Science in China Series G: Physics, Mechanics & Astronomy

www.scichina.com
phys.scichina.com
www.springerlink.com

Evolution of the field quantum entropy and entanglement in a system of multimode light field interacting resonantly with a two-level atom through N_j-degenerate N^{Σ}-photon process

LIU WangYun†, YANG ZhiYong & AN YuYing

School of Technical Physics, Xidian University, Xi'an 710071, China

The time evolution of the field quantum entropy and entanglement in a system of multi-mode coherent light field resonantly interacting with a two-level atom by degenerating the multi-photon process is studied by utilizing the Von Neumann reduced entropy theory, and the analytical expressions of the quantum entropy of the multimode field and the numerical calculation results for three-mode field interacting with the atom are obtained. Our attention focuses on the discussion of the influences of the initial average photon number, the atomic distribution angle and the phase angle of the atom dipole on the evolution of the quantum field entropy and entanglement. The results obtained from the numerical calculation indicate that: the stronger the quantum field is, the weaker the entanglement between the quantum field and the atom will be, and when the field is strong enough, the two subsystems may be in a disentangled state all the time; the quantum field entropy is strongly dependent on the atomic distribution angle, namely, the quantum field and the two-level atom are always in the entangled state, and are nearly stable at maximum entanglement after a short time of vibration; the larger the atomic distribution angle is, the shorter the time for the field quantum entropy to evolve its maximum value is; the phase angles of the atom dipole almost have no influences on the entanglement between the quantum field and the two-level atom. Entangled states or pure states based on these properties of the field quantum entropy can be prepared.

field quantum entropy, Von Neumann reduced entropy, degenerate multi-photon process, quantum entanglement

Received June 15, 2007; accepted February 19, 2008
doi: 10.1007/s11433-008-0134-2
†Corresponding author (email: liu02401073@sina.com)
Supported by the Natural Science Foundation of Shaanxi Province of China (Grant No. 2001SL04) and the Science and Technology Key Task Foundation Item of Shaanxi Province (Grant No. 2002K05-G9)

1 Introduction

The Jaynes-Cummings model (JCM)[1] is an exactly solvable model of a single quantized electromagnetic field mode interacting in rotating wave approximation (RWA) with a single two-level atom in a lossless cavity. In resent years, investigators have paid much attention to the properties of the entropy in JCM and its generalizations due to the quantum entropy theory of the interaction of the field with an atom presented by Phoenix and Knight (P-K)[2−4]. They have shown that the quantum entropy is a very convenient and sensitive operational measure of the purity of the quantum state because it automatically includes all moments of the density operator of the system. The time evolution of the field (atom) quantum entropy reflects the time evolution of the degree of the quantum entanglement between the field and the atom. The higher the quantum entropy is, the greater the quantum entanglement will be. While one of the elementary resources of the quantum information processing[5], such as quantum cryptography[6], quantum computer[7−9] and quantum teleportation[10−12], is the quantum entanglement. Therefore, it is important to quantify the quantum entanglement so as to assess the efficacy of quantum information processing and to prepare the quantum entangled states under well-controlled experimental conditions.

Some investigators have studied the evolution of the field quantum entropy and entanglement of the field-atom in the one-photon JCM[2,3,13,14], two-photon JCM[4,15,16] and multiple-photon JCM[17,18]. Other investigators[19] have examined the evolution of the field quantum entropy and entanglement of the two-mode field interacting resonantly with a two-level atom by non-degenerating the two-photon process. However, up to now, less attention has been paid to the study of the evolution of the field quantum entropy and entanglement of the multi-mode field interacting with a two-level atom. This paper investigates the evolution of the field quantum entropy and entanglement of the multi-mode coherent fields interacting resonantly with a two-level atom via the N_j-degenerate N^{Σ}-photon transition process by regarding the atom as the environment. The quantum entanglement is studied between the multi-mode coherent fields and the two-level atom by utilizing the quantum reduced entropy theory.

2 The model and its solution

We consider a single two-level atom interacting with the multi-mode (q-mode) coherent field through the N_j-degenerate N^{Σ}-photon transition process. The effective Hamiltonian of this system can be written in the RWA as ($\hbar = 1$)

$$\begin{aligned} H &= H_0 + H_I, \\ H_0 &= \omega_0 S_z + \sum_{j=1}^{q} \omega_j a_j^+ a_j, \\ H_I &= g\left[\left(\prod_{j=1}^{q} a_j^{+N_j}\right) S_- + S_+ \left(\prod_{j=1}^{q} a_j^{N_j}\right)\right], \end{aligned} \tag{1}$$

where S_z and $S_{\mp}$ are the usual pseudo-spin operators of the single two-level atom with a transition frequency ω_0; a_j^+ (a_j) is the photon creation (annihilation) operator of the field with frequency ω_j ($j = 1, 2, \cdots, q$); g is the coupling constant between the atom and the q-mode coherent fields and N_j is the degree of degeneration. For simplicity, we consider only the N^{Σ}-photon resonance conditions so that $\omega_0 = \sum_{j=1}^{q} N_j \omega_j$. Using the standard techniques in ref. [2], from the effective Hamiltonian we can obtain the following time evolution operator in the interaction picture:

$$U_I(t) = \mathrm{e}^{-\mathrm{i}H_I t} = \begin{pmatrix} \hat{u}_{11} & -\mathrm{i}\hat{u}_{12} \\ -\mathrm{i}\hat{u}_{21} & \hat{u}_{22} \end{pmatrix}. \tag{2}$$

