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Transverse nonlinear vibrations of a circular spinning disk with a varying rotating speed†

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We analyze the transverse nonlinear vibrations of a rotating flexible disk subjected to a rotating point force with a periodically varying rotating speed. Based on Hamilton's principle, the nonlinear governing equations of motion (coupled equations among the radial, tangential and transverse displacements) are derived for the rotating flexible disk. When the in-plane inertia is ignored and a stress function is introduced, the three nonlinearly coupled partial differential equations are reduced to two nonlinearly coupled partial differential equations. According to Galerkin's approach, a four-degree-of-freedom nonlinear system governing the weakly split resonant modes is derived. The resonant case considered here is 1:1:2:2 internal resonance and a critical speed resonance. The primary parametric resonance for the first-order sin and cos modes and the fundamental parametric resonance for the second-order sin and cos modes are also considered. The method of multiple scales is used to obtain a set of eight-dimensional nonlinear averaged equations. Based on the averaged equations, using numerical simulations, the influence of different parameters on the nonlinear vibrations of the spinning disk is detected. It is concluded that there exist complicated nonlinear behaviors including the periodic, period-n and multi-pulse type chaotic motions for the spinning disk with a varying rotating speed. It is also found that among all parameters, the damping and excitation have great influence on the nonlinear responses of the spinning disk with a varying rotating speed.

circular spinning disk, transverse nonlinear vibration, varying rotating speed, chaotic motion

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1 Introduction

Spinning disks are widely used components in mechanical engineering from early circular saw blades, turbine rotors to recent floppy and memory disk drives. In computer disk drives, excessive transverse vibrations degrade the signal from the read/write head and can destroy the disk by a "head-crash." In the wood cutting process, saw blade vibration results in a greater kerfs loss and can warp the saw permanently through overheating.

It is well-known that each vibration mode of a stationary disk is split into a forward traveling wave (FTW) and a backward traveling wave (BTW) as the disk starts rotating [1–3]. The FTW travels in the same direction as the disk rotation, while the BTW travels in the opposite direction. At certain disk rotation speeds, the frequencies of the BTWs of specific circumferential wave number vanish, which are called critical speeds. Johnson [4] showed that in the presence of a constant, space-fixed transverse load, large amplitude stationary waves are excited at the critical speeds. This phenomenon is called the critical speed resonance.

Initiated by Lamb and Southwell [2] in 1921, early research works on the dynamics of spinning disks were concerned with verifying the critical speed theory and the effects of thermally induced membrane stress field on the natural frequencies of the spinning disk. Southwell [3] continued to analyze the free vibration of a spinning annular plate with clamped inner edge and free outer edge. Nowin-

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ski [5] derived the nonlinear partial differential equations governing large amplitude vibrations of the spinning disks and considered both bending stiffness and mid-plane stretching, based on the small stretch, moderate rotation assumption of the von Karman plate theory. Efstathiades [6] extended the Nowinski's work to the imperfect disks and used the Galerkin's procedure to find the solution. Eversman and Dodson [7] analyzed the nonlinear free vibrations of a centrally clamped spinning circular disk by using a power-series solution of the eigenfunctions. Barash and Chen [8] used a numerical method to study the nonlinear responses of a freely rotating disk. The influence of moving massive loads on stationary circular disks was first studied by Iwan and Stahl [9] and later extended by Iwan and Moeller [10] to the case of spinning elastic disks. Using a series solution, Adams [11] calculated the critical speeds at which a uniform flexible elastic disk is unable to support arbitrary spatially fixed transverse loads.

Spinning disks are usually subjected to the in-plane loading in application. Mote [12,13] studied the free vibration of circular disks subjected to initial in-plane stresses introduced purposefully by rolling or to thermal membrane stresses resulting from the cutting process. The free vibration of a spinning disk under a concentrated radial edge load was first investigated by Carlin and his coworkers [14]. A further study of effects of load parameters, such as friction force, transverse mass, damping and stiffness, and the analogous pitching parameters on the free vibration of a spinning disk was conducted by Chen and Bogy [15]. Recently, Nayfeh et al. [16] analyzed the transverse vibrations of a spinning disk with constant velocity and the stability characteristics of the system. Luo [17] developed an approximate theory of thin plate based on an assumed displacement field, and used it to study a spinning disk. Raman and Mote [18] studied the large-amplitude nonlinear oscillations of an imperfect flexible circular plate spinning near a critical speed resonance and analyzed the near resonant response of a repeated pair of backward and forward traveling waves. Angoshtari and Jalali [19] used a third-order perturbation theory and Melnikov method to prove the existence of chaos in spinning circular disks subjected to a lateral point load.

The spinning speeds of the disk considered in all aforementioned references are constant. However, in the real world, the spinning speeds of the disks usually fluctuate within a small interval. The work given by Kammer and Schlack [20] was the first research concerning this topic. Malhotra et al. [21] used the energy-phase method to investigate the multi-pulse global dynamics of flexible spinning disks which are imperfect and parametrically excited in the spin rate. Eid et al. [22] determined the critical speeds of a moderately thick circular spinning disk using the Mindlin plate theory. DasGupta et al. [23] studied the dynamics of a spinning thin axisymmetric annular disk with an external ring and investigated the effect of the ring on the critical speeds of the disk. Heo et al. [24] investigated the dynamic time responses of a flexible spinning disk of which the axis of rotation is misaligned with the axis of symmetry using the finite element method. Michalek et al. [25] detailed the derivation and solution of the governing equation for linear transverse vibrations of a thin perfectly conducting rotating disk subjected to an axisymmetric in-plane magnetic field having a circumferential flux pattern. Chen et al. [26] extended some former solutions to the case of a spinning disk under stationary edge tractions. Baddour et al. [27] considered the effect of including the in-plane inertia of the disk on the resulting nonlinear dynamics and constructed approximate solutions that capture the new dynamics.

In the present paper, firstly, the nonlinear governing equations of the transverse motion are derived for a clamped-free, rotating thin flexible disk subjected to a rotating point force with a periodically varying rotating speed. The Galerkin's procedure is utilized to obtain a set of ordinary differential equations, which govern the weakly split resonant modes in the case of 1:1:2:2 internal resonance. Then, we employ the method of multiple scales to obtain a set of eight-dimensional averaged equations within the concerns of a critical speed resonance, the primary parametric resonance for the first-order sin and cos modes and the fundamental parametric resonance for the second-order sin and cos modes. Numerical simulations are performed to study the nonlinear dynamic responses of the spinning disk with a varying rotating speed. The effects of different parameters on the nonlinear responses of the spinning disk are detected based on the averaged equations. It is found from the numerical results that there exist the periodic motions, period-n motions and chaotic motions for the spinning disks with a varying rotating speed.

2 Formulation

Consider a flat, circular elastic disk of uniform thickness h and density ρ. The disk is fully clamped at its inner radius a, and is free at the outer radius b. The disk rotates at the angular velocity:

$$\Omega(t) = \Omega_0 + \Omega_1 \cos \omega t, \tag{1}$$

where Ω_0 is the mean angular velocity, Ω_1 is the amplitude of a small periodic perturbation of the spin rate, and $|\Omega_1| \ll \Omega_0$.

The geometry of a spinning disk is shown in Figure 1.

We assume that the disk is subjected to a transverse normal load $q(r, \theta, t)$ and the transverse deflection of the disk is large. A polar coordinate system O-r-θ, embedded in and rotating with the disk, is located at the middle plane of the disk. In the following part, the Hamilton's principle will

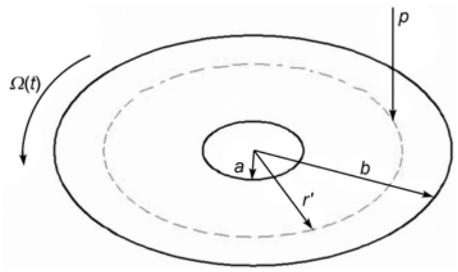

Figure 1 Geometry of a spinning disk.

be used to get the equations of motion and boundary conditions of the spinning disk.

The strain energy per unit volume is denoted by U and written by

$$U=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}, \tag{2}$$

where σ_{ij} and ε_{ij} are the stress and strain, respectively.

Small strains are assumed so that the Hookean stress-strain relationship will be taken. According to the Kirchhoff plate theory, the relationship between the displacement of an arbitrary point u_r, u_θ and u_z and the displacements of the middle surface u, v and w is given as:

$$u_r(r,\theta,z,t)=u(r,\theta,t)-z\frac{\partial w(r,\theta,t)}{\partial r}, \tag{3a}$$

$$u_\theta(r,\theta,z,t)=v(r,\theta,t)-z\frac{\partial w(r,\theta,t)}{r\partial\theta}, \tag{3b}$$

$$u_z(r,\theta,z,t)=w(r,\theta,t). \tag{3c}$$

The relation between strain and displacement is required to express the strain energy in terms of displacements. The von Karman plate theory leads to the following nonlinear strain-displacement expressions:

$$\varepsilon_r=\frac{\partial u_r}{\partial r}+\frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2, \tag{4a}$$

$$\varepsilon_\theta=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{2r^2}\left(\frac{\partial u_r}{\partial\theta}\right)^2, \tag{4b}$$

$$\varepsilon_{r\theta}=\frac{1}{2r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta+r\frac{\partial u_\theta}{\partial r}+\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial\theta}\right). \tag{4c}$$

Using the displacements of the middle surface u, v and w to describe the strain, we obtain

$$\varepsilon_r=\frac{\partial u}{\partial r}-z\frac{\partial^2 w}{\partial r^2}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \tag{5a}$$

$$\varepsilon_\theta=\frac{\partial v}{r\partial\theta}-z\frac{\partial^2 w}{r^2\partial\theta^2}+\frac{u}{r}-z\frac{\partial w}{r\partial r}+\frac{1}{2}\left(\frac{\partial w}{r\partial\theta}\right)^2, \tag{5b}$$

$$\varepsilon_{r\theta}=\frac{\partial u}{r\partial\theta}-z\frac{2\partial^2 w}{r\partial r\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}+z\frac{\partial w}{r^2\partial\theta}+\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta}. \tag{5c}$$

According to the Hook's law, the stresses are represented as:

$$\sigma_r=\frac{E}{1-\gamma^2}\left(\varepsilon_r+\gamma\varepsilon_\theta\right), \tag{6a}$$

$$\sigma_\theta=\frac{E}{1-\gamma^2}\left(\varepsilon_\theta+\gamma\varepsilon_r\right), \tag{6b}$$

$$\sigma_{r\theta}=\frac{E}{2(1+\gamma)}\varepsilon_{r\theta}, \tag{6c}$$

where γ is the Poisson ratio.

The strain energy of the flexible disk is obtained as follows:

$$U=\frac{1}{2}\int_A \boldsymbol{\varepsilon}^{\mathrm{T}}\sigma\,\mathrm{d}A, \tag{7}$$

where A is the area of the disk.

