+\sigma_{1k}(A,\Theta)\xi_k(t),\\&\frac{\mathrm{d}\Theta}{\mathrm{d}t}=s_2(A,\Theta)+\sigma_{2k}(A,\Theta)\xi_k(t),\end{aligned} \tag{3}$$

where

$$\begin{aligned}&s_1(A,\Theta)=-[2\gamma\omega_0+h(A\cos\Phi,-A\omega_0\sin\Phi)]A\sin^2\Phi,\\&s_2(A,\Theta)=-[2\gamma\omega_0+h(A\cos\Phi,-A\omega_0\sin\Phi)]\sin\Phi\cos\Phi,\\&\sigma_{1k}(A,\Theta)=-\frac{f_k}{\omega_0}\sin\Phi,\\&\sigma_{2k}(A,\Theta)=-\frac{f_k}{\omega_0A}\cos\Phi.\end{aligned} \tag{4}$$

Based on the Stratonovich-Khasminskii theorem [25], $A(t)$ converges to a one-dimensional diffusive Markov process as $\delta\to0$. The Itô equation of the limiting diffusion process is of the form:

$$\mathrm{d}A=m(A)\mathrm{d}t+\sigma(A)\mathrm{d}W(t), \tag{5}$$

where $W(t)$ is a unit Wiener process, and $m(A)$ and $\sigma(A)$ are expressed as follows:

$$\begin{aligned}&m(A)=\left\langle s_1+\int_{-\infty}^{0}\left(\left.\frac{\partial\sigma_{1k}}{\partial A}\right|_t\sigma_{1r|t+\tau}+\left.\frac{\partial\sigma_{1k}}{\partial\Theta}\right|_t\sigma_{2r|t+\tau}\right)R_{kr}(\tau)\mathrm{d}\tau\right\rangle,\\&\sigma^2(A)=\left\langle\int_{-\infty}^{\infty}(\sigma_{1k|t}\sigma_{1r|t+\tau})R_{kr}(\tau)\right\rangle,\\&r,k=1,2,\cdots,m,\end{aligned} \tag{6}$$

where $\langle\cdot\rangle_t=\frac{1}{2\pi}\int_0^{2\pi}\cdot\,\mathrm{d}\Phi$ represents the time averaging.

The averaged FPK equation associated with eq. (5) is

$$\frac{\partial p(a,t)}{\partial t}=\frac{\partial}{\partial a}[-m(a)p(a,t)]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2(a)p(a,t)], \tag{7}$$

where $m(a)=m(A)|_{A=a}$, $\sigma^2(a)=\sigma(A)|_{A=a}$. For simplicity, it is assumed that system (1) is initially at rest. The initial condition is $p(a,0)=\hat{\delta}(a)$..

2 An approximate solution of the FPK equation

First, consider the linear oscillator:

$$\begin{aligned}&\ddot{X}(t)+\omega_0^2X(t)+2\gamma\omega_0\dot{X}(t)=f_k\xi_k(t),\\&k=1,2,\cdots,m.\end{aligned} \tag{8}$$

According to sect. 1, the FPK equation of system (8) is

$$\begin{aligned}&\frac{\partial p(a,t)}{\partial t}=\frac{\partial}{\partial a}[-m(a)p(a,t)]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2(a)p(a,t)],\\&m(a)=-\gamma\omega_0a+\frac{\sum_{k=1}^m\sum_{r=1}^mf_kf_r\pi S_{kr}(\omega_0)}{2\omega_0^2a},\\&\sigma^2(a)=\frac{\sum_{k=1}^m\sum_{r=1}^mf_kf_r\pi S_{kr}(\omega_0)}{\omega_0^2}.\end{aligned} \tag{9}$$

Under the restriction $0\leqslant a\leqslant\infty$, the eigenvalues λ_i and the eigenfunctions $E_i(a)$ of eq. (9) are in the following:

$$\begin{aligned}&\lambda_i=2\gamma\omega_0i,\quad i=1,2,\cdots,\\&E_i(a)=\frac{1}{i!}\frac{a}{\sigma_s^2}\exp\left(-\frac{a^2}{2\sigma_s^2}\right)L_i\left(\frac{a^2}{2\sigma_s^2}\right),\\&\sigma_s^2=\frac{\sum_{k=1}^m\sum_{r=1}^mf_kf_r\pi S_{kr}(\omega_0)}{2\gamma\omega_0^3},\end{aligned} \tag{10}$$

in which $L_i(\cdot)$ represents the i-th order Laguerre polynomial.

