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Realization of allowable qeneralized quantum gates

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The most general duality gates were introduced by Long, Liu and Wang and named allowable generalized quantum gates (AGQGs, for short). By definition, an allowable generalized quantum gate has the form of $\mathscr{U} = \sum_{k=0}^{d-1} c_k U_k$, where U_k's are unitary operators on a Hilbert space H and the coefficients c_k's are complex numbers with $|\sum_{k=0}^{d-1} c_k| \leqslant 1$ and $|c_k| \leqslant 1$ for all $k = 0, 1, \ldots, d-1$. In this paper, we prove that an AGQG $\mathscr{U} = \sum_{k=0}^{d-1} c_k U_k$ is realizable, i.e. there are two d by d unitary matrices W and V such that $c_k = W_{0k} V_{k0}$ $(0 \leqslant k \leqslant d-1)$ if and only if $\sum_{k=0}^{d-1} |c_k| \leqslant 1$, in that case, the matrices W and V are constructed.

realizability, allowable generalized quantum gate, Hilbert space, unitary operator, unitary matrix

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1 Introduction

The duality computer, or duality quantum computer has been proposed recently [1], which is a new type of quantum computer exploiting the quantum particle wave duality property. Different from an ordinary quantum computer, a duality computer can offer non-unitary gate operations, hence providing flexibility and ease in constructing computing algorithms. Indeed, a duality computer is a moving quantum computer passing through a multi-slits in general. In a duality computer, there are d slits. In each slit, one can perform different gate operations that are ordinary unitary quantum gates. However, when the wave functions from the slits combine, the resulting operation is the sum of the unitary operations, which is not necessarily unitary. In such a picture, the *duality gate*, or the *generalized gate*, can be written as

$$\sum_{i=0}^{d-1} p_i U_i, \tag{1}$$

where d is the number of slits that the duality computer passes, and p_i is the probability that the duality computer passes through the i-th slit, and

$$\sum_{i=0}^{d-1} p_i = 1. \tag{2}$$

The properties of the duality gates of the form (1) have attracted much attention [2–4]. Some important results have been obtained about the properties of these duality gates and are helpful to studying the capabilities of the duality computer. For example, the set $\mathcal{G}(H)$ of all duality gates of the form (1) on a Hilbert space (state space) H is a convex subset of contract operators on H and its extreme set is just the set $\mathscr{U}(H)$ of all unitary operators on H([4, Corollary 2.2]). Also, in the case where $\dim H < \infty$, $B(H) = \mathbb{R}^+\mathcal{G}(H)$([4, Theorem 2.5]).

A duality quantum computer is a quantum computer that admits two new operations, a divider operator and a combiner operator. The divider operator decomposes the initial wave function into subwaves that are attenuated copies of the initial wave evolving along different paths. Each of the paths can contain quantum gates, represented by unitary operators. After the subwaves pass through the quantum gates, they are collected together by the combiner operator to form a final state. Finally, a measurement is performed on the final state to gain information about the computation. These multiple paths cause additional parallelism in a duality quantum computer and accounts for their superiority over ordinary quantum computers.

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He and Sun in [8] proposed a method for generating the complete circuit of Haar wavelet based MRA by factoring butterfly matrices and conditional perfect shuffle permutation matrices. The pulse sequences of the logic operations used in quantum discrete Fourier transform were designed in [9] for the experiment of nuclear magnetic resonance (NMR), and 2-qubit discrete Fourier transforms are implemented experimentally with NMR. The $\mathbb{C}^M \otimes \mathbb{C}^N$ D-computable state was discussed in ref. [10]. Ref. [11] gave a simple introduction to quantum entanglement, quantum operations and some applications of quantum entanglement and relations between two-qubit entangled states and unitary operations.

Long, Liu and Wang introduced in [7] a type of duality gates, which were called allowable generalized quantum gates (AGQGs, for short) having the form of

$$\mathscr{U} = \sum_{i=0}^{d-1} c_i U_i, \tag{3}$$

where U_k's are unitary operators on a Hilbert space H and the complex coefficients satisfy $|c_i| \leqslant 1$ and $|\sum_{i=0}^{d-1} c_i| \leqslant 1$.