Thus, the strain energy of the disk is given by

$$U=\frac{1}{2}\int_0^{2\pi}\int_{r_1}^{r_2}\left(D_0U_1+D_1U_2+GU_3\right)\mathrm{d}r\mathrm{d}\theta, \tag{8}$$

where

$$D_0=\frac{Eh}{1-\gamma^2},\quad D=\frac{Eh^3}{12\left(1-\gamma^2\right)},\quad G=\frac{E}{2(1+\gamma)}, \tag{9a}$$

$$\begin{aligned}U_1={}&\left(\frac{\partial u}{\partial r}\right)^2+\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{\gamma}{r}\frac{\partial u}{\partial r}\frac{\partial v}{\partial\theta}+\gamma\frac{u}{r}\frac{\partial u}{\partial r}+\gamma\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial\theta}\right)^2\\&+\frac{1}{4}\left(\frac{\partial w}{\partial r}\right)^4+\frac{1}{2}\gamma\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial v}{r\partial\theta}+\frac{1}{2}\gamma\frac{u}{r}\left(\frac{\partial w}{\partial r}\right)^2\\&+\frac{1}{4}\gamma\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial w}{r\partial\theta}\right)^2+\frac{1}{r^2}\left(\frac{\partial v}{\partial\theta}\right)^2+\frac{2u}{r^2}\frac{\partial v}{\partial\theta}\\&+\frac{1}{2r}\left(\frac{\partial w}{\partial\theta}\right)^2\frac{\partial v}{\partial\theta}+\frac{\gamma}{r}\frac{\partial v}{\partial\theta}\frac{\partial u}{\partial r}+\frac{1}{2r}\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial v}{\partial\theta}+\frac{u^2}{r^2}\\&+\frac{1}{2}\frac{u}{r^3}\left(\frac{\partial w}{\partial\theta}\right)^2+\frac{\gamma u}{r}\frac{\partial u}{\partial r}+\frac{1}{2}\frac{\gamma u}{r}\left(\frac{\partial w}{\partial r}\right)^2\\&+\frac{1}{2r^3}\left(\frac{\partial w}{\partial\theta}\right)^2\frac{\partial v}{\partial\theta}+\frac{u}{2r^3}\left(\frac{\partial w}{\partial\theta}\right)^2+\frac{1}{4r^4}\left(\frac{\partial w}{\partial\theta}\right)^4\\&+\frac{\gamma}{2r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\frac{\partial u}{\partial r}+\frac{\gamma}{4r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\left(\frac{\partial w}{\partial r}\right)^2,\end{aligned} \tag{9b}$$

$$\begin{aligned}U_2={}&\frac{\partial^4 w}{\partial r^4}+\frac{\gamma}{r^2}\frac{\partial^4 w}{\partial r^2\partial\theta^2}+\gamma\frac{\partial^3 w}{r\partial r^3}+\frac{1}{r^4}\left(\frac{\partial^2 w}{\partial\theta^2}\right)^2+\frac{2}{r^3}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial w}{\partial r}\\&+\frac{\gamma}{r^2}\frac{\partial^2 w}{\partial\theta^2}\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{\gamma}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial w}{\partial r},\end{aligned} \tag{9c}$$

$$U_3 = h\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \frac{h}{r}\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial r} - 2\frac{hv}{r^2}\frac{\partial u}{\partial \theta} + 2\frac{h}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial^2 w}{\partial r\partial \theta}$$
$$+\frac{4h^3}{r^2}\left(\frac{\partial^2 w}{\partial r\partial \theta}\right)^2 - \frac{2h^3}{r^3}\frac{\partial^2 w}{\partial r\partial \theta}\frac{\partial w}{\partial \theta} + \frac{h}{r}\frac{\partial v}{\partial r}\frac{\partial u}{\partial \theta} + h\left(\frac{\partial v}{\partial r}\right)^2$$
$$-2\frac{hv}{r}\frac{\partial v}{\partial r} + \frac{2h}{r}\frac{\partial v}{\partial r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{hv^2}{r^2} - \frac{2hv}{r}\frac{\partial^2 w}{\partial r\partial \theta}$$
$$+\frac{h^3}{r^4}\left(\frac{\partial w}{\partial \theta}\right)^2 + \frac{h}{r^2}\left(\frac{\partial^2 w}{\partial r\partial \theta}\right)^2. \tag{9d}$$

We suppose that e_r, e_θ and e_z are unit vectors in the r, θ and z directions, respectively, where the direction of e_θ points to the direction of θ increasing. Suppose that S_1 is a space-fixed coordinate system and S_2 is a co-rotating system which is fixed to the disk. The angular velocity vector of the co-rotating system is $\boldsymbol{\omega} = \Omega e_z$. Therefore, the time derivatives of the co-rotating unit vectors are

$$\frac{\mathrm{d}e_r}{\mathrm{d}t} = \boldsymbol{\omega}\times e_r = \Omega e_\theta, \tag{10a}$$

$$\frac{\mathrm{d}e_\theta}{\mathrm{d}t} = \boldsymbol{\omega}\times e_\theta = -\Omega e_r, \tag{10b}$$

$$\frac{\mathrm{d}e_z}{\mathrm{d}t} = \boldsymbol{\omega}\times e_z = 0. \tag{10c}$$

If the original position of a particle on the disk is described as $s_0 = re_r + ze_z$, the deformed position of the particle can be written as:

$$s = (r+u_r)e_r + u_\theta e_\theta + (z+u_z)e_z. \tag{11}$$

The velocity of the particle is given by

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = (r\Omega + u_r\Omega + \dot{u}_\theta)e_\theta$$
$$+(\dot{u}_r - u_\theta\Omega - z\Omega)e_r + \dot{u}_z e_z. \tag{12}$$

The kinetic energy of the disk is obtained as:

$$T = \frac{1}{2}\rho\int_0^{2\pi}\int_{r_1}^{r_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} v\cdot v\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z, \tag{13}$$

where

$$v\cdot v = (r\Omega + u_r\Omega + \dot{u}_\theta)^2 + (\dot{u}_r - u_\theta\Omega - z\Omega)^2 + \dot{u}_z^{\,2},$$

since the vectors are orthogonal.

Thus, the kinetic energy of the disk is expressed as follows:

$$T = \frac{1}{2}\rho\int_0^{2\pi}\int_{r_1}^{r_2}\left(hT_1 + \frac{1}{3}h^3T_2\right)\mathrm{d}\theta\,\mathrm{d}r, \tag{14}$$

where

$$T_1 = \left(\frac{\partial u}{\partial t}\right)^2 - 2v\Omega\frac{\partial u}{\partial t} + v^2\Omega^2 + \left(\frac{\partial v}{\partial t}\right)^2 + 2\Omega r\frac{\partial v}{\partial t}$$
$$+2\Omega u\frac{\partial v}{\partial t} + \Omega^2 r^2 + 2\Omega^2 ru + \Omega^2 u^2 + \left(\frac{\partial w}{\partial t}\right)^2, \tag{15}$$

$$T_2 = \frac{1}{2}\left(\sigma_2 + \frac{1}{4}b_{12}\right)x_5 - \frac{1}{4}b_{15}\left(x_1^{\,2} + x_2^{\,2}\right)x_5$$
$$-\frac{1}{4}b_{16}\left(x_3^{\,2} + x_4^{\,2}\right)x_5 - \frac{3}{8}b_{14}\left(x_5^2 + x_6^2\right)x_5. \tag{16}$$

The non-conservative virtual work done by the transverse force p in the z direction is of the form:

$$W = \int_0^{2\pi}\int_{r_1}^{r_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} p\delta w\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}z. \tag{17}$$

Therefore, the equation of motion and the boundary conditions for a clamped-free, rotating thin flexible disk subjected to a rotating point force and with a periodically varying rotating speed are derived by using the Hamilton's principle:

$$\delta\int_{t_1}^{t_2}(T - U + W)\,\mathrm{d}t = 0, \tag{18}$$

where t_1 and $x_6 = x_{60} + \delta_6$ are arbitrary time.

Introducing eqs. (8), (13) and (17) into eq. (18), the equations of motion are obtained for a clamped-free, rotating thin flexible disk as follows:

$$\rho h\left[\frac{\partial^2 u}{\partial t^2} - v\frac{\partial \Omega}{\partial t} - \Omega^2(r+u) - 2\Omega\frac{\partial v}{\partial t}\right]$$
$$= D_0\left[\frac{(1+\gamma)}{2r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{\partial^2 u}{\partial r^2} - \frac{(3-\gamma)}{2r^2}\frac{\partial v}{\partial \theta}\right.$$
$$\left.-\frac{(1+\gamma)}{2r^3}\left(\frac{\partial w}{\partial \theta}\right)^2 - \frac{u}{r^2} + \frac{\partial u}{r\partial r} + \frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} + \frac{(1+\gamma)}{2r^2}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial r\partial\theta}\right]$$
$$+\frac{h}{r}G\left[\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial^3 w}{\partial r\partial\theta} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial \theta^2}\right], \tag{19a}$$

$$\rho h\left[\frac{\partial^2 v}{\partial t^2} + (r+u)\frac{\partial \Omega}{\partial t} - \Omega^2 v + 2\Omega\frac{\partial u}{\partial t}\right]$$
$$= D_0\left[\frac{(1+\gamma)}{2r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial\theta} + \frac{(3-\gamma)}{2r^2}\frac{\partial u}{\partial \theta} + \frac{(1+\gamma)}{2r}\frac{\partial^2 u}{\partial r\partial\theta}\right.$$
$$\left.+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^3}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial \theta^2}\right]$$
$$+\frac{h}{r}G\left[r\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial r} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial r^2}\right], \tag{19b}$$

$$\rho h\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w - D_0\left[\frac{1+\gamma}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r} + \frac{\partial^2 u}{\partial r^2}\frac{\partial w}{\partial r} + \frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}\right.$$

$$+\frac{1}{2r}\left(\frac{\partial w}{\partial r}\right)^3+\frac{3}{2}\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+\frac{\gamma}{r}\frac{\partial v}{\partial \theta}\frac{\partial^2 w}{\partial r^2}+\frac{\gamma}{r}u\frac{\partial^2 w}{\partial r^2}$$

$$-\frac{\gamma}{2r^3}\left(\frac{\partial w}{\partial \theta}\right)^2\frac{\partial w}{\partial r}+\frac{2\gamma}{r^2}\frac{\partial w}{\partial \theta}\frac{\partial^2 w}{\partial \theta\partial r}\frac{\partial w}{\partial r}+\frac{\gamma}{2}\left(\frac{\partial w}{r\partial \theta}\right)^2\frac{\partial^2 w}{\partial r^2}$$

$$+\frac{1}{r^3}\frac{\partial^2 v}{\partial \theta^2}\frac{\partial w}{\partial \theta}+\frac{1}{r^3}\frac{\partial v}{\partial \theta}\frac{\partial^2 w}{\partial \theta^2}+\frac{1}{r^3}u\frac{\partial^2 w}{\partial \theta^2}-\frac{3}{2r^4}\left(\frac{\partial w}{\partial \theta}\right)^2\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{\gamma}{r^2}\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial \theta^2}+\frac{\gamma}{2r^2}\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial \theta^2}\Bigg]-Gh\Bigg[\frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial^2 w}{\partial \theta^2}$$

$$+\frac{1}{r}\frac{\partial^2 v}{\partial r^2}\frac{\partial w}{\partial \theta}+\frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial^2 w}{\partial \theta\partial r}+\frac{v}{r^3}\frac{\partial w}{\partial \theta}-\frac{v}{r^2}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r\partial \theta}\frac{\partial w}{\partial \theta}$$

$$-\frac{1}{r^2}\frac{\partial^3 w}{\partial r^2\partial \theta}\frac{\partial w}{\partial \theta}+\frac{2}{r^2}\left(\frac{\partial^2 w}{\partial r\partial \theta}\right)^2+\frac{\partial^2 u}{r^2\partial \theta^2}\frac{\partial w}{\partial r}+\frac{\partial u}{r^2\partial \theta}\frac{\partial^2 w}{\partial r\partial \theta}$$

$$+\frac{1}{r}\frac{\partial v}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial v}{\partial \theta}\frac{\partial w}{\partial r}-\frac{1}{r^2}v\frac{\partial^2 w}{\partial r\partial \theta}+\frac{1}{r^2}\frac{\partial^3 w}{\partial r\partial \theta^2}\frac{\partial w}{\partial r}\Bigg]$$

$$-\left(D_0\gamma+Gh\right)\left(\frac{1}{r}\frac{\partial^2 v}{\partial \theta\partial r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta\partial r}\frac{\partial w}{\partial r}\right)$$

$$-\left(Gh-D_0\right)\frac{1}{r^3}\frac{\partial u}{\partial \theta}\frac{\partial w}{\partial \theta}=p. \tag{19c}$$

The associated boundary conditions are given by the following equations:

at $r=a$, $w=0$,

$$\frac{\partial w}{\partial t}=0,\quad \frac{\partial^2 \varphi}{\partial r^2}-\gamma\left(\frac{1}{r}\frac{\partial \varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right)=0,$$
$$\frac{\partial^3 \varphi}{\partial r^3}+\frac{1}{r}\frac{\partial^2 \varphi}{\partial r^2}-\frac{1}{r^2}\frac{\partial \varphi}{\partial r}+\frac{2+\gamma}{r^2}\frac{\partial^3 \varphi}{\partial r\partial \theta^2}-\frac{3-\gamma}{r^3}\frac{\partial^2 \varphi}{\partial \theta^2}=0; \tag{20}$$

at $r=b$,

$$\frac{\partial^2 w}{\partial r^2}+\frac{\gamma}{r}\frac{\partial w}{\partial r}+\frac{\gamma}{r^2}\frac{\partial^2 w}{\partial \theta^2}=0,$$
$$\frac{\partial}{\partial r}\left(\nabla^2 w\right)+\frac{1-\gamma}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)=0, \tag{21}$$
$$\frac{\partial \varphi}{\partial r}+\frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta^2}=0,\quad \frac{\partial^2 \varphi}{\partial r\partial \theta}-\frac{1}{r}\frac{\partial \varphi}{\partial \theta}=0,$$

where ∇^2 is the two-dimensional Laplace operator in the polar coordinate.