With the properties of the Laguerre polynomial, it can be proved that $E_i(a)$ satisfies the pronominalization condition:

$$\int_0^\infty \frac{E_r(a)E_k(a)}{E_0(a)}\mathrm{d}a = \delta_{kr}, \tag{11}$$

where δ_{kr} is the Kronecker delta symbol.

Based on the Galerkin method, first, the approximate solution of eq. (7) can be expressed as [21]:

$$p(a,t) = \sum_{i=0}^{\infty}(\mathrm{e}^{-\lambda_i t} + z_i(t))E_i(a), \tag{12}$$

where $z_i(t)$ are functions of time to be determined. Substituting eq. (12) into eq. (7), the residual error is available:

$$\begin{aligned} R_{\text{error}} &= \sum_{i=0}^{\infty}(-\lambda_i \mathrm{e}^{-\lambda_i t} + \dot{z}_i(t))E_i(a) \\ &-\sum_{i=0}^{\infty}\left[(\mathrm{e}^{-\lambda_i t} + z_i(t))\left(-\frac{\mathrm{d}(m(a)E_i(a))}{\mathrm{d}a}\right.\right. \\ &\left.\left.+\frac{1}{2}\frac{\mathrm{d}^2(\sigma^2(a)E_i(a))}{\mathrm{d}a^2}\right)\right]. \end{aligned} \tag{13}$$

In actual numerical calculation, the series need to be properly truncated. If N+1 (i=0,1,$\cdots$,N) terms are adopted, N+1 unknown $z_i(t)$ will be determined.

According to the Galerkin method, these $z_i(t)$ can be evaluated by making the projection of the residual error R_{error} on a set of independent functions be zero. Selecting $E_i(a)/E_0(a)$ as weighting functions, the condition above can be expressed as:

$$\int_0^\infty \frac{E_i(a)}{E_0(a)}R_{\text{error}}[a, \boldsymbol{z}(t)]\mathrm{d}a = 0. \tag{14}$$

Substituting eq. (13) into eq. (14) together with the properties of Laguerre polynomial, a set of linear first-order ordinary differential equations governing $\boldsymbol{z}(t)$ can be deduced:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{F}\boldsymbol{z}(t) + \boldsymbol{D}(t) \tag{15}$$

$\boldsymbol{F}$ and $\boldsymbol{D}$ represent a matrix of order (N+1)×(N+1) and a (N+1)-dimensional vector, respectively. Eq. (17) can be numerically solved via Runge-Kutta method.

Besides, the statistic moments of the response envelope can be obtained. The j-th order moment can be written as:

$$m_j(t) = E[a^j(t)] = \sum_{i=0}^{N}\left[(\mathrm{e}^{-\lambda_i t} + z_i(t))\int_0^\infty a^j E_i(a)\mathrm{d}a\right]. \tag{16}$$

3 Example

Consider the Van der pol-Rayleigh system subjected to an external colored noise

$$\begin{aligned} &\ddot{x}(t) + \omega_0^2 x + 2\gamma\omega_0\dot{x}(t) + 2\gamma\omega_0[-2 + \varepsilon(x^2(t) \\ &+\dot{x}^2(t))]\dot{x}(t) = \xi(t), \end{aligned} \tag{17}$$

where ε is a constant, ω_0 and γ are the same as those in eq. (1). $\xi(t)$ is a colored noise with zero mean and rational spectral density:

$$S(\omega) = \frac{D}{\pi\left[(\omega^2 - \omega_1^2) + 4\zeta^2\omega_1^2\omega^2\right]}, \tag{18}$$

where D, ω_1, ζ are constants. ε and D are assumed of the same order of δ in eq. (1). $\xi(t)$ can be regarded as the output of the second order filter $\ddot{\xi} + 2\omega_1\zeta\dot{\xi} + \omega_1^2\xi = W_g(t)$. $W_g(t)$ is a Gaussian white noise with intensity D.