Motivated by the duality quantum gate circuit given by Long's in [7], we introduce the following notation.

Definition 1.1 An AGQG of the form (3) is said to be realizable if there are two d by d unitary matrices W and V such that

$$c_k = W_{0k} V_{k0}, k = 0, 1, \ldots, d-1. \tag{4}$$

A realizable AGQG of the form (3) with condition (4) can be illustrated by Figure 1.

At first the initial state of the quantum computer plus the qudit system is $|\Psi\rangle|0\rangle$. Then we perform the unitary operation V on the auxiliary qudit, and this transforms the state $|\Psi\rangle|0\rangle$ into

$$|\Psi\rangle V|0\rangle = \sum_{i=0}^{d-1} V_{i0}|\Psi\rangle|i\rangle. \tag{5}$$

Then performing the auxiliary qudit controlled operation on the quantum computer, one has the following state

$$\sum_{i=0}^{d-1} V_{i0} U_i|\Psi\rangle|i\rangle. \tag{6}$$

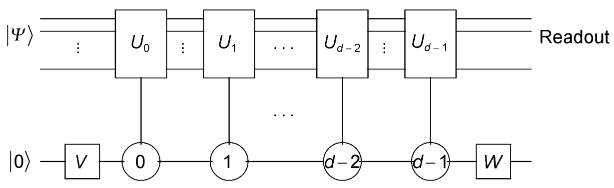

Figure 1 Duality quantum gate circuit, where $|\Psi\rangle$ is the target state and $|0\rangle$ is the d-level controlling qudit. The circles represent the d-level controlling qudit, and the squares represent unitary operations. Unitary $U_0, U_1, \ldots, U_{d-1}$ are activated only when the controlling qudit holds the respective values indicated in circles.

This simulates the simultaneous operations on the duality computer at different slits. To combine the wave functions, one performs another unitary operation W, and the result of this operation is

$$\sum_{i=0}^{d-1} V_{i0} U_i|\Psi\rangle W|i\rangle = \sum_{k=0}^{d-1} \mathscr{U}_k|\Psi\rangle|k\rangle, \tag{7}$$

where

$$\mathscr{U}_k = \sum_{i=0}^{d-1} c_i^k U_i (k = 0, 1, \ldots, d-1), \tag{8}$$

and

$$c_i^k = W_{ki} V_{i0} (i, k = 0, 1, \ldots, d-1). \tag{9}$$

Especially, $c_i = c_i^0 = W_{0i} V_{i0} (i = 0, 1, \ldots, d-1)$.

As a special case of AGQGs, the restricted allowable generalized quantum gates (RAGQGs) were discussed in ref. [12], which have the form of (3) with $0 < \sum_{k=0}^{d-1} |c_k| \leqslant 1$ and some properties of the set $\mathcal{RAGQG}(H)$ of all RAGQGs on H are established there.

In this paper, we will show that an AGQG of the form (3) is realizable if and only if the coefficients c_i's satisfy the following condition:

$$\sum_{i=0}^{d-1} |c_i| \leqslant 1. \tag{10}$$

In that case, the matrices W and V will be constructed.

2 Main results

Let $r \in \mathbb{C}$, $\{a_j\}_{j=1}^n \subset \mathbb{C}$. In this section, we denote by $E(\{a_j\}_{j=1}^n, r)$ the system of equations

$$\begin{cases} \sum_{j=1}^n x_j = r; \\ \sum_{j=1}^n y_j = r; \\ x_i y_i = a_i^2 (i = 1, 2, \ldots, n) \end{cases}$$

and write $E(\{a_j\}_{j=1}^n, 1) = E(\{a_j\}_{j=1}^n)$.

Lemma 2.1 Let $n \in \mathbb{N}$, and

$$x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})^T \in \mathbb{C}^n,$$

$$y^{(1)} = (y_1^{(1)}, y_2^{(1)}, \ldots, y_n^{(1)})^T \in \mathbb{C}^n$$

be unit vectors. Then there exist unitary matrices $W = (w_{ij}) \in M_n(\mathbb{C})$ and $V = (v_{ij}) \in M_n(\mathbb{C})$ such that

$$x_j^{(1)} = w_{1j} (j = 1, 2, \ldots, n) \text{ and } y_i^{(1)} = v_{i1} (i = 1, 2, \ldots, n). \tag{11}$$

Moreover, if in addition $x^{(1)}$ (resp., $y^{(1)}$) $\in \mathbb{R}^n$, we can choose the matrices W (resp., V) $\in M_n(\mathbb{R})$ with condition (11).