The internal forces per unit length of the middle surface is represented as [25]:

$$q_r=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_r \mathrm{d}z$$
$$=D_0\left[\frac{\partial u}{\partial r}+\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{\gamma}{r}\frac{\partial v}{\partial \theta}+\frac{\gamma}{r}u+\frac{\gamma}{2}\left(\frac{\partial w}{r\partial \theta}\right)^2\right], \tag{22}$$

$$q_\theta=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_\theta \mathrm{d}z$$
$$=D_0\left[\frac{\partial v}{r\partial \theta}+\frac{u}{r}-\frac{1}{2}\left(\frac{\partial w}{r\partial \theta}\right)^2+\gamma\frac{\partial u}{\partial r}+\frac{\gamma}{2}\left(\frac{\partial w}{\partial r}\right)^2\right], \tag{23}$$

$$q_{r\theta}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{r\theta}\mathrm{d}z=Gh\left(\frac{\partial u}{r\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial^2 w}{\partial r\partial \theta}\right). \tag{24}$$

To make the equations of motion simpler, we omit the in-plane inertias terms, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 v}{\partial t^2}$, as well as the Coriolis terms, $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ from eqs. (19a) and (19b). With a small displacement, we replace (r+u) with r and omit $v\frac{\partial \Omega}{\partial t}$ and $v\Omega^2$. Then, eqs. (19a) and (19b) can be rewritten as:

$$\frac{\partial q_r}{\partial r}+\frac{1}{r}\frac{\partial q_{r\theta}}{\partial \theta}+\frac{q_r-q_\theta}{r}+\rho r\Omega^2=0, \tag{25}$$

$$\frac{\partial q_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{2q_{r\theta}}{r}=\rho r\frac{\partial \Omega}{\partial t}, \tag{26}$$

which actually are the well-known equations governing the in-plane stresses q_r, q_θ and $q_{r\theta}$ of the disk due to rotation.

Eq. (19c) is rewritten as:

$$\rho h\frac{\partial^2 w}{\partial t^2}+D\nabla^4 w-\frac{\partial}{r\partial r}\left(rq_r\frac{\partial w}{\partial r}+q_{r\theta}\frac{\partial w}{r\partial \theta}\right)$$
$$-\frac{\partial}{r\partial \theta}\left(q_{r\theta}\frac{\partial w}{\partial r}+q_\theta\frac{\partial w}{r\partial \theta}\right)=p. \tag{27}$$

Eqs. (25) and (26) are identically satisfied if we derive the stress component from a stress function Φ in the following manner:

$$q_r=\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}+q_r^*, \tag{28a}$$

$$q_\theta=\frac{\partial^2 \Phi}{\partial r^2}+q_\theta^*, \tag{28b}$$

$$q_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)+q_{r\theta}^*. \tag{28c}$$

Consequently, the resulting stress field is the sum of two parts in which one refers to the axisymmetric stresses induced due to the centripetal acceleration in a symmetric disk, and the other refers to the stresses induced due to the angular acceleration. Therefore, we have

$$q_r^* = q_{r1}^* + q_{r2}^*, \tag{29}$$

$$q_\theta^* = q_{\theta 1}^* + q_{\theta 2}^*, \tag{30}$$

$$q_{r\theta}^* = q_{r\theta 1}^* + q_{r\theta 2}^*, \tag{31}$$

where the expressions of q_{r1}^*, q_{r2}^*, $q_{\theta 1}^*$, $q_{r\theta 2}^*$, $q_{r\theta 1}^*$ and $q_{r\theta 2}^*$ are given as [19,29]:

$$\begin{aligned}
q_{r1}^* &= -\frac{\rho h b^2 \Omega^2 (3+\gamma)}{8}\left[\left(\frac{r^2}{b^2}-1\right)+\beta_1\left(1-\frac{b^2}{r^2}\right)\right],\\
q_{\theta 1}^* &= -\frac{\rho h b^2 \Omega^2 (3+\gamma)}{8}\left[\left(\frac{1+3\gamma}{3+\gamma}\frac{r^2}{b^2}-1\right)+\beta_1\left(1+\frac{b^2}{r^2}\right)\right],\\
q_{r\theta 1}^* &= 0,\ q_{r2}^* = q_{\theta 2}^* = 0,\\
q_{r\theta 2}^* &= \beta_2 h\left(\frac{b}{r}\right)^2 + \frac{1}{4}\rho h r^2 \frac{\partial \Omega}{\partial t},
\end{aligned} \tag{32}$$

with

$$\beta_1 = \frac{a^2}{b^2}\left(\frac{1-\gamma}{3+\gamma}\right)\left(\frac{\left(3+\gamma-(1+\gamma)\dfrac{a^2}{b^2}\right)}{(1+\gamma)+(1-\gamma)\dfrac{a^2}{b^2}}\right)$$

and

$$\beta_2 = -\frac{1}{4}\rho b^2 \frac{\partial \Omega}{\partial t}. \tag{33}$$

Substituting eq. (28) into eq. (27) and dropping the stars, we obtain the following nonlinear governing equations of the transverse motion for the disk subjected to the transverse load and with viscous damping.

$$\begin{aligned}
&\rho h \ddot{w} + c\dot{w} + D\nabla^4 w - \frac{1}{r}\frac{\partial}{\partial r}\left(r q_r \frac{\partial w}{\partial r} + q_{r\theta}\frac{\partial w}{\partial \theta}\right)\\
&-\frac{1}{r}\frac{\partial}{\partial \theta}\left(q_{r\theta}\frac{\partial w}{\partial r} + q_\theta \frac{\partial w}{\partial \theta}\frac{1}{r}\right)\\
&= \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) + \frac{\partial^2 \varphi}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\\
&-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) + p(r,\theta,t),
\end{aligned} \tag{34}$$

where ∇^4 is the biharmonic operator in the polar coordinate and c is the damping coefficient.

Eliminating u_r and u_θ from eq. (4) and replacing u_z with w, we obtain the following equation:

$$\begin{aligned}
&\frac{\partial \varepsilon_r}{\partial r} - \frac{1}{r}\frac{\partial^2 \varepsilon_r}{\partial^2 \theta} - 2\frac{\partial \varepsilon_\theta}{\partial r} - r\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial \gamma_{r\theta}}{\partial \theta} + \frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta}\\
&= -\frac{1}{2r^3}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{2}\frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2}\frac{\partial^3 w}{\partial \theta^2 \partial r}.
\end{aligned} \tag{35}$$

According to eqs. (28a)–(28c) and (35) and the stress strain relation of the elastic thin plane, the following form of the compatibility equation for the strain components can be obtained

$$\nabla^4 \varphi = Eh\left[\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]. \tag{36}$$

The dimensionless variables and parameters are introduced as follows:

$$\begin{aligned}
&r' = \frac{r}{b},\ a' = \frac{a}{b},\ r_m' = \frac{r_m}{b},\ w' = \frac{b}{h^2}w,\\
&t' = t\sqrt{\frac{D}{\rho h b^4}},\ \varphi' = \frac{b^2}{Eh^5}\varphi,\ c' = \frac{b^4}{24(1-\gamma^2)\sqrt{\rho h^5 D}}c,\\
&p' = \frac{b^7}{12(1-\gamma^2)Dh^4}p,\ \Omega' = \Omega b^2\sqrt{\frac{\rho h}{D}}.
\end{aligned} \tag{37}$$

Substituting eq. (37) into eqs. (34) and (36) and dropping the primes for convenience yield a concise representation of the nonlinear transverse responses of the spinning disk as follows:

$$\begin{aligned}
&\ddot{w} + \nabla^4 w - \frac{1}{r}\frac{\partial}{\partial r}\left(r q_r \frac{\partial w}{\partial r} + q_{r\theta}\frac{\partial w}{\partial \theta}\right)\\
&-\frac{1}{r}\frac{\partial}{\partial \theta}\left(q_{r\theta}\frac{\partial w}{\partial r} + q_\theta \frac{\partial w}{\partial r}\frac{1}{r}\right)\\
&= \frac{12(1-\nu^2)h^2}{b^2}\left(-2c\dot{w} + \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right)\right.\\
&\left.+\frac{\partial^2 \varphi}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right)\\
&+\frac{12(1-\nu^2)h^2}{b^2}\left(-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) + p(r,\theta,t)\right),
\end{aligned} \tag{38}$$

$$\nabla^4 \varphi = \left[\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right]. \tag{39}$$

Relative to the corotating basis, the transverse excitation is a rotating point force having the form:

$$P(r,\theta,t) = \delta(r-r_0)\delta(\theta-\Omega t)\frac{P_0}{r}, \tag{40}$$

where δ is the Dirac delta function and r_0 is the point of application of the point force and $a < r_0 < b$, P_0 is the amplitude of forcing.

3 Galerkin projection

Eqs. (38) and (39) are the partial differential equations with varying coefficients and are unable to be solved analytically. A particular solution to eq. (38) is assumed to be in the form of a Fourier sine and cosine series expansion as follows [17]:

$$w = W_{mn}(r)(\eta(t)\cos n\theta + \xi(t)\sin n\theta), \tag{41}$$

where m and n ($n>0$) are the number of nodal circles and nodal diameters, η and ξ are the generalized coordinates, and $W_{mn}(r)$ is the linear transverse free vibrations modes.

We consider the first two-order cos and sin modes of the spinning disk so that the transverse nonlinear oscillations take the form:

$$\begin{aligned} w(r,\theta,t) = {} & W_1(\eta_1\cos\theta + \xi_1\sin\theta) \\ & + W_2(\eta_2\cos 2\theta + \xi_2\sin 2\theta). \end{aligned} \tag{42}$$

Substituting eq. (42) into eq. (39) yields the new form of the compatibility equation for the strain components:

$$\begin{aligned} \nabla^4\varphi = {} & \frac{1}{2}\chi_1\left(\eta_1^2+\xi_1^2\right) \\ & + \chi_2\left[\frac{1}{2}\left(\xi_1^2-\eta_1^2\right)\cos 2\theta - \eta_1\xi_1\sin 2\theta\right] \\ & + \frac{1}{2}\chi_3\left(\eta_2^2+\xi_2^2\right) \\ & + \chi_4\left[\frac{1}{2}\left(\xi_2^2-\eta_2^2\right)\cos 4\theta - \eta_2\xi_2\sin 4\theta\right] \\ & + \chi_5\left(\eta_1\eta_2\sin\theta\sin 2\theta - \eta_1\xi_2\sin\theta\cos 2\theta\right. \\ & \left. - \xi_1\eta_2\cos\theta\sin 2\theta + \xi_1\xi_2\cos\theta\cos 2\theta\right) \\ & + \chi_6\left(\eta_1\eta_2\cos\theta\cos 2\theta + \eta_1\xi_2\cos\theta\sin 2\theta\right. \\ & \left. + \xi_1\eta_2\sin\theta\cos 2\theta + \xi_1\xi_2\sin\theta\sin 2\theta\right), \end{aligned} \tag{43}$$

where

$$\begin{aligned} \chi_1 &= \frac{1}{r^2}W_1''^2 - \frac{2}{r^3}W_1'W_1 + \frac{1}{r^4}W_1' - \frac{1}{r}W_1'' + \frac{1}{r^2}W_1''W_1, \\ \chi_2 &= \frac{1}{r^2}W_1'^2 - \frac{2}{r^3}W_1'W_1 + \frac{1}{r^4}W_1' + \frac{1}{r}W_1''W_1' - \frac{1}{r^2}W_1''W_1, \\ \chi_3 &= \frac{4}{r^2}W_2'^2 - \frac{8}{r^3}W_2'W_2 + \frac{4}{r^4}W_2^2 + \frac{1}{r^2}W_2''W_2 - \frac{1}{r}W_2''W_2', \\ \chi_4 &= \frac{4}{r^2}W_2'^2 - \frac{8}{r^3}W_2'W_2 + \frac{4}{r^4}W_2^2 - \frac{1}{r^2}W_2''W_2 + \frac{1}{r}W_2''W_2', \\ \chi_5 &= \frac{4}{r^2}W_1'W_2' - \frac{8}{r^3}\left(W_1'W_2 + W_2'W_1\right) + \frac{4}{r^4}W_1W_2, \\ \chi_6 &= \frac{4}{r^2}W_1''W_2 + \frac{1}{r^2}W_2''W_1. \end{aligned} \tag{44}$$

