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Quantum entanglement of the binomial field interacting with a cascade three-level atom beyond the rotating wave approximation

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In this paper, the quantum entanglement between a single mode binomial field and a cascade three-level atom is calculated mechanically without the rotating wave approximation. The numerical results indicate that the quantum entanglement at the first few periods is reduced notably due to the fact that the atom is initially in the superposition state. With increasing field parameter η, the period of the entanglement evolution becomes obvious and the quantum decoherence phenomenon emerges in a short time.

binomial state field, quantum entanglement, field entropy

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1 Introduction

The quantum entanglement plays an important role in quantum communications and quantum computations. However, it is essential to find a reliable way to measure the entanglement. Since Phoenix and Knight [1,2] introduced the entropy theory into the realm of quantum optics, entropy has demonstrated its superiority in explaining correlations of light and the atom as well as the quantum effects in the interaction between them. In particular, entropy is found to be a far better measure of the entanglement between the light field and the atom [1–5].

The binomial state was introduced in 1985 by Stoler et al. [6] to represent states intermediate between the number and the coherent states. Dattoli et al. [7] found in 1987 that the binomial state could occur in the free electron laser. Since then, many different properties of the binomial states, such as squeezing [8] and anti-bunching [9], have been studied. A single mode binomial state can be written as $|\eta M\rangle = \sum_{n=0}^{M} \beta_n^M |n\rangle$, with $0<\eta<1$, and $M>0$, where $\beta_n^M = \{M!\eta^n(1-\eta)^{M-n}/[n!(M-n)!]\}^{1/2}$ [9]. M is the maximum photon number of the binomial state and η is the parameter of the binomial state. This state is called an intermediate state which reduces to a vacuum state when $\eta=0$, and to a number state when $\eta=1$. In the limit of $M\to\infty$, $\eta\to 0$, it becomes a coherent state, and $\bar{n} = \eta M=\alpha^2$ is the average photon number.

Recently, the interaction between the atom and the field is widely studied [10–14]. Compared with the two-level atom, the three-level atom has different transition characteristics. More interesting quantum phenomena have been observed from the three-level atom [15–17]. Therefore, it is necessary to deeply study the three-level atom.

A common approximation which is widely used in the realm of quantum optics is the rotating wave approximation (RWA), but sometimes the RWA could lead to the symmetry violation [18]. With the improvement of experimental conditions and techniques, the coupling of the cavity field to an atom has increased notably [19,20]. In this case,

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counter-rotating terms are no longer neglectable [21–24]. In this paper, the quantum entanglement between a single mode binomial field and a cascade three-level atom is calculated numerically in the framework of a complete quantum theory and without the RWA. The influences of different initial conditions on the quantum entanglement are discussed.

2 Model and solutions

We consider a three-level cascade type atom as shown in Figure 1. The atomic levels are labeled as the ground state $|1\rangle$, the middle state $|2\rangle$ and the upper state $|3\rangle$. The transitions between ground state $|1\rangle$ and upper state $|3\rangle$ are forbidden. The Hamiltonian H' without the RWA is given by ($\hbar=1$) [25],

$$H' = H_f + H_a + H_I, \tag{1}$$

where

$$H_f = \omega_0 a^+ a, \tag{2}$$

$$H_a = \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2| + \omega_3 |3\rangle\langle 3|, \tag{3}$$

$$H_I = u(a^+ + a)\left(|2\rangle\langle 3| + |3\rangle\langle 2|\right) + v(a^+ + a)\left(|2\rangle\langle 1| + |1\rangle\langle 2|\right). \tag{4}$$

where, $a^+(a)$ is the cavity field creation (annihilation) operator, ω_1, ω_2 and ω_3 are the eigen-frequencies for the atomic states, respectively, ω_0 is the frequency of the single mode and the atom-field coupling constants which are defined by $u = g\sin\theta$ and $v = g\cos\theta$. In order to obtain the diagonalizable Hamiltonian, and to perform an SO(3) transformation on the Hamiltonian, we introduce a unitary matrix

$$\boldsymbol{U} = \begin{bmatrix} \sqrt{2}\sin\theta/2 & -\sqrt{2}/2 & \sqrt{2}\cos\theta/2 \\ \sqrt{2}\sin\theta/2 & \sqrt{2}/2 & \sqrt{2}\cos\theta/2 \\ -\cos\theta & 0 & \sin\theta \end{bmatrix}, \tag{5}$$

where θ is the angel of the transformation, in this paper θ=45°.