Here we have written the operators as

$$\begin{aligned} &\hat{u}_{11} = \cos(\hat{A}gt), \qquad \hat{u}_{12} = \left(\prod_{j=1}^{q} a_j^{N_j}\right)\frac{\sin(\hat{B}gt)}{\hat{B}}, \\ &\hat{u}_{21} = \left(\prod_{j=1}^{q} a_j^{+N_j}\right)\frac{\sin(\hat{A}gt)}{\hat{A}}, \qquad \hat{u}_{22} = \cos(\hat{B}gt), \\ &\hat{A} = \left[\left(\prod_{j=1}^{q} a_j^{N_j}\right)\left(\prod_{j=1}^{q} a_j^{+N_j}\right)\right]^{1/2}, \quad \hat{B} = \left[\left(\prod_{j=1}^{q} a_j^{+N_j}\right)\left(\prod_{j=1}^{q} a_j^{N_j}\right)\right]^{1/2}. \end{aligned} \tag{3}$$

We consider the situation that the single two-level atom is in a coherent superposition state of the excited state $|e\rangle$ and the ground state $|g\rangle$ at the initial time $t=0$, that is

$$|\psi_a(0)\rangle = \cos\left(\frac{\theta}{2}\right)|e\rangle + \mathrm{e}^{-\mathrm{i}\phi}\sin\left(\frac{\theta}{2}\right)|g\rangle, \tag{4}$$

while the field is in the q-mode coherent state, namely

$$\begin{aligned} &|\psi_f(0)\rangle = |\alpha_1, \alpha_2, \cdots, \alpha_q\rangle = \sum_{n_1, n_2, \cdots, n_q = 0}^{\infty} P(n_1)P(n_2)\cdots P(n_q)|n_1, n_2, \cdots, n_q\rangle, \\ &P(n_j) = \exp\left(-\frac{\bar{n}_j}{2}\right)\frac{\alpha_j^{n_j}}{\sqrt{n_j\,!}}, \end{aligned} \tag{5}$$

where $0 \leqslant \theta < \pi$ with θ denoting the atomic distribution angle, $0 \leqslant \phi < 2\pi$ with ϕ being the phase angle of the atom dipole, $\alpha_j = \sqrt{\bar{n}_j}\exp(\mathrm{i}\varphi_j)$, where $\bar{n}_j$ and φ_j ($0 \leqslant \varphi_j < 2\pi$) represent the initial average photon number and the direction angle of the excitation for j ($j = 1, 2, \cdots, q$) mode, respectively. So, the initial state of the system can be written as

$$|\psi(0)\rangle = |\psi_a(0)\rangle \otimes |\psi_f(0)\rangle. \tag{6}$$

At any time $t > 0$ the state vector of the system is given by

$$|\psi(t)\rangle = U_I(t)|\psi(0)\rangle = |C\rangle|e\rangle + |D\rangle|g\rangle, \tag{7}$$

with

$$\begin{aligned} |C\rangle = &\sum_{n_1, n_2, \cdots, n_q = 0}^{\infty}\left[\prod_{j=1}^{q} q(n_j)\right]\exp\left[\mathrm{i}\left(\sum_{j=1}^{q} n_j\varphi_j\right)\right]\left[\cos\frac{\theta}{2}\cos(\Omega_{n_j}^{+} t)|n_1, n_2, \cdots, n_q\rangle\right. \\ &\left. -\mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\sin\frac{\theta}{2}\sin(\Omega_{n_j}^{-} t)|n_1 - N_1, n_2 - N_2, \cdots, n_j - N_j, \cdots, n_q - N_q\rangle\right], \end{aligned} \tag{8}$$

$$|D\rangle = \sum_{n_1, n_2, \cdots, n_q = 0}^{\infty}\left[\prod_{j=1}^{q} q(n_j)\right]\exp\left[\mathrm{i}\left(\sum_{j=1}^{q} n_j\varphi_j\right)\right]\left[-\mathrm{i}\cos\frac{\theta}{2}\sin(\Omega_{n_j}^{+} t)\right.$$