Proof Take unit vectors

$$x^{(2)}, x^{(3)}, \ldots, x^{(n)}, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in \mathbb{C}^n$$

such that $\{x^{(i)}\}_{i=1}^n$ and $\{y^{(i)}\}_{i=1}^n$ are orthonormal bases for $\mathbb{C}^n$. Let

$$W=(w_{ij})=\begin{bmatrix} x^{(1)\mathrm{T}} \\ x^{(2)\mathrm{T}} \\ \cdots \\ x^{(n)\mathrm{T}} \end{bmatrix}, V=(v_{ij})=[y^{(1)},y^{(2)},\ldots,y^{(n)}]. \quad (12)$$

Then $W^*W=V^*V=I_n$ with

$$x_j^{(1)}=w_{1j}(j=1,2,\ldots,n) \text{ and } y_i^{(1)}=v_{i1}(i=1,2,\ldots,n).$$

If in addition $x^{(1)}\in\mathbb{R}^n$, then we can choose unit vectors

$$x^{(2)},x^{(3)},\ldots,x^{(n)}\in\mathbb{R}^n,$$

such that $\{x^{(i)}\}_{i=1}^n$ is an orthonormal basis for $\mathbb{R}^n$. Then the matrix $W\in M_n(\mathbb{R})$ defined as in (12) is a real unitary matrix. If in addition $y^{(1)}\in\mathbb{R}^n$, then we can choose unit vectors $y^{(2)},y^{(3)},\ldots,y^{(n)}$ such that $\{y^{(i)}\}_{i=1}^n$ is an orthonormal basis for $\mathbb{R}^n$. Then the matrix $V\in M_n(\mathbb{R})$ defined as in (12) is a real unitary matrix. This completes the proof.

Theorem 2.2 Let $n\geqslant 2, \{c_j\}_{j=1}^n\subset\mathbb{C}$, and $0\leqslant |c_j|:=a_j\leqslant 1, 0\leqslant\sum_{j=1}^n a_j:=r\leqslant 1$. If there exists $j_0\in\{1,2,\ldots,n\}$ such that $c_{j_0}=0$, then there exists $x=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$ and $y=(y_1,y_2,\ldots,y_n)^{\mathrm{T}}\in\mathbb{C}^n$ satisfying $\|x\|=\|y\|=1$ and $x_jy_j=c_j(j=1,2,\ldots,n)$.

Proof We write $c_j=a_j\mathrm{e}^{\mathrm{i}\theta_j}(j=1,2,\ldots,n)$, where θ_j's are real. Take $\{\varepsilon_j\}_{j=1}^n\subset\mathbb{R}^+\cup\{0\}$ with $\varepsilon_{j_0}=0$, $a_j+\varepsilon_j>0(j\neq j_0)$ and $\sum_{j=1}^n\varepsilon_j=1-r$. For all $j=1,2,\ldots,n$, define

$$y_j=\sqrt{a_j+\varepsilon_j},\ x_j=\begin{cases}\sqrt{\dfrac{a_j^2}{a_j+\varepsilon_j}}\mathrm{e}^{\mathrm{i}\theta_j}, & j\neq j_0;\\ \sqrt{1-\sum_{j\neq j_0}\dfrac{a_j^2}{a_j+\varepsilon_j}}, & j=j_0.\end{cases} \quad (13)$$

Then

$$\|y\|^2=\sum_{j=1}^n|y_j|^2=\sum_{j=1}^n(a_j+\varepsilon_j)=\sum_{j=1}^n a_j+\sum_{j=1}^n\varepsilon_j=r+(1-r)=1,$$

$$\|x\|^2=\sum_{j=1}^n|x_j|^2=\sum_{j\neq j_0}\frac{a_j^2}{a_j+\varepsilon_j}+\left(1-\sum_{j\neq j_0}\frac{a_j^2}{a_j+\varepsilon_j}\right)=1,$$

and $x_jy_j=c_j(j=1,2,\ldots,n)$.