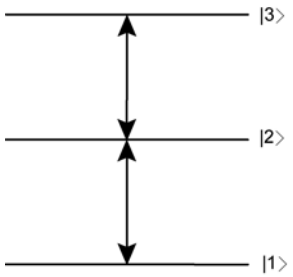

Figure 1 Three-level cascade type atom.

Using the transformation of $\boldsymbol{H} = \boldsymbol{U}H'\boldsymbol{U}^{-1}$, we obtain

$$\boldsymbol{H} = \begin{bmatrix} h_{11} + \omega_0 A^+ A - g^2/\omega_0 & h_{12} & h_{13} \\ h_{21} & h_{22} + \omega_0 B^+ B - g^2/\omega_0 & h_{23} \\ h_{31} & h_{32} & h_{33} + \omega_0 a^+ a \end{bmatrix}, \tag{6}$$

where we have introduced two displacement transformations $A = a - g/\omega_0$ and $B = a + g/\omega_0$, and we have defined $h_{11} = h_{22} = (\omega_3 \sin^2\theta + \omega_2 + \omega_1 \cos^2\theta)/2$, $h_{12} = h_{21} = (\omega_3 \sin^2\theta - \omega_2 + \omega_1 \cos^2\theta)/2$, $h_{33} = \omega_3 \cos^2\theta + \omega_1 \sin^2\theta$, $h_{13} = h_{23} = h_{31} = h_{32} = -\sqrt{2}\cos\theta\sin\theta(\omega_3 - \omega_1)/2$. The wave function can be expanded in complete basis $|n\rangle_A$, $|n\rangle_B$ and $|n\rangle$, respectively, where $|n\rangle_A$ and $|n\rangle_B$ are the Fock states of the new bosonic operators $A(A^+)$ and $B(B^+)$, $|n\rangle$ is a Fock state. Therefore, the wave function can be written as [26–28]

$$|\psi\rangle = \begin{bmatrix} \sum_{n=0}^{N} c_n |n\rangle_A \\ \sum_{n=0}^{N} d_n |n\rangle_B \\ \sum_{n=0}^{N} f_n |n\rangle \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{N} c_n \frac{1}{\sqrt{n!}} \left(a^+ - \frac{g}{\omega_0}\right)^n \exp\left(\frac{g}{\omega_0} a^+ - \frac{g^2}{2\omega_0^2}\right)|0\rangle \\ \sum_{n=0}^{N} d_n \frac{1}{\sqrt{n!}} \left(a^+ + \frac{g}{\omega_0}\right)^n \exp\left(-\frac{g}{\omega_0} a^+ - \frac{g^2}{2\omega_0^2}\right)|0\rangle \\ \sum_{n=0}^{N} f_n \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle \end{bmatrix}. \tag{7}$$

Putting eqs. (6) and (7) into the Schrödinger equation and left multiplying states by ${}_A\langle m|$, ${}_B\langle m|$ and $\langle m|$, respectively, we obtain

$$\left(h_{11} + \omega_0 m - \frac{g^2}{\omega_0}\right) c_m + h_{12} \sum_{n=0}^{N} d_n \, {}_A\langle m|n\rangle_B + h_{13} \sum_{n=0}^{N} f_n \, {}_A\langle m|n\rangle = E c_m, \tag{8}$$

$$h_{21} \sum_{n=0}^{N} c_n \, {}_B\langle m|n\rangle_A + \left(h_{22} + \omega_0 m - \frac{g^2}{\omega_0}\right) d_m + h_{23} \sum_{n=0}^{N} f_n \, {}_B\langle m|n\rangle = E d_m, \tag{9}$$