Based on the procedure proposed in sect. 2, the coefficients of the averaged FPK equation of the system eq. (17) are

$$\begin{aligned} &m(a) = -\gamma\omega_0 a + \varepsilon\gamma\omega_0 a^3\left(\frac{1}{4} + \frac{3}{4}\omega_0^2\right) + \frac{\pi S(\omega_0)}{2a\omega_0^2}, \\ &\sigma^2(a) = \frac{\pi S(\omega_0)}{\omega_0^2}, \end{aligned} \tag{19}$$

with

$$\begin{aligned} &\lambda_i = 2\gamma\omega_0 i, \\ &E_i(a) = \frac{1}{i!}\frac{a}{\sigma_s^2}\exp\left(-\frac{a^2}{2\sigma_s^2}\right)L_i\left(\frac{a^2}{2\sigma_s^2}\right), \\ &\sigma_s^2 = \frac{\pi S(\omega_0)}{2\gamma\omega_0^3}. \end{aligned} \tag{20}$$

In the case of white noise, the stationary variance $\sigma_s^2 = \dfrac{D}{2\gamma\omega_0^3}$. Following the procedure eqs. (12)–(15), together with the properties of Laguerre polynomial, a set of linear first-order ordinary differential equations governing $\boldsymbol{z}(t)$ can be written as follows:

$$\begin{aligned} \dot{z}_i(t) &= -\lambda_i z_i(t) + 4\gamma\omega_0 i z_i(t) \\ &+\frac{\varepsilon\gamma\omega_0 + 3\varepsilon\gamma\omega_0^2}{4}\left\{G_1 + 3G_2 + \sum_{r=0}^{N} z_r(t)\right. \\ &\left.\int_0^\infty a^2\frac{E_i}{E_0}[(r+1)E_{r+1} - 2(1-r)E_r]\mathrm{d}a\right\} \\ &-2\gamma\omega_0\left\{G_3 + G_4 + 2\sum_{r=0}^{N}(r+1)z_r(t)\delta_{i,r+1}\right\}, \\ r &= 0,1,2,\cdots,N, \end{aligned} \tag{21}$$

where

$$\begin{aligned} &G_1 = \int_0^\infty a^3\frac{E_i}{E_0}\frac{\partial p_l(a,t)}{\partial a}\mathrm{d}a, \quad G_2 = \int_0^\infty a^2\frac{E_i}{E_0}p_l(a,t)\mathrm{d}a, \\ &G_3 = \int_0^\infty \frac{E_i}{E_0}p_l(a,t)\mathrm{d}a, \quad G_4 = \int_0^\infty a\frac{E_i}{E_0}\frac{\partial p_l(a,t)}{\partial a}\mathrm{d}a, \end{aligned} \tag{22}$$

$$p_l(a,t)=\frac{a}{\sigma_t^2(t)}\exp\left[-\frac{a^2}{2\sigma_t^2(t)}\right],\ \sigma_t^2=\sigma_s^2[1-\exp(-2\mu_0\omega_0 t)].$$

Figure 1 describes the response of the amplitude of eq. (1) at t=50 s. It is shown that even for large value of the nonlinear parameter ε, 16 terms are enough for us to get a good agreement with digital data.

Figures 2–4 present the nonstationary PDF values of the amplitude of system (17) at different time instants corresponding to different sets of parameters respectively. It can be seen that even the parameters of the colored noise are different. The results obtained via the proposed method match well those obtained by MCS at each time instant. It is also shown that the PDF becomes cliffier as the correlation time of the colored noise becomes longer.

Figures 5 and 6 display the time evolution of the first-order and the second-order moments of system (19) envelope corresponding to different sets of parameters. It can be seen that the results obtained by the proposed procedure are in good agreement with the digital simulation when N=15, even when N=2 (Figure 6). From Figures 5 and 6 one also can see that the system will experience a dynamic process and then reach the steady state.