Theorem 2.3 Let $n\geqslant 2, c_j=|c_j|\mathrm{e}^{\mathrm{i}\theta_j}\in\mathbb{C}(j=1,2,\ldots,n)$, and $\sum_{j=1}^n|c_j|=1$. Then there exist $x=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}, y=(y_1,y_2,\ldots,y_n)^{\mathrm{T}}\in\mathbb{C}^n$ satisfying $\|x\|=\|y\|=1$ and $x_jy_j=c_j(j=1,2,\ldots,n)$.

Proof Let

$$x=(\sqrt{|c_1|},\sqrt{|c_2|},\ldots,\sqrt{|c_n|}),$$

$$y=(\sqrt{|c_1|}\mathrm{e}^{\mathrm{i}\theta_1},\sqrt{|c_2|}\mathrm{e}^{\mathrm{i}\theta_2},\ldots,\sqrt{|c_n|}\mathrm{e}^{\mathrm{i}\theta_n}).$$

Then $\|x\|=\|y\|=1$ and $x_jy_j=c_j(j=1,2,\ldots,n)$.

Theorem 2.4 Let $c_1,c_2\in\mathbb{C}\setminus\{0\}$ and $0<|c_1|:=a, |c_2|:=b<1$. Then $a+b\leqslant 1 \Leftrightarrow$ there exist unitary matrices $W=(w_{ij})\in M_2(\mathbb{R})$ and $V=(v_{ij})\in M_2(\mathbb{C})$ such that $c_i=w_{1i}v_{i1}(i=1,2)$.

Proof Suppose that there exist unitary matrices $W=(w_{ij})\in M_2(\mathbb{R})$ and $V=(v_{ij})\in M_2(\mathbb{C})$ such that $c_i=w_{1i}v_{i1}(i=1,2)$, then

$$a+b=|c_1|+|c_2|=|w_{11}v_{11}|+|w_{12}v_{21}|\leqslant\sqrt{|w_{11}|^2+|w_{12}|^2}\sqrt{|v_{11}|^2+|v_{21}|^2}=1.$$

Let $a+b\leqslant 1$. Consider the following equation

$$f(m)=m^2-m(a^2+1-b^2)+a^2=0. \quad (14)$$

Observe that

$$f(0)=a^2>0, f(a)=-a((1-a)^2-b^2)\leqslant 0, f(1)=b^2>0,$$

thus there exist m_1,m_2 with $0<m_1\leqslant a\leqslant m_2<1$ such that

$$f(m_1)=f(m_2)=0.$$

Let

$$m=m_1,\ \ n=1-m_1,\ \ x=m_2,\ \ y=1-m_2.$$

Then $m,n,x,y\in(0,1)$, and

$$m+n=1,\ \ mx=m_1m_2=a^2,\ \ x+y=1,$$

$$ny=(1-m_1)(1-m_2)=1-(m_1+m_2)+m_1m_2=1-(a^2+1-b^2)+a^2=b^2.$$

Suppose that $c_1=a\mathrm{e}^{\mathrm{i}\theta_1}, c_2=a\mathrm{e}^{\mathrm{i}\theta_2}$. Let

$$W=\begin{pmatrix}\sqrt{x} & \sqrt{y}\\ -\sqrt{y} & \sqrt{x}\end{pmatrix},\ V=\begin{pmatrix}\sqrt{m}\mathrm{e}^{\mathrm{i}\theta_1} & -\sqrt{n}\mathrm{e}^{-\mathrm{i}\theta_2}\\ \sqrt{n}\mathrm{e}^{\mathrm{i}\theta_2} & \sqrt{m}\mathrm{e}^{-\mathrm{i}\theta_1}\end{pmatrix}.$$

It is easy to check that

$$WW^*=W^*W=VV^*=V^*V=I,$$

and

$$w_{11}v_{11}=\sqrt{x}\sqrt{m}\mathrm{e}^{\mathrm{i}\theta_1}=\sqrt{xm}\mathrm{e}^{\mathrm{i}\theta_1}=a\mathrm{e}^{\mathrm{i}\theta_1}=c_1,$$

$$w_{12}v_{21}=\sqrt{y}\sqrt{n}\mathrm{e}^{\mathrm{i}\theta_2}=\sqrt{yn}\mathrm{e}^{\mathrm{i}\theta_2}=b\mathrm{e}^{\mathrm{i}\theta_2}=c_2.$$

The proof is completed.