$$h_{31} \sum_{n=0}^{N} c_n \langle m|n\rangle_A + h_{32} \sum_{n=0}^{N} d_n \langle m|n\rangle_B + (h_{33} + \omega_0 m) f_m = E f_m, \tag{10}$$

where ${}_A\langle m|n\rangle = \langle m|n\rangle_B = (-1)^m D_{mn}(g/\omega_0)$, ${}_A\langle m|n\rangle_B = (-1)^m D_{mn}(2g/\omega_0)$, ${}_B\langle m|n\rangle_A = (-1)^{n-m}\,{}_A\langle m|n\rangle_B$, ${}_B\langle m|n\rangle = (-1)^{n-m}\langle m|n\rangle_B$, $\langle m|n\rangle_A = (-1)^{n-m}\langle m|n\rangle_B$, with $D_{mn}(x)= \mathrm{e}^{-\frac{x^2}{2}}\sum_{r=0}^{\min(m,n)}\frac{(-1)^r\sqrt{m!n!}x^{m+n-2r}}{r!(m-r)!(n-r)!}$, $x=g/\omega_0$. The eigenvalues E and eigenvectors with coefficients $\{c_n^{(i)}\}$, $\{d_n^{(i)}\}$ and $\{f_n^{(i)}\}$ $(i=1,\cdots 3N+3)$ can be exactly solved by diagonalizing the above $(3N+3)\times(3N+3)$ matrix numerically.

Assuming that the cavity field is initially in the binomial field and the atom is in the coherent superposition state, we have $C_a|1\rangle + C_b|2\rangle + C_c|3\rangle$, where C_a, C_b and C_c are the superposition coefficients of atomic energy levels. If the atom is initially in the ground state (C_a=1, C_b=C_c=0), the wave function can be written as

$$\begin{aligned}
&|t=0\rangle \\
&=\sum_{n=0}^{M}\beta_n^M|n\rangle|1\rangle \\
&=\sum_{n=0}^{M}\beta_n^M|n\rangle\left(U_{13}|3'\rangle+U_{23}|2'\rangle+U_{33}|1'\rangle\right) \\
&=\sum_{i=0}^{3N}k_i\left[\sum_{n=0}^{N}\left(c_n^{(i)}|n\rangle_A|3'\rangle+d_n^{(i)}|n\rangle_B|2'\rangle+f_n^{(i)}|n\rangle|1'\rangle\right)\right], \quad (11)
\end{aligned}$$

where $|3'\rangle$, $|2'\rangle$ and $|1'\rangle$ are the bases after transformation. Left multiplying the state by ${}_A\langle m|$, ${}_B\langle m|$ and $\langle m|$, respectively, we obtain

$$\sum_{i=0}^{3N}k_i c_m^{(i)} = U_{13}\sum_{n=0}^{M}\beta_n^M\,{}_A\langle m|n\rangle, \quad (12)$$

$$\sum_{i=0}^{3N}k_i d_m^{(i)} = U_{23}\sum_{n=0}^{M}\beta_n^M\,{}_B\langle m|n\rangle, \quad (13)$$

$$\sum_{i=0}^{3N}k_i f_m^{(i)} = U_{33}\beta_m^M. \quad (14)$$

By solving the non-homogeneous linear eqs. (12)–(14), we can obtain $\{k_i\}$ $(i=1,\cdots 3N+3)$. The wave function at any time can be written as

$$|\psi(t)\rangle = \sum_{i=0}^{3N}k_i\,\mathrm{e}^{-\mathrm{i}E_i t}\left[\sum_{n=0}^{N}\left(c_n^{(i)}|n\rangle_A|3'\rangle+d_n^{(i)}|n\rangle_B|2'\rangle+f_n^{(i)}|n\rangle|1'\rangle\right)\right]. \quad (15)$$