We write the solution of eq. (43) which satisfies the boundary conditions (23)–(28) in the following form:

$$\begin{aligned} \varphi = {} & \frac{1}{2}\psi_1\left(\eta_1^2+\xi_1^2\right) + \psi_2\left[\frac{1}{2}\left(\xi_1^2-\eta_1^2\right)\cos 2\theta - \eta_1\xi_1\sin 2\theta\right] \\ & + \frac{1}{2}\psi_3\left(\eta_2^2+\xi_2^2\right) + \psi_4\left[\frac{1}{2}\left(\xi_2^2-\eta_2^2\right)\cos 4\theta - \eta_2\xi_2\sin 4\theta\right] \\ & + \psi_5\left(\eta_1\eta_2\sin\theta\sin 2\theta - \eta_1\xi_2\sin\theta\cos 2\theta\right. \\ & \left. - \xi_1\eta_2\cos\theta\sin 2\theta + \xi_1\xi_2\cos\theta\cos 2\theta\right). \end{aligned} \tag{45}$$

Moreover, the functions ψ_i $(i = 1, \cdots, 5)$ are determined by the following equations:

Function ψ_1:

$$\begin{aligned} & \psi_1'''' + \frac{2}{r}\psi_1''' + \frac{3}{r^2}\psi_1'' + \frac{2}{r^3}\psi_1' + \frac{1}{r^4}\psi_1 = \chi_1, \\ & \psi_1'' - \frac{\nu}{a}\psi_1' = 0,\ \psi_1''' + \frac{1}{a}\psi_1'' - \frac{1}{a^2}\psi_1' = 0,\ \psi_1' = 0. \end{aligned} \tag{46}$$

Function ψ_2:

$$\begin{aligned} & \psi_2'''' + \frac{2}{r}\psi_2''' + \frac{3}{r^2}\psi_2'' + \frac{2}{r^3}\psi_2' + \frac{1}{r^4}\psi_2 = \chi_2, \\ & \psi_2'' - \frac{\nu}{a}\psi_2' + \frac{1}{a^2}\psi_2 = 0, \\ & \psi_2''' + \frac{1}{a}\psi_2'' + \frac{1+\nu}{a^2}\psi_2' - \frac{3+\nu}{a^3}\psi_2 = 0, \\ & \psi_2' + \frac{1}{b}\psi_2 = 0,\ \psi_2' - \frac{1}{b}\psi_2 = 0. \end{aligned} \tag{47}$$

Function ψ_3:

$$\begin{aligned} & \psi_3'''' + \frac{2}{r}\psi_3''' + \frac{3}{r^2}\psi_3'' + \frac{2}{r^3}\psi_3' + \frac{1}{r^4}\psi_3 = \chi_3, \\ & \psi_3'' - \frac{\nu}{a}\psi_3' = 0,\ \psi_3''' + \frac{1}{a}\psi_3'' - \frac{1}{a^2}\psi_3' = 0,\ \psi_3' = 0. \end{aligned} \tag{48}$$

Function ψ_4:

$$\begin{aligned} & \psi_4'''' + \frac{2}{r}\psi_4''' + \frac{3}{r^2}\psi_4'' + \frac{2}{r^3}\psi_4' + \frac{1}{r^4}\psi_4 = \chi_4, \\ & \psi_4'' - \frac{\nu}{a}\psi_4' + \frac{1}{a^2}\psi_4 = 0, \\ & \psi_4''' + \frac{1}{a}\psi_4'' + \frac{1+\nu}{a^2}\psi_4' - \frac{3+\nu}{a^3}\psi_4 = 0, \\ & \psi_4' + \frac{1}{b}\psi_4 = 0,\ \psi_4' - \frac{1}{b}\psi_4 = 0. \end{aligned} \tag{49}$$

Function ψ_5:

$$\begin{aligned} & \psi_5'''' + \frac{2}{r}\psi_5''' + \frac{1}{r^2}\psi_5'' - \frac{10}{r^2}\psi_5'' - \frac{10}{r^3}\psi_5' + \frac{41}{r^4}\psi_5 = \chi_5, \\ & \frac{8}{r^2}\psi_5'' - \frac{10}{r^3}\psi_5' - \frac{40}{r^4}\psi_5 = \chi_6,\ \psi_5'' - \frac{\nu}{a}\psi_5' + \frac{1}{a^2}\psi_5 = 0, \\ & \psi_5''' + \frac{1}{a}\psi_5'' + \frac{1+\nu}{a^2}\psi_5' - \frac{3+\nu}{a^3}\psi_5 = 0, \\ & \psi_5' + \frac{1}{b}\psi_5 = 0,\ \psi_5' - \frac{1}{b}\psi_5 = 0. \end{aligned} \tag{50}$$

After determining the functions ψ_i $(i=1,\cdots,5)$, we can obtain the solution to the compatibility equation for the strain components. When we substitute eqs. (42) and (45) into eq. (38), take inner products of eq. (38) with the eigenfunctions of the cos and sin modes and use the orthonormality, the governing equations of motion for the spinning disk are simplified to the following four-degree-of-freedom nonlinear system:

$$\begin{aligned}&\ddot{\eta}_1+\omega_1{}^2\eta_1+c_1\dot{\eta}_1+a_{11}\eta_1\cos\omega t+\frac{1}{2}a_{12}\eta_1\cos 2\omega t\\&+a_{13}\xi_1\sin\omega t-a_{14}\eta_1^3-a_{15}\eta_1\xi_1^2-a_{16}\eta_1\eta_2^2-a_{17}\eta_1\xi_2^2\\&=f\cos\Omega t,\end{aligned}\tag{51a}$$

$$\begin{aligned}&\ddot{\xi}_1+\omega_1{}^2\xi_1+c_1\dot{\xi}_1+a_{11}\xi_1\cos\omega t+\frac{1}{2}a_{12}\xi_1\cos 2\omega t\\&-a_{13}\eta_1\sin\omega t-a_{24}\xi_1^3-a_{25}\xi_1\eta_1^2-a_{26}\xi_1\eta_2^2-a_{27}\xi_1\xi_2^2\\&=f\sin\Omega t,\end{aligned}\tag{51b}$$

$$\begin{aligned}&\ddot{\eta}_2+\omega_2{}^2\eta_2+c_2\dot{\eta}_2+b_{11}\eta_2\cos\omega t+\frac{1}{2}b_{12}\eta_2\cos 2\omega t\\&+b_{13}\xi_2\sin\omega t-b_{14}\eta_2^3-b_{15}\eta_2\eta_1^2-b_{16}\eta_2\xi_1^2-b_{17}\eta_2\eta_2^2\\&=f\cos 2\Omega t,\end{aligned}\tag{51c}$$

$$\begin{aligned}&\ddot{\xi}_2+\omega_2{}^2\xi_2+c_2\dot{\xi}_2+b_{11}\xi_2\cos\omega t+\frac{1}{2}b_{12}\xi_2\cos 2\omega t\\&-b_{13}\eta_2\sin\omega t-b_{24}\xi_2^3-b_{25}\xi_2\eta_1^2-b_{26}\xi_2\eta_2^2-b_{27}\xi_2\xi_1^2\\&=f\sin 2\Omega t,\end{aligned}\tag{51d}$$

where the coefficients are obtained. For the sake of saving space, the coefficients are omitted herein.

In pursuit of a system which is suitable for the application of the method of multiple scales, the small parameter ε is introduced. Therefore, four-degree-of-freedom nonlinear system with small parameter ε is obtained as follows:

$$\begin{aligned}&\ddot{\eta}_1+\varepsilon c_1\dot{\eta}_1+\omega_1{}^2\eta_1+\varepsilon a_{11}\eta_1\cos\omega t+\frac{1}{2}\varepsilon a_{12}\eta_1\cos 2\omega t\\&+\varepsilon a_{13}\xi_1\sin\omega t-\varepsilon a_{14}\eta_1^3-\varepsilon a_{15}\eta_1\xi_1^2-\varepsilon a_{16}\eta_1\eta_2^2\\&-\varepsilon a_{17}\eta_1\xi_2^2=\varepsilon f\cos\Omega t,\end{aligned}\tag{52a}$$

$$\begin{aligned}&\ddot{\xi}_1+\varepsilon c_1\dot{\xi}_1+\omega_1{}^2\xi_1+\varepsilon a_{11}\xi_1\cos\omega t+\frac{1}{2}\varepsilon a_{12}\xi_1\cos 2\omega t\\&-\varepsilon a_{13}\eta_1\sin\omega t-\varepsilon a_{24}\xi_1^3-\varepsilon a_{25}\xi_1\eta_1^2-\varepsilon a_{26}\xi_1\eta_2^2\\&-\varepsilon a_{27}\xi_1\xi_2^2=\varepsilon f\sin\Omega t,\end{aligned}\tag{52b}$$

$$\begin{aligned}&\ddot{\eta}_2+\varepsilon c_2\dot{\eta}_2+\omega_2{}^2\eta_2+\varepsilon b_{11}\eta_2\cos\omega t+\frac{1}{2}\varepsilon b_{12}\eta_2\cos 2\omega t\\&+\varepsilon b_{13}\xi_2\sin\omega t-\varepsilon b_{14}\eta_2^3-\varepsilon b_{15}\eta_2\eta_1^2-\varepsilon b_{16}\eta_2\xi_1^2\\&-\varepsilon b_{17}\eta_2\eta_2^2=\varepsilon f\cos 2\Omega t,\end{aligned}\tag{52c}$$

$$\begin{aligned}&\ddot{\xi}_2+\varepsilon c_2\dot{\xi}_2+\omega_2{}^2\xi_2+\varepsilon b_{11}\xi_2\cos\omega t+\frac{1}{2}\varepsilon b_{12}\xi_2\cos 2\omega t\\&-\varepsilon b_{13}\eta_2\sin\omega t-\varepsilon b_{24}\xi_2^3-\varepsilon b_{25}\xi_2\eta_1^2-\varepsilon b_{26}\xi_2\eta_2^2\\&-\varepsilon b_{27}\xi_2\xi_1^2=\varepsilon f\sin 2\Omega t.\end{aligned}\tag{52d}$$

4 Perturbation analysis

We employ the method of multiple scales [28] to analyze the solution to eq. (52). We assume an expansion of the solution in the form:

$$\eta_1(t,\varepsilon)=x_{10}(T_0,T_1)+\varepsilon x_{11}(T_0,T_1)+\cdots,\tag{53}$$

$$\eta_2(t,\varepsilon)=x_{20}(T_0,T_1)+\varepsilon x_{21}(T_0,T_1)+\cdots,\tag{54}$$

$$\xi_1(t,\varepsilon)=y_{10}(T_0,T_1)+\varepsilon y_{11}(T_0,T_1)+\cdots,\tag{55}$$

$$\xi_2(t,\varepsilon)=y_{20}(T_0,T_1)+\varepsilon y_{21}(T_0,T_1)+\cdots,\tag{56}$$

where $T_0=t$ and $T_1=\varepsilon t$.

Then, the differential operators can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial T_0}\frac{\partial T_0}{\partial t}+\frac{\partial}{\partial T_1}\frac{\partial T_1}{\partial t}+\cdots=D_0+\varepsilon D_1+\cdots,\tag{57}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=(D_0+\varepsilon D_1+\cdots)^2=D_0{}^2+2\varepsilon D_0D_1+\cdots,\tag{58}$$

where $D_0=\partial/\partial T_0$ and $D_1=\partial/\partial T_1$.

We consider the case of 1:1:2:2 internal resonance for the first two order of sin and cos modes and the critical speed resonance. The primary parametric resonance for the first order of sin and cos mode and the fundamental parametric resonance for the second order of sin and cos mode are also considered. In this case, there are the following relations:

$$\begin{aligned}&\omega_1\approx\frac{1}{2}\omega_2,\ \omega_1{}^2=\frac{1}{4}\omega^2+\varepsilon\sigma_1,\\&\omega_2^2=\omega^2+\varepsilon\sigma_2,\ \omega_n^2\approx n^2\Omega^2\ (n=1,2),\end{aligned}\tag{59}$$

where σ_1 and σ_2 are two detuning parameters.