Lemma 2.5 Let $a_1,a_2,r>0$. Then $a_1+a_2\leqslant r\Leftrightarrow E(\{a_j\}_{j=1}^2,r)$ has a positive solution.

Proof Suppose that $E(\{a_j\}_{j=1}^2,r)$ has a positive solution (x_1,x_2,y_1,y_2). Then

$$a_1+a_2=\sqrt{x_1y_1}+\sqrt{x_2y_2}\leqslant\sqrt{x_1+x_2}\sqrt{y_1+y_2}=r.$$

Let $a_1+a_2\leqslant r$. Consider the following function

$$g(m)=rm^2-m(a_1^2+r^2-a_2^2)+ra_1^2. \quad (15)$$

Observe that

$$g(0) = ra_1^2 > 0,$$
$$g(a_1) = -a_1((r-a_1)^2 - a_2^2) \leqslant 0,$$
$$g(r) = ra_2^2 > 0,$$

thus there exist m_1, m_2 with $0 < m_1 \leqslant a_1 \leqslant m_2 < r$ such that

$$g(m_1) = g(m_2) = 0.$$

Let

$$x_1 = m_1, \quad x_2 = r - m_1, \quad y_1 = m_2, \quad y_2 = r - m_2.$$

Then $x_1, x_2, y_1, y_2 \in (0, r)$, and

$$x_1 + x_2 = r, \quad x_1 y_1 = m_1 m_2 = a_1^2, \quad y_1 + y_2 = r,$$
$$y_1 y_2 = (r - m_1)(r - m_2) = r^2 - (m_1 + m_2) + m_1 m_2 = r^2 - (a_1^2 + r^2 - a_2^2) + a_1^2 = a_2^2.$$

This shows that (x_1, x_2, y_1, y_2) is a positive solution of $E(\{a_j\}_{j=1}^2, r)$. This completes the proof.

Theorem 2.6 Let $n \geqslant 2$, $\{a_j\}_{j=1}^n \subset \mathbb{R}^+$. Then $\sum_{j=1}^n a_j \leqslant 1 \Leftrightarrow E(\{a_j\}_{j=1}^n)$ has a positive solution $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$.

Proof Use induction.

(1) When $n = 2$, by Lemma 2.5, we know that $a_1 + a_2 \leqslant 1 \Leftrightarrow E(\{a_j\}_{j=1}^2)$ has a positive solution.

(2) Suppose that the conclusion holds for $n = k$. Let $n = k+1$, $\{a_j\}_{j=1}^{k+1} \subset \mathbb{R}^+$.

Claim 1 $\sum_{j=1}^{k+1} a_j \leqslant 1 \Leftrightarrow$ there exists $z_0 \in (a_{k+1}^2, 1)$ such that

$$\sum_{j=1}^{k} a_j \leqslant \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}}. \qquad (16)$$

To see this, we first suppose that there exists $z_0 \in (a_{k+1}^2, 1)$ such that (16) holds. Then

$$\begin{aligned}(z_0 - a_{k+1})^2 \geqslant 0 \Leftrightarrow & z_0 - z_0^2 - a_{k+1}^2 + a_{k+1}^2 z_0 \\ & \leqslant a_{k+1}^2 z_0 + z_0 - 2a_{k+1} z_0 \\ & \Leftrightarrow (1-z_0)(z_0 - a_{k+1}^2) \leqslant (1-a_{k+1})^2 z_0 \\ & \Leftrightarrow \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}} \leqslant 1 - a_{k+1}.\end{aligned}$$

It follows from (16) that

$$\sum_{j=1}^{k} a_j \leqslant \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}} \leqslant 1 - a_{k+1},$$

thus $\sum_{j=1}^{k+1} a_j \leqslant 1$.