3 Quantum entanglement

The interaction between the cavity field and the atom can lead to the quantum entanglement. We use the field entropy as a measure of the entanglement between the field and the atom. The entropy is defined as $S=-\mathrm{Tr}\boldsymbol{\rho}\ln\boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the density matrix for a given system [29]. The reduced density matrix of the atom can be written as

$$\begin{aligned}
\boldsymbol{\rho}_a &= \mathrm{Tr}_f\boldsymbol{\rho} \\
&=\sum_{ij=0}^{3N}k_j^*k_i\mathrm{e}^{-\mathrm{i}(E_i-E_j)t}\begin{bmatrix}
\sum_{m=0}^{N}c_m^{(i)}c_m^{(j)*} & \sum_{nm=0}^{N}c_n^{(i)}d_m^{(j)*}\,{}_B\langle m|n\rangle_A & \sum_{nm=0}^{N}c_n^{(i)}f_m^{(j)*}\langle m|n\rangle_A \\
\sum_{nm=0}^{N}d_n^{(i)}c_m^{(j)*}\,{}_A\langle m|n\rangle_B & \sum_{m=0}^{N}d_m^{(i)}d_m^{(j)*} & \sum_{nm=0}^{N}d_n^{(i)}f_m^{(j)*}\langle m|n\rangle_B \\
\sum_{nm=0}^{N}f_n^{(i)}c_m^{(j)*}\,{}_A\langle m|n\rangle & \sum_{nm=0}^{N}f_n^{(i)}d_m^{(j)*}\,{}_B\langle m|n\rangle & \sum_{m=0}^{N}f_m^{(i)}f_m^{(j)*}
\end{bmatrix}. \quad (16)
\end{aligned}$$

From eq. (11) we know the initial state is a pure state, so the entropy of the system is zero and does not change with time. According to Araki and Lieb inequality [2]

$$|S_a - S_f| \leqslant S \leqslant |S_a + S_f|, \quad (17)$$

the atom and field entropies are equal during their evolutions. So we find [30]

$$\begin{aligned}
S_f(t) &= S_a(t) \\
&= -\mathrm{Tr}\{\boldsymbol{\rho}_a(t)\ln\boldsymbol{\rho}_a(t)\} \\
&= -\sum_k \lambda_k \ln\lambda_k, \quad (18)
\end{aligned}$$

where λ_k $(k=1,2,3)$ are the eigenvalues of the atomic reduced density matrix.

4 Results and discussion

The results of our calculation are summarized in the following. When the atom is initially in the ground state (C_a=1, C_b=C_c=0) for ω_0=100g, M=100, the entanglement in general oscillates rapidly as a function of time. For small η (η=0.05), the initial cavity field approximates a coherent state. The mean value of the entanglement increases quickly to its maximum value as can be seen in Figure 2(a).

When η increases, the cavity field changes from an intermediate state to a Fock state. The period of entanglement evolution becomes clear as is shown in Figures 2(b)–2(d). For η=0.9, we observe from Figure 2(d) that disentanglement between the cavity field and the atom may occur. But