Substituting eqs. (53)–(59) into eq. (52) and balancing the coefficients of like power of ε on the left-hand and right-hand sides of the equations, we obtain the differential equations as follows:

order ε^0 :

$$D_0^2x_{10}+\frac{1}{4}\omega^2x_{10}=0,\tag{60a}$$

$$D_0^2y_{10}+\frac{1}{4}\omega^2y_{10}=0,\tag{60b}$$

$$D_0^2x_{20}+\omega^2x_{20}=0,\tag{60c}$$

$$D_0^2y_{20}+\omega^2y_{20}=0;\tag{60d}$$

order ε^1 :

$$D_0^2 x_{11} + \frac{1}{4}\omega^2 x_{11} = -2D_0 D_1 x_{10} - \sigma_1 x_{10} - c_1 D_0 x_{10} - a_{11} x_{10} \cos\omega t - \frac{1}{2} a_{12} x_{10} \cos 2\omega t - a_{13} y_{10} \sin\omega t + a_{14} x_{10}^3 + a_{15} x_{10} {y_{10}}^2 + a_{16} x_{10} x_{20}^2 + a_{17} x_{10} y_{20}^2 + f \cos\frac{1}{2}\omega t, \quad (61a)$$

$$D_0^2 y_{11} + \frac{1}{4}\omega^2 y_{11} = -2D_0 D_1 y_{10} - \sigma_1 y_{10} - c_1 D_0 y_{10} - a_{11} y_{10} \cos\omega t - \frac{1}{2} a_{12} y_{10} \cos 2\omega t + a_{13} x_{10} \sin\omega t + a_{24} y_{10}^3 + a_{25} x_{10}^2 y_{10} + a_{26} y_{10} x_{20}^2 + a_{27} y_{10} y_{20}^2 + f \sin\frac{1}{2}\omega t, \quad (61b)$$

$$D_0^2 x_{21} + \omega^2 x_{21} = -2D_0 D_1 x_{20} - \sigma_2 x_{20} - c_2 D_0 x_{20} - b_{11} x_{20} \cos\omega t - \frac{1}{2} b_{12} x_{20} \cos 2\omega t - b_{13} y_{20} \sin\omega t + b_{14} x_{20}^3 + b_{15} x_{20} x_{10}^2 + b_{16} x_{20} y_{10}^2 + b_{17} x_{20} y_{20}^2 + f \cos\omega t, \quad (61c)$$

$$D_0^2 y_{21} + \omega^2 y_{21} = -2D_0 D_1 y_{20} - \sigma_2 y_{20} - c_2 D_0 y_{20} - b_{11} y_{20} \cos\omega t - \frac{1}{2} b_{12} y_{20} \cos 2\omega t + b_{13} x_{20} \sin\omega t + b_{24} y_{20}^3 + b_{25} y_{20} x_{10}^2 + b_{26} y_{20} x_{20}^2 + b_{27} y_{20} y_{10}^2 + f \sin\omega t. \quad (61d)$$

The solutions to eq. (61) in the complex form can be written as:

$$\begin{aligned} x_{10} &= A_1(T_1) e^{\frac{1}{2}\mathrm{i}\omega T_0} + \bar{A}_1(T_1) e^{-\frac{1}{2}\mathrm{i}\omega T_0}, \\ y_{10} &= A_2(T_2) e^{\frac{1}{2}\mathrm{i}\omega T_0} + \bar{A}_2(T_2) e^{-\frac{1}{2}\mathrm{i}\omega T_0}, \\ x_{20} &= A_3(T_3) e^{\mathrm{i}\omega T_0} + \bar{A}_3(T_3) e^{-\mathrm{i}\omega T_0}, \\ y_{20} &= A_4(T_4) e^{\mathrm{i}\omega T_0} + \bar{A}_4(T_4) e^{-\mathrm{i}\omega T_0}, \end{aligned} \quad (62)$$

where $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$ and $\bar{A}_4$ are the complex conjugates of A_1, A_2, A_3 and A_4, respectively.

Substituting eq. (62) into eq. (61) yields

$$D_0^2 x_{11} + \frac{1}{4} x_{11} = \Big(-\mathrm{i}D_1 A_1 - \sigma_1 A_1 - \frac{1}{2}\mathrm{i}c_1 A_1 - \frac{1}{2} a_{11} \bar{A}_1 - \frac{1}{2} a_{12} A_1 - \frac{1}{2}\mathrm{i}a_{13} \bar{A}_2 + 3a_{14} {A_1}^2 \bar{A}_1 + 2a_{15} A_1 A_2 \bar{A}_2 + a_{15} \bar{A}_1 {A_2}^2 + 2a_{16} A_1 A_3 \bar{A}_3 + 2a_{17} A_1 A_4 \bar{A}_4 + \frac{1}{2} f \Big) e^{\frac{1}{2}\mathrm{i}\omega T_0} + \mathrm{cc} + \mathrm{NST}, \quad (63a)$$

$$D_0^2 y_{11} + \frac{1}{4} y_{11} = \Big(-\mathrm{i}D_1 A_2 - \sigma_1 A_2 - \frac{1}{2}\mathrm{i}c_1 A_2 - \frac{1}{2} a_{11} \bar{A}_2 - \frac{1}{2} a_{12} A_2 + \frac{1}{2}\mathrm{i}a_{13} \bar{A}_1 + 3a_{24} A_2^2 \bar{A}_2 + 2a_{25} A_1 \bar{A}_1 A_2 + a_{25} A_1^2 \bar{A}_2 + 2a_{26} A_2 A_3 \bar{A}_3 + 2a_{27} A_2 A_4 \bar{A}_4 + \frac{1}{2}\mathrm{i}f \Big) e^{\frac{1}{2}\mathrm{i}\omega T_0} + \mathrm{cc} + \mathrm{NST}, \quad (63b)$$

$$D_0^2 x_{21} + x_{21} = \Big(-2\mathrm{i}D_1 A_3 - \sigma_2 A_3 - \mathrm{i}c_2 A_3 - \frac{1}{2} b_{12} A_3 - \frac{1}{4} b_{12} \bar{A}_3 + 3b_{14} A_3^2 \bar{A}_3 + 2b_{15} A_1 \bar{A}_1 A_3 + 2b_{16} A_2 \bar{A}_2 A_3 + b_{17} \bar{A}_3 A_4^2 + 2b_{17} A_3 A_4 \bar{A}_4 + \frac{1}{2} f \Big) e^{\mathrm{i}\omega T_0} + \mathrm{cc} + \mathrm{NST}, \quad (63c)$$

$$D_0^2 y_{21} + y_{21} = \Big(-2\mathrm{i}D_1 A_4 - \sigma_2 A_4 - \mathrm{i}c_2 A_4 - \frac{1}{2} b_{12} A_4 - \frac{1}{4} b_{12} \bar{A}_4 + 3b_{24} A_4^2 \bar{A}_4 + 2b_{25} A_1 \bar{A}_1 A_4 + 2b_{26} A_3 \bar{A}_3 A_4 + b_{26} A_3^2 \bar{A}_4 + 2b_{27} A_2 \bar{A}_2 A_4 + \frac{1}{2}\mathrm{i}f \Big) e^{\mathrm{i}\omega T_0} + \mathrm{cc} + \mathrm{NST}, \quad (63d)$$

where *cc* represents the parts of the complex conjugate of the function on the right-hand side of eq. (63) and NST represents the terms that do not produce secular terms.

For convenience in the following analysis, let $\omega = 1$. Eliminating the secular terms from eq. (63) yields:

$$D_1 A_1 = \frac{3}{4} a_{24} \left({x_3}^2 + {x_4}^2 \right) x_4 + \frac{1}{2} a_{26} \left({x_5}^2 + {x_6}^2 \right) x_4 + \frac{1}{2} a_{27} \left(x_7^2 + x_8^2 \right) x_4 + f_2 - \mathrm{i}a_{15} \bar{A}_1 {A_2}^2 - 2\mathrm{i}a_{16} A_1 A_3 \bar{A}_3 - 2\mathrm{i}a_{17} A_1 A_4 \bar{A}_4 - \frac{1}{2}\mathrm{i}f, \quad (64a)$$

$$D_1 A_2 = -\frac{1}{2} c_1 A_2 + \mathrm{i}\sigma_1 A_2 + \frac{1}{2}\mathrm{i}a_{11} \bar{A}_2 + \frac{1}{2} a_{13} \bar{A}_1 - 3\mathrm{i}a_{24} {A_2}^2 \bar{A}_2 - 2\mathrm{i}a_{25} A_1 \bar{A}_1 A_2 - \mathrm{i}a_{25} A_1^2 \bar{A}_2 - 2\mathrm{i}a_{26} A_2 A_3 \bar{A}_3 - 2\mathrm{i}a_{27} A_2 A_4 \bar{A}_4 + \frac{1}{2} f, \quad (64b)$$

$$D_1 A_3 = -\frac{1}{2} c_2 A_3 + \frac{1}{2}\mathrm{i}\sigma_2 A_3 + \frac{1}{8}\mathrm{i}b_{12} \bar{A}_3 - \frac{3}{2}\mathrm{i}b_{14} A_3^2 \bar{A}_3 - \mathrm{i}b_{15} A_1 \bar{A}_1 A_3 - \mathrm{i}b_{16} A_2 \bar{A}_2 A_3 - \frac{1}{2}\mathrm{i}b_{17} \bar{A}_3 A_4^2 - \mathrm{i}b_{17} A_3 A_4 \bar{A}_4 - \frac{1}{4}\mathrm{i}f, \quad (64c)$$

$$D_1A_4 = -\frac{1}{2}c_4A_4 + \frac{1}{2}\mathrm{i}\sigma_2 A_4 + \frac{1}{8}\mathrm{i}b_{12}\bar{A}_4 - \frac{3}{2}\mathrm{i}b_{24}A_4^2\bar{A}_4 - \mathrm{i}b_{25}A_1\bar{A}_1A_4 - \mathrm{i}b_{26}A_3\bar{A}_3A_4 - \frac{1}{2}\mathrm{i}b_{26}A_3^2\bar{A}_4 - \mathrm{i}b_{27}A_2\bar{A}_2A_4 + \frac{1}{4}f. \quad (64\mathrm{d})$$

We express the functions A_1, A_2, A_3 and A_4 in the following form:

$$\begin{aligned} A_1(T_1) &= \frac{1}{2}\left[x_1(T_1) + \mathrm{i}x_2(T_1)\right], \\ A_2(T_1) &= \frac{1}{2}\left[x_3(T_1) + \mathrm{i}x_4(T_1)\right], \\ A_3(T_1) &= \frac{1}{2}\left[x_5(T_1) + \mathrm{i}x_6(T_1)\right], \\ A_4(T_1) &= \frac{1}{2}\left[x_7(T_1) + \mathrm{i}x_8(T_1)\right]. \end{aligned} \quad (65)$$

Substituting eq. (65) into eq. (64), then, we obtain the averaged equations in the Cartesian form as follows:

$$\dot{x}_1 = -\frac{1}{2}c_1x_1 - \left(\sigma_1 - \frac{1}{2}a_{11}\right)x_2 + \frac{3}{4}a_{14}\left(x_1^2 + x_2^2\right)x_2 + \frac{1}{4}a_{15}\left(x_3^2 + 3x_4^2\right)x_2 + \frac{1}{2}a_{16}\left(x_5^2 + x_6^2\right)x_2 + \frac{1}{2}a_{17}\left(x_7^2 + x_8^2\right)x_2 - \frac{1}{2}a_{13}x_3 + \frac{1}{2}a_{15}x_1x_3x_4, \quad (66\mathrm{a})$$

$$\dot{x}_2 = -\frac{1}{2}c_2x_5 - \frac{1}{2}\left(\sigma_2 - \frac{1}{4}b_{12}\right)x_6 + \frac{1}{4}b_{15}\left(x_1^2 + x_2^2\right)x_6 + \frac{1}{4}b_{16}\left(x_3^2 + x_4^2\right)x_6 - \frac{1}{2}a_{16}\left(x_5^2 + x_6^2\right)x_1 - \frac{1}{2}a_{17}\left(x_7^2 + x_8^2\right)x_1 - \frac{1}{2}a_{15}x_2x_3x_4 + \frac{1}{2}a_{13}x_4 - f_1, \quad (66\mathrm{b})$$

$$\dot{x}_3 = -\frac{1}{2}c_1x_3 - \left(\sigma_1 - \frac{1}{2}a_{11}\right)x_4 + \frac{1}{2}a_{13}x_1 + \frac{1}{2}a_{25}x_1x_2x_3 + \frac{1}{4}a_{25}\left(x_1^2 + 3x_2^2\right)x_4 + \frac{3}{4}a_{24}\left(x_3^2 + x_4^2\right)x_4 + \frac{1}{2}a_{26}\left(x_5^2 + x_6^2\right)x_4 + \frac{1}{2}a_{27}\left(x_7^2 + x_8^2\right)x_4 + f_2 \quad (66\mathrm{c})$$