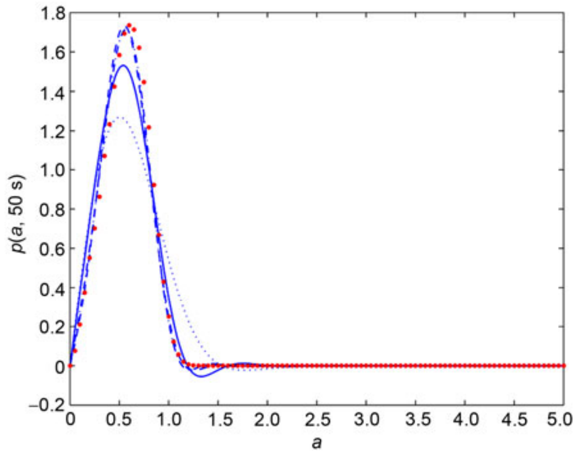

Figure 1 (Color online) Nonstationary probability density of the amplitude of system eq. (17) at t=50 s for different values of N by the proposed method and MCS. γ=0.02, ω_0=1, ε=5.84, D=1, ω_1=3, ζ=0.5. ······ N=5, ------ N=8, --- N=15, -·-·- N=20, ● MCS.

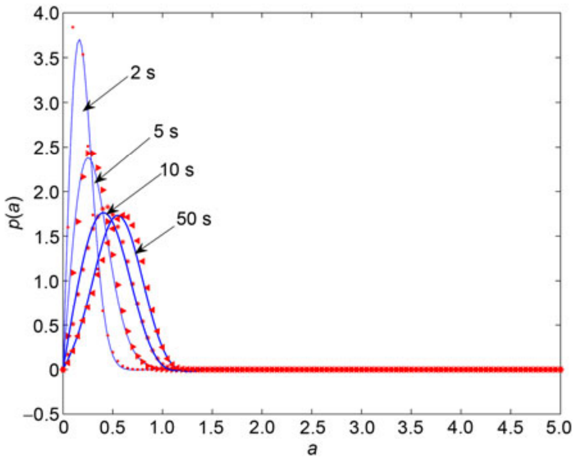

Figure 2 (Color online) Nonstationary probability density of the amplitude of system eq. (17) at different time instants. γ=0.02, ω_0=1, ε=5.84, D=1, ω_1=3, ζ=0.5, N=15. —— the proposed method, ●►*◄MCS.

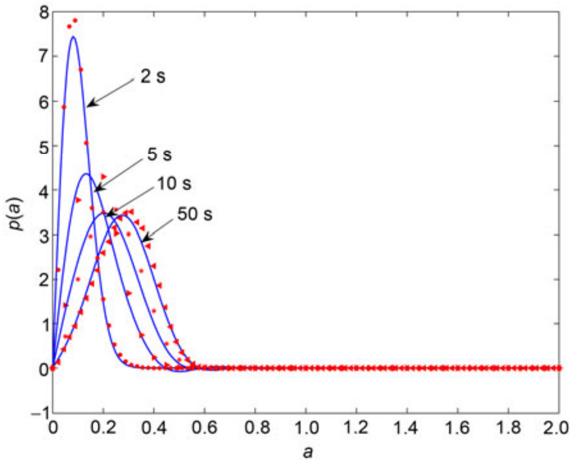

Figure 3 (Color online) Nonstationary probability density of the amplitude of system eq. (17) at different time instants. γ=0.02, ω_0=1, ε=24, D=2, ω_1=5, ζ=0.5, N=15. —— the proposed method, ●►*◄ MCS.

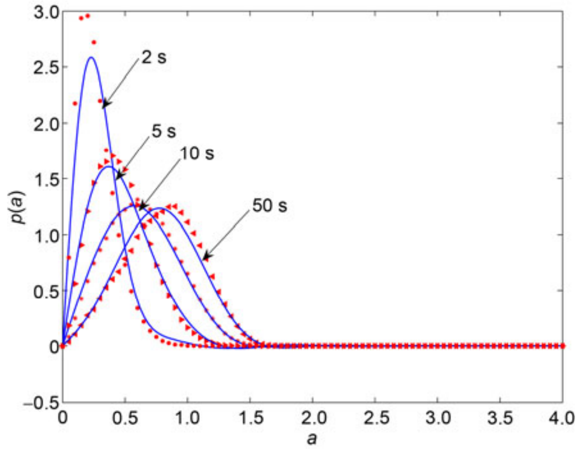

Figure 4 (Color online) Nonstationary probability density of the amplitude of system eq. (17) at different time instants. γ=0.02, ω_0=1, ε=2.92, D=2, ω_1=3, ζ=0.5, N=15. —— the proposed method, ●►*◄ MCS.