Conversely, we suppose that $\sum_{j=1}^{k+1} a_j \leqslant 1$. Consider the following function

$$f(m) = m^2 - m\left[a_{k+1}^2 + 1 - \left(\sum_{j=1}^{k} a_j\right)^2\right] + a_{k+1}^2. \qquad (17)$$

Observe that

$$f(0) = a_{k+1}^2 > 0, \quad f(1) = \left(\sum_{j=1}^{k} a_j\right)^2 > 0,$$

$$f(a_{k+1}) = -a_{k+1}\left[(1 - a_{k+1})^2 - \left(\sum_{j=1}^{k} a_j\right)^2\right] \leqslant 0,$$

thus there exist m_1, m_2 with $0 < m_1 \leqslant a_{k+1} \leqslant m_2 < 1$ such that

$$f(m_1) = f(m_2) = 0.$$

So, the set of all solutions of the inequality $f(m) \leqslant 0$ is $[m_1, m_2] \subset (0, 1)$. Take

$$z_0 \in [a_{k+1}, m_2] \subset [m_1, m_2] \subset (0, 1),$$

then $0 < a_{k+1}^2 < a_{k+1} \leqslant z_0 < 1$ and $f(z_0) \leqslant 0$, that is,

$$f(z_0) = z_0^2 - z_0\left[a_{k+1}^2 + 1 - \left(\sum_{j=1}^{k} a_j\right)^2\right] + a_{k+1}^2 \leqslant 0. \qquad (18)$$

It follows from (18) that

$$\sum_{j=1}^{k} a_j \leqslant \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}}.$$

Claim 2 Let $z_0 \in (a_{k+1}^2, 1)$. Then the following statements are equivalent.

(1) $\sum_{j=1}^{k} a_j \leqslant \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}}$.

(2) The system of equations

$$\begin{cases} \sum_{j=1}^{k} x_j = 1 - z_0; \\ \sum_{j=1}^{k} y_j = 1 - \dfrac{a_{k+1}^2}{z_0}; \\ x_1 y_1 = a_1^2; \\ \quad \cdots \\ x_k y_k = a_k^2, \end{cases}$$

that is,

$$\begin{cases} \sum_{j=1}^{k} \dfrac{x_j}{1 - z_0} = 1; \\ \sum_{j=1}^{k} \dfrac{y_j}{1 - \frac{a_{k+1}^2}{z_0}} = 1; \\ \dfrac{x_1}{1 - z_0} \dfrac{y_1}{1 - \frac{a_{k+1}^2}{z_0}} = \dfrac{a_1^2 z_0}{(1-z_0)(z_0 - a_{k+1}^2)}; \\ \quad \cdots \\ \dfrac{x_k}{1 - z_0} \dfrac{y_k}{1 - \frac{a_{k+1}^2}{z_0}} = \dfrac{a_k^2 z_0}{(1-z_0)(z_0 - a_{k+1}^2)} \end{cases} \qquad (19)$$

has a positive solution $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k)$.

Note that

$$\sum_{j=1}^{k} a_j \leqslant \sqrt{\frac{(1-z_0)(z_0 - a_{k+1}^2)}{z_0}}$$

$$\Leftrightarrow \sum_{j=1}^{k} \sqrt{\frac{a_j^2 z_0}{(1-z_0)(z_0-a_{k+1}^2)}} \leqslant 1.$$

By the assumption that the conclusion holds for $n = k$, we know that Claim 2 holds.

It is obvious that $E(\{a_j\}_{j=1}^{k+1})$ has a positive solution if and only if there exists $z_0 \in (a_{k+1}^2, 1)$ such that (19) has a positive solution. Hence, Theorem 2.6 is true for all $n \geqslant 2$ by using Claim 1 and Claim 2. This completes the proof.

Remark 1 Let $n \geqslant 2, \{a_j\}_{j=1}^n \subset \mathbb{R}^+, \sum_{j=1}^n a_j < 1$. From the proof of Theorem 2.4 and 2.6, we get a positive solution of $E(\{a_j\}_{j=1}^n)$.