$$\dot{x}_4 = -\frac{1}{2}a_{13}x_2 + \left(\sigma_1 + \frac{1}{2}a_{11}\right)x_3 - \frac{1}{4}a_{25}\left(3x_1^2 + x_2^2\right)x_3 - \frac{3}{4}a_{24}\left(x_3^2 + x_4^2\right)x_3 - \frac{1}{2}a_{26}\left(x_5^2 + x_6^2\right)x_3 - \frac{1}{2}a_{27}\left(x_7^2 + x_8^2\right)x_3 - \frac{1}{2}a_{25}x_1x_2x_4 - \frac{1}{2}c_1x_4, \quad (66\mathrm{d})$$

$$\dot{x}_5 = -\frac{1}{2}c_2x_5 - \frac{1}{2}\left(\sigma_2 - \frac{1}{4}b_{12}\right)x_6 + \frac{1}{4}b_{15}\left(x_1^2 + x_2^2\right)x_6 + \frac{1}{4}b_{16}\left(x_3^2 + x_4^2\right)x_6 + \frac{3}{8}b_{14}\left(x_5^2 + x_6^2\right)x_6 + \frac{1}{8}b_{17}\left(x_7^2 + 3x_8^2\right)x_6 + \frac{1}{4}b_{17}x_5x_7x_8, \quad (66\mathrm{e})$$

$$\dot{x}_6 = \frac{1}{2}\left(\sigma_2 + \frac{1}{4}b_{12}\right)x_5 - \frac{1}{4}b_{15}\left(x_1^2 + x_2^2\right)x_5 - \frac{1}{4}b_{16}\left(x_3^2 + x_4^2\right)x_5 - \frac{3}{8}b_{14}\left(x_5^2 + x_6^2\right)x_5 - \frac{1}{8}b_{17}\left(3x_7^2 + x_8^2\right)x_5 - \frac{1}{2}c_2x_6 - \frac{1}{4}b_{17}x_6x_7x_8 - \frac{1}{2}f_3, \quad (66\mathrm{f})$$

$$\dot{x}_7 = -\frac{1}{2}c_2x_7 + \frac{1}{4}b_{26}x_5x_6x_7 - \frac{1}{2}\left(\sigma_2 - \frac{1}{4}b_{12}\right)x_8 + \frac{1}{4}b_{25}\left(x_1^2 + x_2^2\right)x_8 + \frac{1}{4}b_{27}\left(x_3^2 + x_4^2\right)x_8 + \frac{1}{8}b_{26}\left(x_5^2 + 3x_6^2\right)x_8 + \frac{3}{8}b_{24}\left(x_7^2 + x_8^2\right)x_8 + \frac{1}{2}f_4, \quad (66\mathrm{g})$$

$$\dot{x}_8 = \frac{1}{2}\left(\sigma_2 + \frac{1}{4}b_{12}\right)x_7 - \frac{1}{4}b_{25}\left(x_1^2 + x_2^2\right)x_7 - \frac{1}{4}b_{27}\left(x_3^2 + x_4^2\right)x_7 - \frac{1}{8}b_{26}\left(3x_5^2 + x_6^2\right)x_7 - \frac{3}{8}b_{24}\left(x_7^2 + x_8^2\right)x_7 - \frac{1}{2}c_2x_8 - \frac{1}{4}b_{26}x_5x_6x_8. \quad (66\mathrm{h})$$

5 Numerical simulation

In this section, the fourth-order Runge-Kutta method is utilized to numerically analyze the nonlinear dynamic behaviors of the spinning disk based on the averaged eq. (66). We focus on the complex nonlinear dynamics of the system and the influence of different parameters on the motions of the spinning disk.

When we change the parameters of the averaged equations to analyze the nonlinear responses of the spinning disk, we find that there exist the periodic, periodic-*n* and chaotic motions of the spinning disk, and the motion of the system is sensitive to the parameter *f* which is the external excitation. To detect the influence of the external excitation *f* on the motion of the spinning disk, we obtain the bifurcation diagram for x_1 and x_5 via the external excitation *f*, as shown in Figure 2. In Figure 2, the external excitation *f* changes from 45 to 56, and the other parameters and the initial conditions in the numerical simulation are chosen as $a_{11} = 30.53$, $a_{12} = -36.64$, $a_{13} = -188.81$, $a_{14} = 10.5$, $a_{15} = 2.5$, $a_{16} = 21.96$, $a_{17} = 5.75$, $a_{24} = 135.97$, $a_{25} = 20.62$, $a_{26} = 75.14$, $a_{27} = -7.37$, $c_1 = 0.405$, $c_2 = 0.304$, $b_{12} = 18.99$, $b_{14} = -33.92$, $b_{15} = -25.4$, $b_{16} = 38.78$, $b_{17} = 1.84$, $b_{24} = 67.1$, $b_{25} = -149.45$, $b_{26} = 122.4$, $b_{27} = 91.84$,

$\sigma_1 = 31.05$, $\sigma_2 = 96$, $x_{10} = 10$, $x_{20} = 0.5$, $x_{30} = 1$, $x_{40} = 2$, $x_{50} = 3$, $x_{60} = 4$, $x_{70} = 8$ and $x_{80} = 6$.

From Figure 2, it is found that the motions of the spinning disk mainly are the periodic responses except two narrow chaotic windows. The first chaotic window appears when the external excitation f is located near 46. The second chaotic window appears when the external excitation f is located near 50. To have a better observation of the motion of the spinning disk around the first chaotic window, we give an enlarged drawing of the bifurcation diagram of the external excitation f with $45 \leqslant f \leqslant 48.5$, as shown in Figure 3. It is seen from Figure 3 that the motion of the spinning disk changes from the periodic motion to the chaotic motion suddenly without the well-known process of period doubling bifurcation.

When the external excitation f equals 45.5 and the other parameters are the same as those in Figure 2, Figure 4 indicates that there exists the periodic motion of the spinning disk. Figures 4(a)–4(d) represent the phase portraits on the planes (x_1, x_2), (x_3, x_4), (x_5, x_6) and (x_7, x_8), respectively. Figures 4(e)–4(h) give the waveforms on the planes (t, x_1), (t, x_3), (t, x_5) and (t, x_7), respectively. Figures 4(i)–4(l) represent the power spectrum pictures. Figures 4(m) and 4(n) give three-dimensional phase portraits in the spaces (x_1, x_2, x_3) and (x_5, x_6, x_7), respectively.

When the external excitation f equals 46, the motion of the spinning disk is chaotic, as shown in Figure 5. When the external excitation f changes to 49.532, Figure 6 illustrates that the chaotic motion of the spinning disk occurs. It is observed from Figures 5 and 6 that the phase portraits, waveforms and spectrum pictures with the same parameter are consistent with the bifurcation diagram given in Figure 2. It is found from Figures 5(m) and 6(m) that the multi-pulse chaotic motions of the spinning disk occur.

To analyze the influence of the damping on the motion for the spinning disk, we obtain the bifurcation diagram for x_1 and x_5 via the damping parameter c_2, as shown in Figure 7.

In Figure 7, the damping c_2 changes from 0.001 to 1.5. The other parameters and the initial conditions are chosen as $a_{11} = -186.13$, $a_{12} = 39.04$, $a_{13} = -329.03$, $a_{14} = 10.5$, $a_{15} = 2.5$, $a_{16} = 21.96$, $a_{17} = 5.75$, $a_{24} = 135.97$, $a_{25} = 20.62$, $a_{26} = 75.14$, $a_{27} = -3.31$, $c_1 = 0.543$, $b_{12} = 17.36$, $b_{14} = -41.96$, $b_{15} = -27.35$, $b_{16} = 56.96$, $b_{17} = 9.88$, $b_{24} = -18.05$, $b_{25} = 73.8$, $b_{26} = 122.4$, $b_{27} = 91.84$, $\sigma_1 = 31.05$,

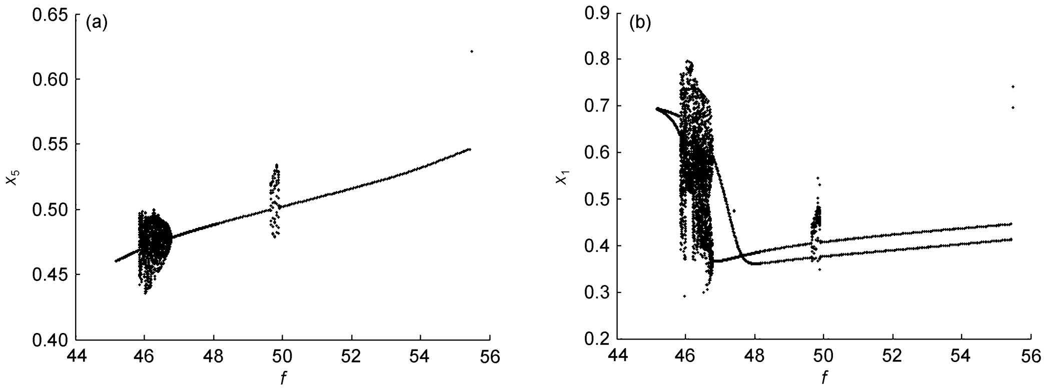

Figure 2 The bifurcation diagram of the excitation f is given for the spinning disk when $45 \leqslant f \leqslant 56$.

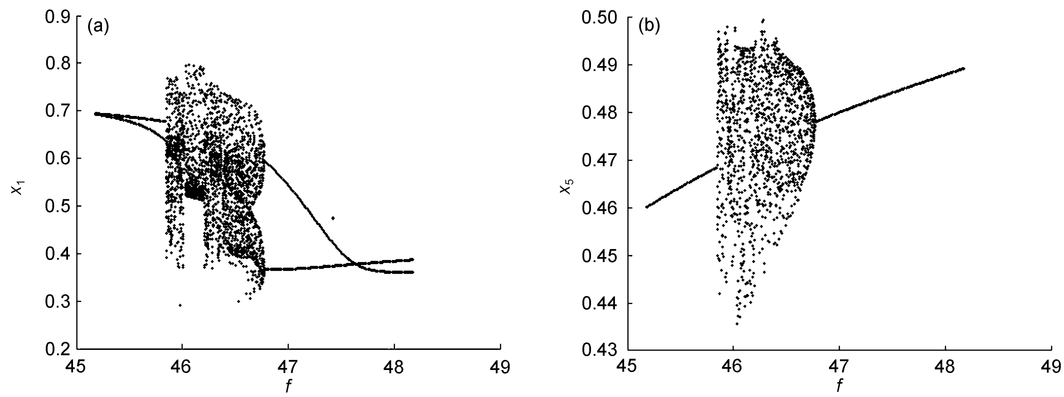

Figure 3 The bifurcation diagram of the excitation f is given for the spinning disk when $45 \leqslant f \leqslant 48.5$.