4 Conclusion

In the present paper, the nonstationary PDF of a class of nonlinear oscillators subjected to external colored noise is considered. The averaged Itô equation is obtained by the stochastic averaging method. The approximate solution of the FPK equation becomes available by applying the Galerkin method. The proposed procedure is then applied to an example at last. The results obtained by the procedure proposed in the paper are in good agreement with the digital

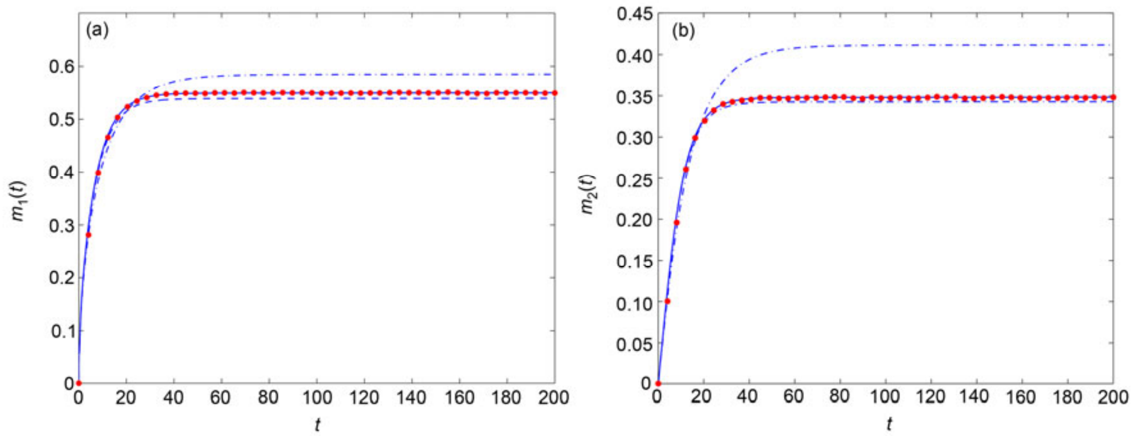

Figure 5 (Color online) Time evolution of the first-order (a) and the second-order (b) moments of the response envelope. γ=0.02, ω_0=1, ε=5.84, D=1, ω_1=3, ζ=0.5. ----- N=2, – – – N=8, ------ N=15, ● MCS.

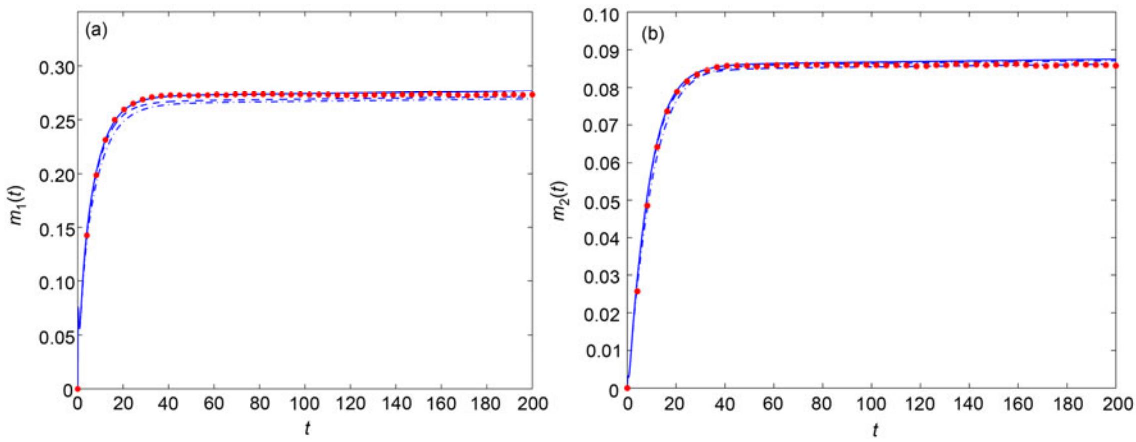

Figure 6 (Color online) Time evolution of the first-order (a) and the second-order(b) moments of the response envelope. γ=0.02, ω_0=1, ε=24, D=2, ω_1=5, ζ=0.5. ----- N=2, – – – N=8, ------ N=15, ● MCS.

simulation. The significant effect of the colored noise on the response is discussed as well.

It is noted that this procedure can be extended to the non-Gaussian noise case. This will be the subject of our future work.

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