Case 1 $n = 2$. In this case, $(x_1, x_2, y_1, y_2) = (m, 1-m, \frac{a_1^2}{m}, 1-\frac{a_1^2}{m})$ is a positive solution of $E(\{a_j\}_{j=1}^2)$, where

$$m = \frac{1}{2}\left[(a_1^2+1-a_2^2) - \sqrt{(a_1^2+1-a_2^2)^2-4a_1^2}\,\right].$$

Case 2 $n \geqslant 3$. In this case,

$$\begin{aligned}&(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\\ &\quad= \left(z, r-z, a_3, \ldots, a_n, \frac{a_1^2}{z}, r-\frac{a_1^2}{z}, a_3, \ldots, a_n\right)\end{aligned}$$

is a positive solution of $E(\{a_j\}_{j=1}^n)$, where

$$r = 1 - \sum_{j=3}^{n} a_j,$$

$$z = \frac{1}{2r}\left[(a_1^2+r^2-a_2^2) - \sqrt{(a_1^2+r^2-a_2^2)^2-4a_1^2r^2}\,\right].$$

Theorem 2.7 Let $n \geqslant 2, \{c_j\}_{j=1}^n \in \mathbb{C}\setminus\{0\}$ and $0 < |c_j| := a_j < 1 (j = 1, 2, \ldots, n)$. Then $\sum_{j=1}^n a_j \leqslant 1$ if and only if there are unitary matrices $W = (w_{ij}) \in M_n(\mathbb{R})$ and $V = (v_{ij}) \in M_n(\mathbb{C})$ such that $c_j = w_{1j}v_{j1} (j = 1, 2, \ldots, n)$.

Proof We write $c_j = |c_j|e^{i\theta_j} = a_j e^{i\theta_j} (j = 1, 2, \ldots, n)$, where θ_j is an argument of c_j for each $j = 1, 2, \ldots, n$.

Suppose that there are unitary matrices $W = (w_{ij}) \in M_n(\mathbb{R})$ and $V = (v_{ij}) \in M_n(\mathbb{C})$ such that $c_j = w_{1j}v_{j1} (j = 1, 2, \ldots, n)$, then

$$\begin{aligned}\sum_{j=1}^n a_j &= \sum_{j=1}^n |c_j| = \sum_{j=1}^n |w_{1j}v_{j1}|\\ &\leqslant \sqrt{\sum_{j=1}^n |w_{1j}|^2}\sqrt{\sum_{j=1}^n |v_{j1}|^2} = 1.\end{aligned}$$

Conversely, let $\sum_{j=1}^n a_j \leqslant 1$. It follows from Theorem 2.6 that $E(\{a_j\}_{j=1}^n)$ has a positive solution, say $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)$. Let

$$x^{(1)} = (\sqrt{\alpha_1}, \sqrt{\alpha_2}, \ldots, \sqrt{\alpha_n})^{\mathrm{T}} \in \mathbb{R}^n,$$

$$y^{(1)} = (\sqrt{\beta_1}e^{i\theta_1}, \sqrt{\beta_2}e^{i\theta_2}, \ldots, \sqrt{\beta_n}e^{i\theta_n})^{\mathrm{T}} \in \mathbb{C}^n.$$

Then $x^{(1)}$ and $y^{(1)}$ are unit vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. Take unit vectors $x^{(2)}, x^{(3)}, \ldots, x^{(n)} \in \mathbb{R}^n, y^{(2)}, y^{(3)}, \ldots, y^{(n)} \in \mathbb{C}^n$ such that $\{x^{(i)}\}_{i=1}^n$, $\{y^{(i)}\}_{i=1}^n$ are orthonormal bases for $\mathbb{R}^n$, $\mathbb{C}^n$, respectively. Let W and V be given by (12). Then $WW^* = V^*V = I_n$, hence W, V are unitary matrices with

$$c_j = a_j e^{i\theta_j} = \sqrt{\alpha_n}(\sqrt{\beta_n}e^{i\theta_n}) = w_{1j}v_{j1} (j = 1, 2, \ldots, n)$$

since $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)$ is a positive solution of $E(\{a_j\}_{j=1}^n)$.