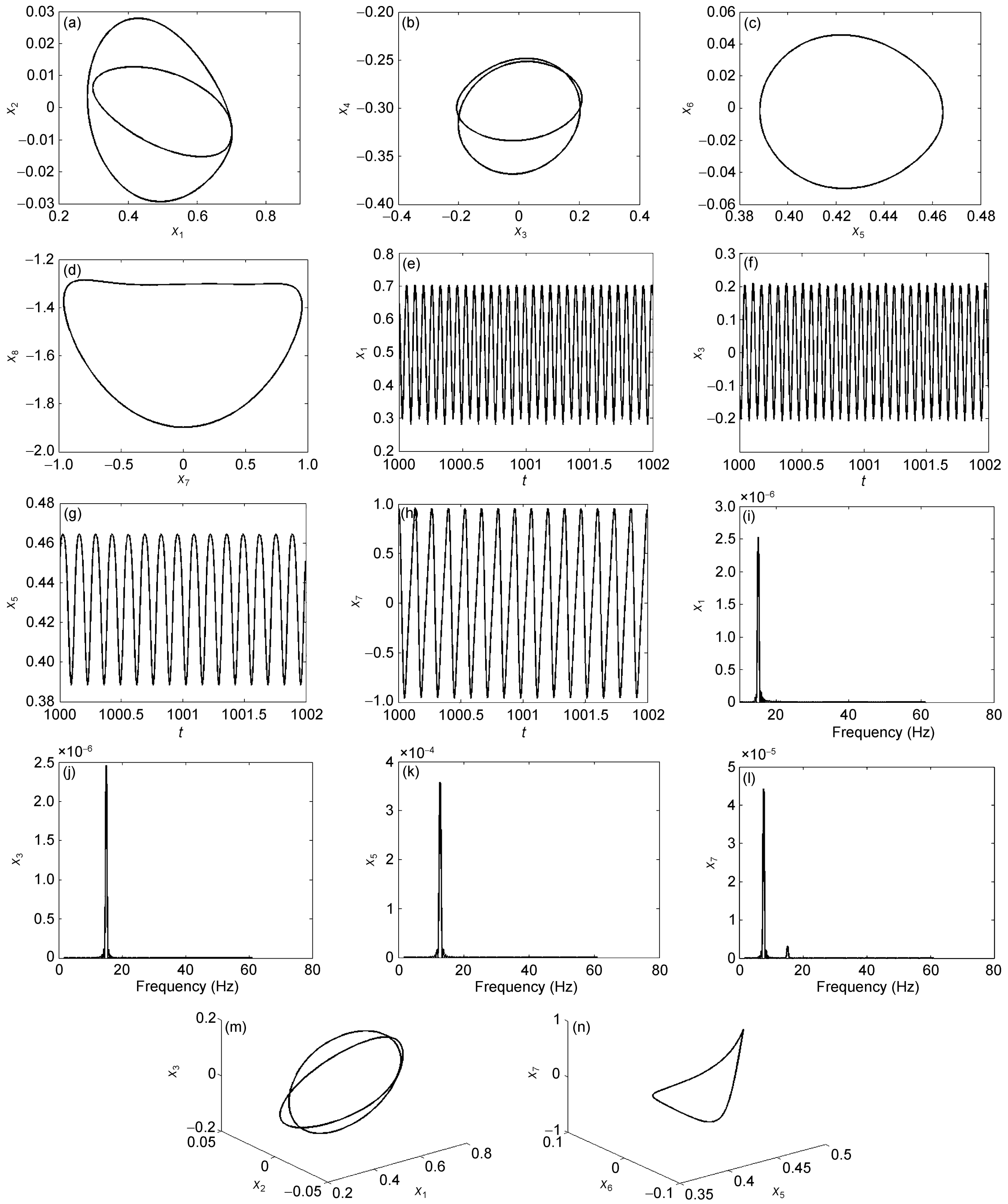

Figure 4 The periodic motion is given for the spinning disk when f=45.5.

$\sigma_2 = 0.1$, $f = 9.1$, $x_{10} = 10$, $x_{20} = 0.5$, $x_{30} = 1$, $x_{40} = 2$, $x_{50} = 3$, $x_{60} = 4$, $x_{70} = 8$ and $x_{80} = 6$. When the damping parameter c_2 equals 0.05, it is found that the chaotic motion of the spinning disk occurs, as shown in Figure 8. When the damping parameter c_2 equals 0.5, the periodic motion of the spinning disk exists, as shown in Figure 9. When c_2 equals 1.5, Figure 10 demonstrates that there exists the periodic motion.

From Figure 7, we can see that when c_2<0.35 the motion

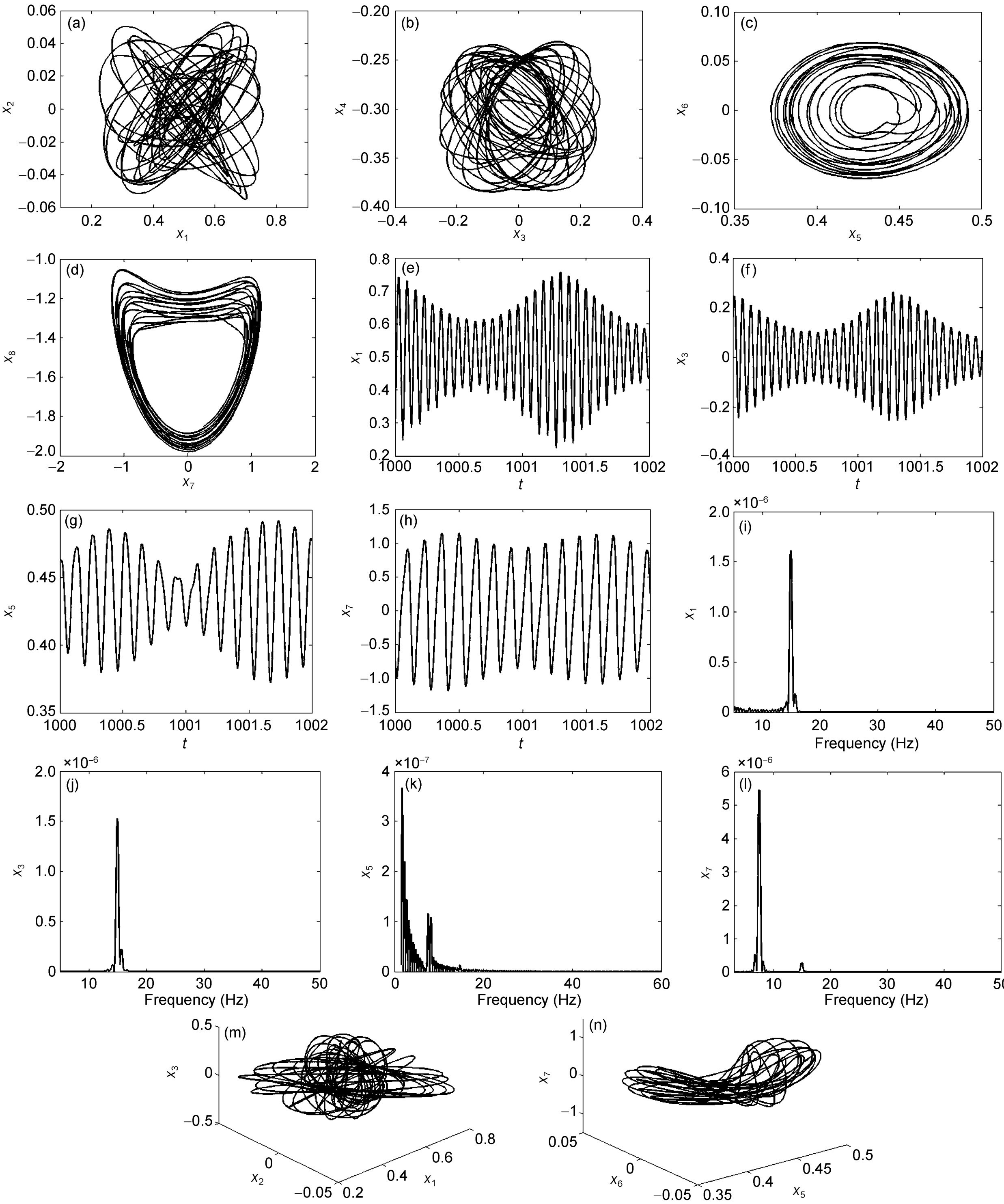

Figure 5 The chaotic motion is given for the spinning disk when f=46.

of the spinning disk is chaotic. When c_2>0.35, the motion of the spinning disk changes to periodic motion. It is concluded that the damping parameter c_2 of the second-order mode has a major effect on the nonlinear responses of the spinning disk. The nonlinear responses of the spinning disk tend to be stable with the damping parameter increasing.

6 Conclusions

In this paper, we investigate the nonlinear oscillations and chaotic dynamics of a spinning imperfect disk subjected to a transverse concentrated load and with a periodically varying rotating speed. The partial differential governing equations

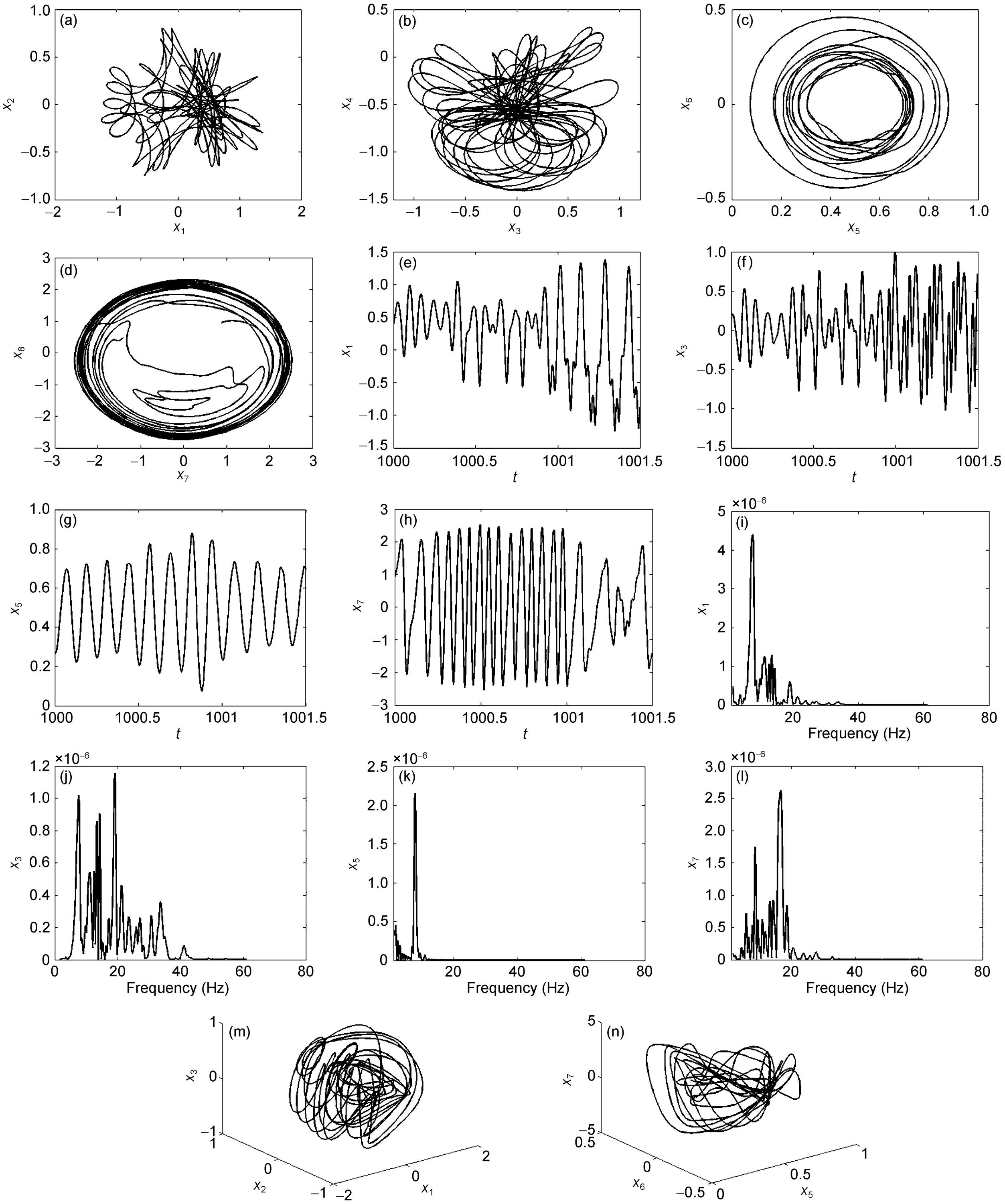

Figure 6 The chaotic motion is given for the spinning disk when $f = 49.532$.

of motion for the spinning disk are firstly derived by using Hamilton's principle. When the in-plane inertia is ignored and a stress function is introduced, the three nonlinearly coupled partial differential equations are reduced to two nonlinearly coupled partial differential equations. Then, the Galerkin's procedure is utilized to obtain the ordinary differential equations governing the split resonant modes onto a case of 1:1:2:2 internal resonance. The method of multiple scales is used to obtain a set of eight- dimensional averaged equations within the concerns of the critical speed resonance, the primary parametric resonance for the first-order sin and cos modes and the fundamental parametric reso-

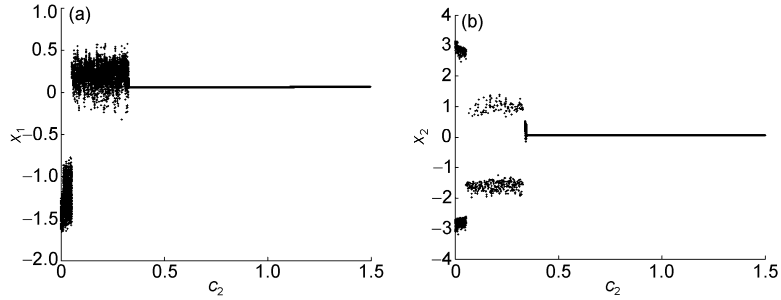

Figure 7 The bifurcation diagram of the excitation c_2 is given for the spinning disk when $0.001\leqslant c_2\leqslant 1.5$.