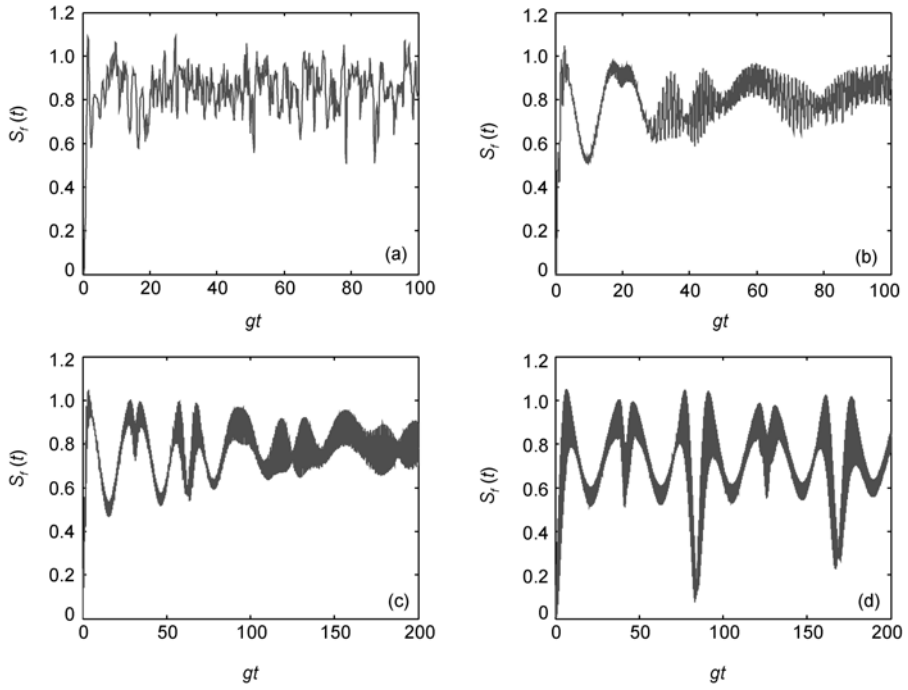

Figure 2 Entanglement evolution when the atom is initially in the ground state for M=100. (a) η=0.05, (b) η=0.2, (c) η=0.5, (d) η=0.9.

such disentanglement can not sustain and disappears as time goes on. One common feature found from these studies is that for all η values, the entanglement reaches its maximum of about 1.1 in a short time.

Next, we study the evolution of entanglement when the atom is initially in superposition states. When the initial state is a superposition of $|2\rangle$ and $|3\rangle$ levels of equal probability $\left(C_a = 0,\ C_b = C_c = \sqrt{2}/2\right)$, the results are plotted in Figure 3. When the initial state is a superposition of all the three levels with equal probability $\left(C_a = C_b = C_c = \sqrt{3}/3\right)$, the entanglement evolution is given in Figure 4.

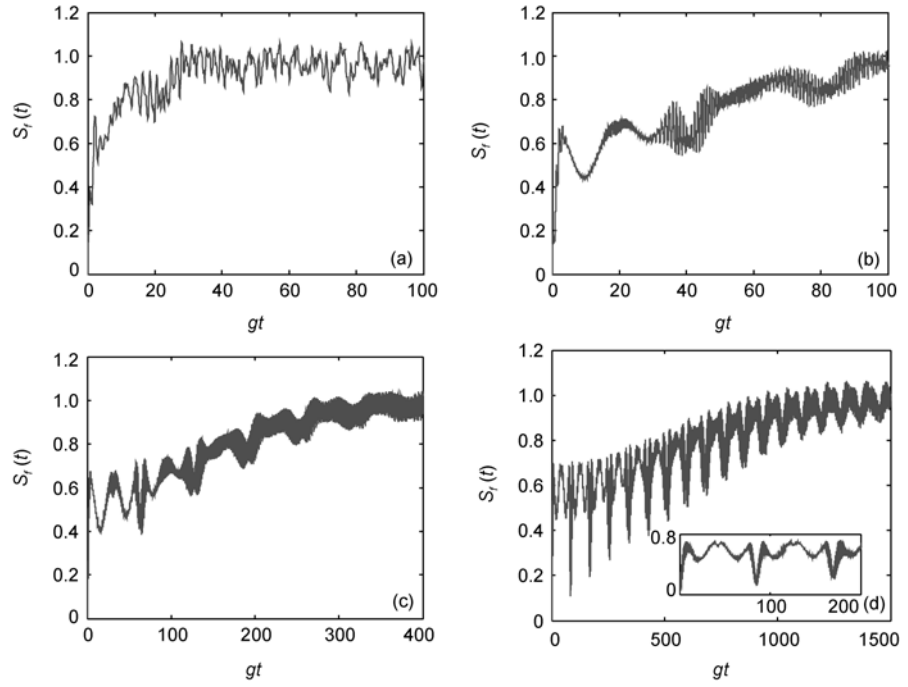

Figure 3 The entanglement evolution when the atom is initially in the superposition state of $|2\rangle$ and $|3\rangle$ for M=100, ω_0=100g. (a) η=0.05, (b) η=0.2, (c) η=0.5, (d) η=0.9.