Remark 2 Let $n \geqslant 2, \{c_j\}_{j=1}^n \subset \mathbb{C}$ and $c_j = |c_j|e^{i\theta_j} = a_j e^{i\theta_j} (j = 1, 2, \ldots, n), \sum_{j=1}^n a_j \leqslant 1$. By Remark 1 and Theorem 2.2, 2.4 and 2.7, we can get unitary matrices $W = (w_{ij}) \in M_n(\mathbb{R})$ and $V = (v_{ij}) \in M_n(\mathbb{C})$ such that $c_j = w_{1j}v_{j1} (j = 1, 2, \ldots, n)$.

Case 1 $\{c_j\}_{j=1}^n \subset \mathbb{C}\setminus\{0\}$.

When $n = 2$, we compute from Theorem 2.4 that

$$W = \begin{pmatrix} \sqrt{m} & \sqrt{1-m} \\ -\sqrt{1-m} & \sqrt{m} \end{pmatrix},$$

$$V = \begin{pmatrix} \sqrt{\frac{a_1^2}{m}}e^{i\theta_1} & -\sqrt{1-\frac{a_1^2}{m}}e^{-i\theta_2} \\ \sqrt{1-\frac{a_1^2}{m}}e^{i\theta_2} & \sqrt{\frac{a_1^2}{m}}e^{-i\theta_1} \end{pmatrix},$$

where $m = \left[(a_1^2+1-a_2^2) - \sqrt{(a_1^2+1-a_2^2)^2-4a_1^2}\,\right]/2$.

When $n \geqslant 3$, we compute from Theorem 2.7 that

$$W = \begin{pmatrix} z & r-z & a_3 & \cdots & a_n \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2n} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \cdots & w_{nn} \end{pmatrix},$$

$$V = \begin{pmatrix} \frac{a_1^2}{z}e^{i\theta_1} & v_{12} & v_{13} & \cdots & v_{1n} \\ r-\frac{a_1^2}{z}e^{i\theta_2} & v_{22} & v_{23} & \cdots & v_{2n} \\ a_3e^{i\theta_3} & v_{32} & v_{33} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_ne^{i\theta_n} & v_{n2} & v_{n3} & \cdots & v_{nn} \end{pmatrix},$$

where

$$r = 1 - \sum_{j=3}^{n} a_j,$$

$$z = \frac{1}{2r}\left[(a_1^2+r^2-a_2^2) - \sqrt{(a_1^2+r^2-a_2^2)^2-4a_1^2r^2}\,\right],$$

and $\{w_{ij}\}_{i=2,j=1}^n \subset \mathbb{R}, \{v_{ij}\}_{i=1,j=2}^n \subset \mathbb{C}$ such that $W \in M_n(\mathbb{R})$, $V \in M_n(\mathbb{C})$ are unitary matrices.

Case 2 There exists $j_0 \in \{1, 2, \ldots, n\}$ such that $c_{j_0} = 0$.

$$W = (w_{ij}),\ V = (v_{ij})$$

with

$$w_{1j} = \sqrt{a_j + \varepsilon_j}\,,\ v_{j1} = \begin{cases} \sqrt{\dfrac{a_j^2}{a_j + \varepsilon_j}}\mathrm{e}^{\mathrm{i}\theta_j}, & j \neq j_0; \\ \sqrt{1 - \sum_{j \neq j_0} \dfrac{a_j^2}{a_j + \varepsilon_j}}\,, & j = j_0, \end{cases}$$

where $\{\varepsilon_j\}_{j=1}^n \subset \mathbb{R}^+ \cup \{0\}$ satisfy $\varepsilon_{j_0} = 0,\ a_j + \varepsilon_j > 0 (j \neq j_0)$, $\sum_{j=1}^n \varepsilon_j = 1 - \sum_{j=1}^n a_j$, and $\{w_{ij}\}_{i=2,j=1}^n \subset \mathbb{R}, \{v_{ij}\}_{i=1,j=2}^n \subset \mathbb{C}$ such that $W \in M_n(\mathbb{R}), V \in M_n(\mathbb{C})$ are unitary matrices.

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