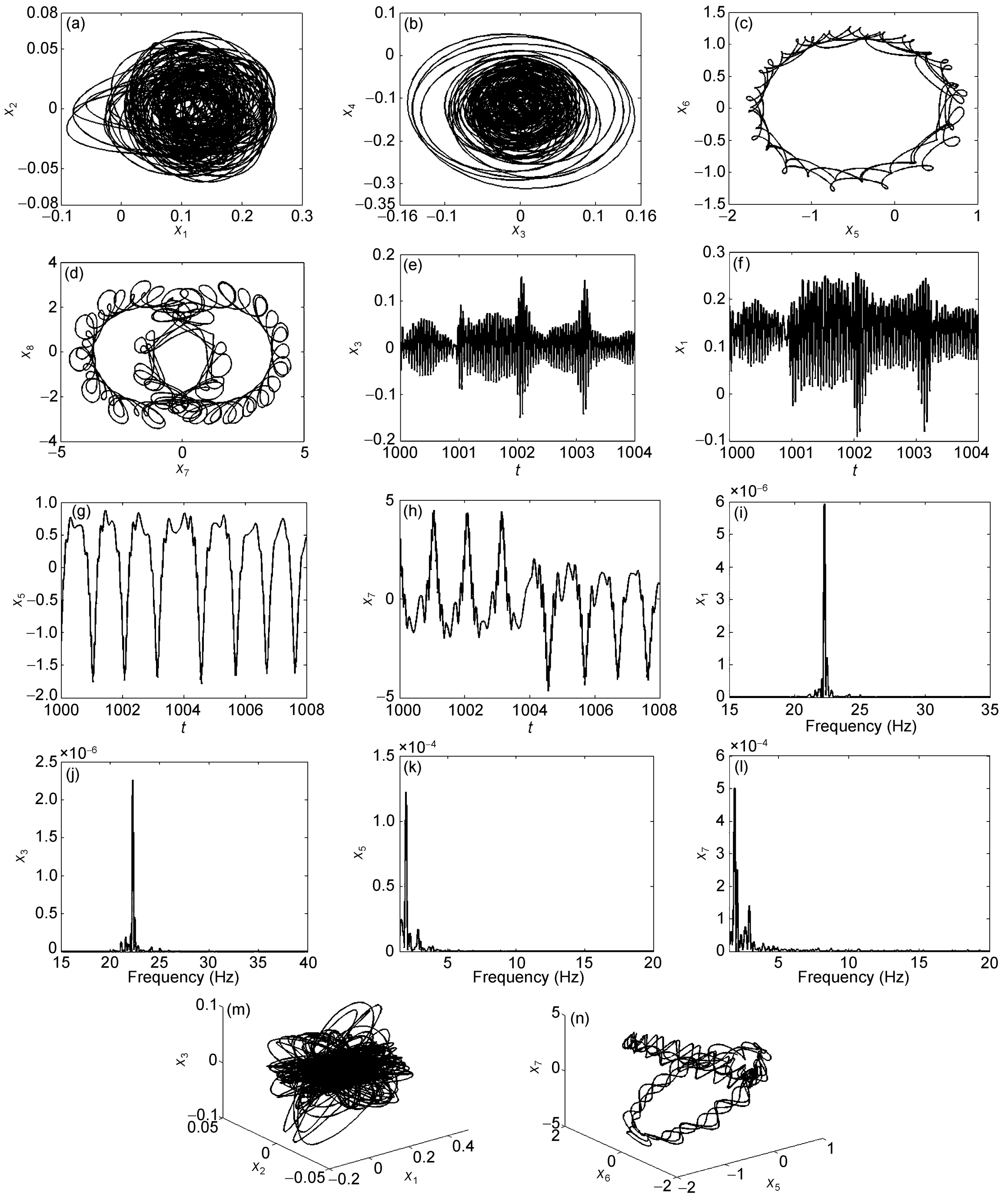

Figure 8 The chaotic motion is given for the spinning disk when c_2=0.05.

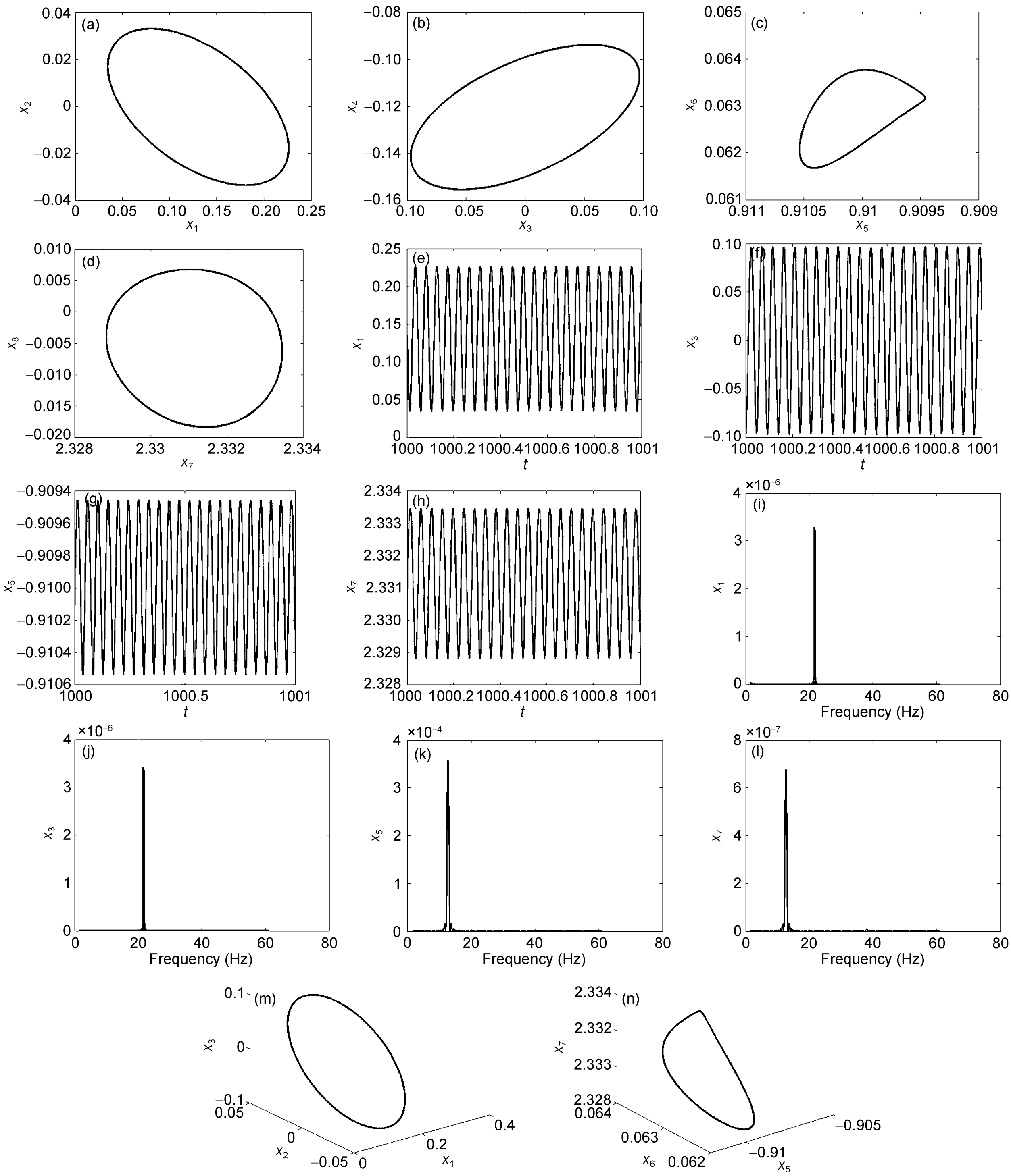

Figure 9 The periodic motion is given for the spinning disk when c_2=0.5.

nance for the second-order sin and cos modes. Numerical simulations are performed to study the nonlinear dynamic responses of the spinning disk, and detect the effects of different parameters on the nonlinear responses based on the averaged equations.

It can be concluded from the numerical results that there exist complex nonlinear behaviors for the spinning disk with a periodically varying rotating speed. The multi-pulse chaotic motions, periodic and period-n motions can occur for the spinning disk. We find that the motions of the spinning disk are sensitive to the external excitation f and the damping parameter c_2 of the second-order mode. From the

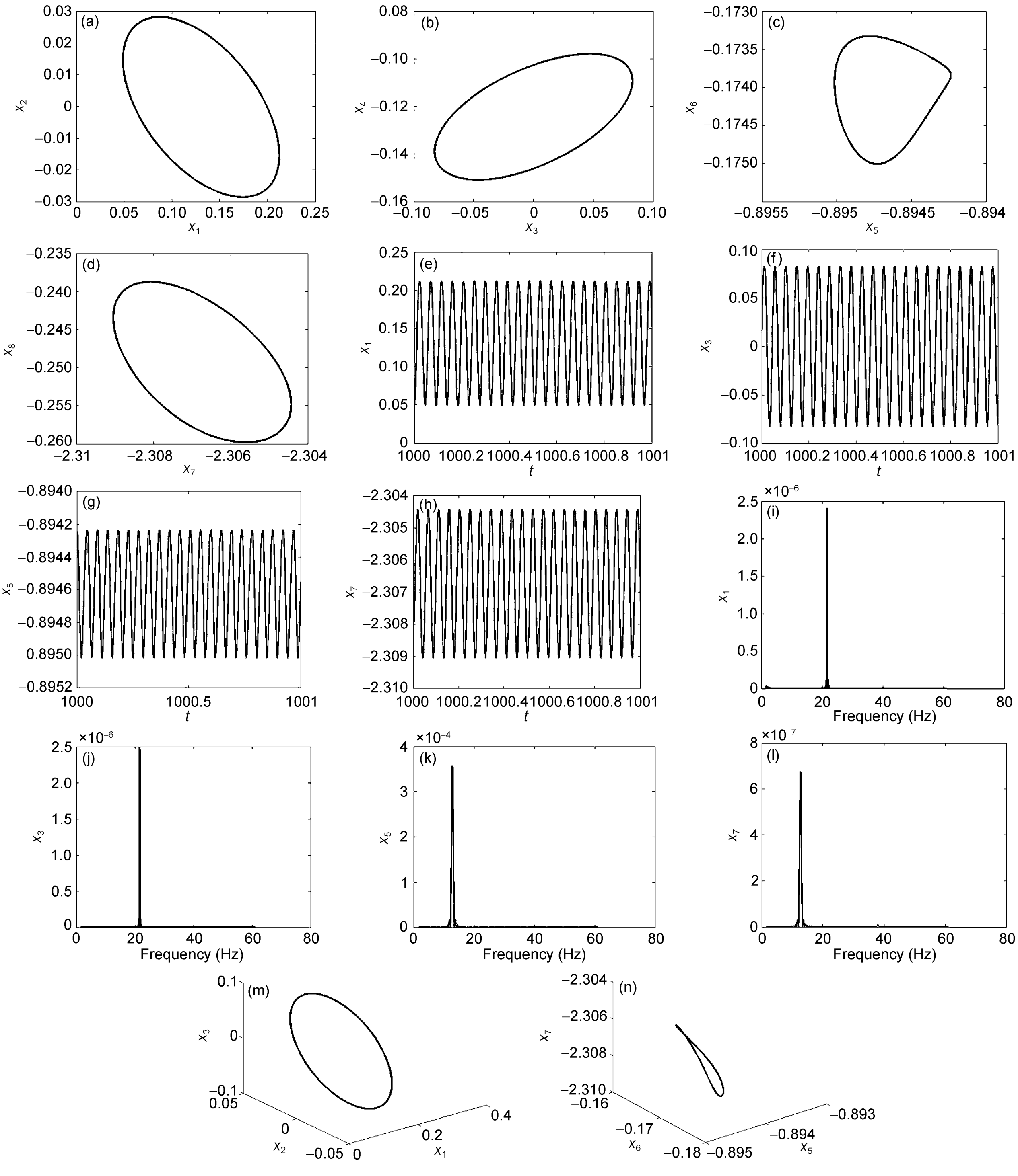

Figure 10 The periodic motion is given for the spinning disk when c_2=1.5.

obtained bifurcation diagram of the external excitation f in Figure 2, it is seen that with the chosen parameters, the motions of the spinning disk mainly are the periodic responses except two narrow chaotic windows in this case. The motion of the spinning disk changes from the periodic motion to the chaotic motion suddenly without the process of period doubling bifurcation. It is also observed from the obtained bifurcation diagram of the damping parameter c_2 that the motions of the spinning disk change from chaotic to periodic. It can be concluded that the damping parameter c_2 of the second-order mode has a significant effect on the nonlinear responses of the spinning disk. The motions of the spinning disk tend to be stable with the damping parameter increasing.

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