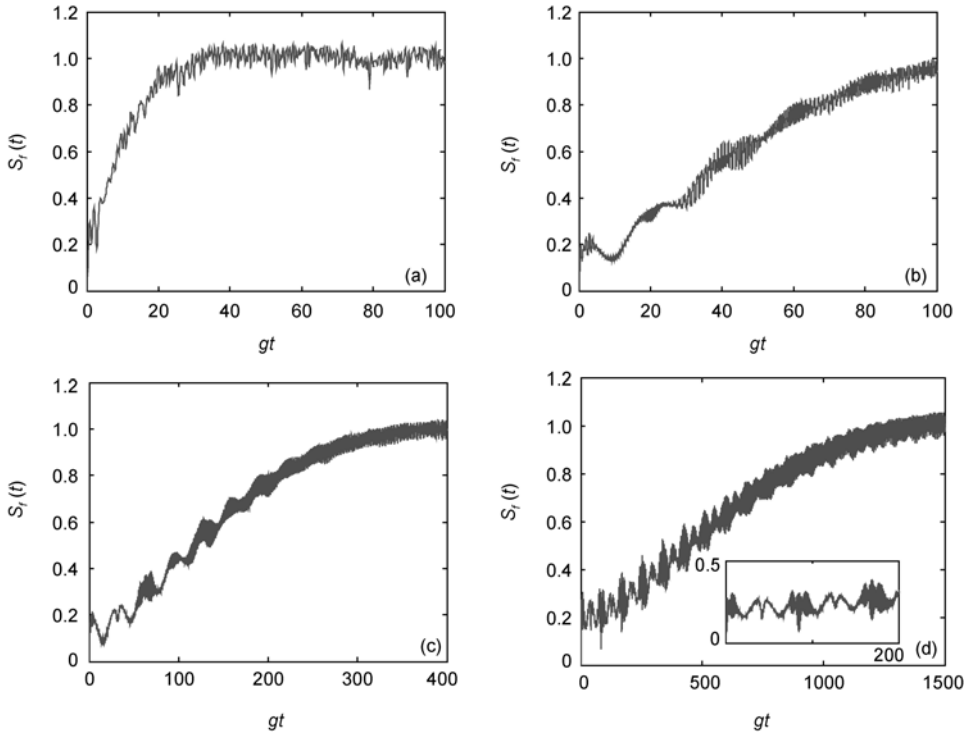

Figure 4 The entanglement evolution when the atom is initially in the superposition state of all the three levels for M=100, ω_0=100g. (a) η=0.05, (b) η=0.2, (c) η=0.5, (d) η=0.9.

Because of the superposition initial states in these cases, the initial maximum entanglement is significantly weaker than that of the previous case. It is about 0.7 in Figure 3 and only 0.3 in Figure 4, while it is about 1.1 in Figure 2. The common feature in both cases is that for η=0.05, the mean entanglement increases quickly at the beginning and becomes saturated at $gt \approx 30$. For larger η, however, it is observed that it takes longer time for the mean entanglement to reach the maximum value due to the superposition of the atomic levels. Of particular interest is the appearance of disentanglement phenomena at the first few periods in Figures 3(d) and 4(d), where the atom is initially in the superposition state. The disentanglement will disappear as time goes on.

The result of ref. [31] is in the case of RWA. By comparison with ref. [31] it can be found that without the RWA, with the increase of η the curve of the entanglement evolution shows small zigzag-shaped oscillation, which disappears in the case of RWA.

5 Conclusion

In this paper, the quantum entanglement between a single mode binomial field and a cascade three-level atom is calculated numerically in the framework of a complete quantum theory and without the RWA. The influences of the different initial states and different sizes of the parameter η on the quantum entanglement are discussed. The results show that if the atom is initially in the ground state, the quantum entanglement will reach the maximum value in a short time. In the case of the atom initially in the superposition state, the degree of the entanglement decreases notably at the early stage. For the two cases, with the increases of η the disentanglement may appear and the period of entanglement evolution becomes clear.

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