$$\times\left|n_1+N_1,n_2+N_2,\cdots,n_j+N_j,\cdots,n_q+N_q\right\rangle+\mathrm{e}^{-\mathrm{i}\phi}\sin\frac{\theta}{2}\cos(\Omega_{n_j}^{-}t)\left|n_1,n_2,\cdots,n_q\right\rangle\Bigg], \tag{9}$$

where

$$q(n_j)=\exp\left(-\frac{\overline{n}_j}{2}\right)\frac{\overline{n}_j^{(n_j/2)}}{\sqrt{n_j\,!}},$$

$$\Omega_{n_j}^{+}=g\left\{\prod_{j=1}^{q}\left[\frac{(n_j+N_j)!}{n_j\,!}\right]\right\}^{1/2} \quad \text{and} \quad \Omega_{n_j}^{-}=g\left\{\prod_{j=1}^{q}\left[\frac{n_j\,!}{(n_j-N_j)!}\right]\right\}^{1/2} \tag{10}$$

are the parameters relevant to the Rabi frequency of the atom.

Therefore, at any time $t>0$, the density matrix for the system is given by

$$\rho(t)=\left|\psi(t)\right\rangle\left\langle\psi(t)\right|=\begin{pmatrix}\left|C\right\rangle\left\langle C\right| & \left|C\right\rangle\left\langle D\right|\\ \left|D\right\rangle\left\langle C\right| & \left|D\right\rangle\left\langle D\right|\end{pmatrix}, \tag{11}$$

and the reduced density matrix of the q-mode coherent field is given by

$$\rho_f(t)=tr_a[\rho(t)]=\left|C\right\rangle\left\langle C\right|+\left|D\right\rangle\left\langle D\right|. \tag{12}$$

By using the standard techniques in ref. [2], it can be shown that the eigenvalues of the reduced density operator $\hat{\rho}_f(t)$ are given by

$$\pi_f^{\pm}(t)=\left\langle C|C\right\rangle\pm\exp(\mp\delta)\left|\left\langle C|D\right\rangle\right|=\left\langle D|D\right\rangle\pm\exp(\pm\delta)\left|\left\langle D|C\right\rangle\right|, \tag{13}$$

where

$$\delta=\sinh^{-1}\left[\frac{1}{2\left|\left\langle C|D\right\rangle\right|}(\left\langle C|C\right\rangle-\left\langle D|D\right\rangle)\right], \tag{14}$$

$$\begin{aligned}\left\langle C|C\right\rangle=\sum_{n_1,n_2,\cdots,n_q=0}^{\infty}&\left\{\left(\prod_{j=1}^{q}\left|q(n_j)\right|^2\right)\left[\cos^2\left(\frac{\theta}{2}\right)\cos^2(\Omega_{n_j}^{+}t)+\sin^2\left(\frac{\theta}{2}\right)\sin^2(\Omega_{n_j}^{-}t)\right]\right.\\&\left.+\left[\prod_{j=1}^{q}q(n_j-N_j)q(n_j)\right]\sin\theta\cos(\Omega_{n_j}^{-}t)\sin(\Omega_{n_j}^{-}t)\sin\left[\left(\sum_{j=1}^{q}N_j\varphi_j\right)-\phi\right]\right\},\end{aligned} \tag{15}$$

$$\begin{aligned}\left\langle D|D\right\rangle=\sum_{n_1,n_2,\cdots,n_q=0}^{\infty}&\left\{\left(\prod_{j=1}^{q}\left|q(n_j)\right|^2\right)\left[\cos^2\left(\frac{\theta}{2}\right)\sin^2(\Omega_{n_j}^{+}t)+\sin^2\left(\frac{\theta}{2}\right)\cos^2(\Omega_{n_j}^{-}t)\right]\right.\\&\left.-\left[\prod_{j=1}^{q}q(n_j+N_j)q(n_j)\right]\sin\theta\cos(\Omega_{n_j}^{+}t)\sin(\Omega_{n_j}^{+}t)\sin\left[\left(\sum_{j=1}^{q}N_j\varphi_j\right)-\phi\right]\right\},\end{aligned} \tag{16}$$

$$\begin{aligned}\left\langle C|D\right\rangle=\sum_{n_1,n_2,\cdots,n_q=0}^{\infty}&\left\{\left(\prod_{j=1}^{q}\left|q(n_j)\right|^2\right)\left[\frac{1}{2}\mathrm{e}^{-\mathrm{i}\phi}\sin\theta\cos(\Omega_{n_j}^{+}t)\cos(\Omega_{n_j}^{-}t)\right]+\mathrm{i}\mathrm{e}^{-\mathrm{i}\sum_{j=1}^{q}N_j\varphi_j}\right.\\&\times\left[\left(\prod_{j=1}^{q}q(n_j)q(n_j-N_j)\right)\sin^2\left(\frac{\theta}{2}\right)\sin(\Omega_{n_j}^{-}t)\cos(\Omega_{n_j-N_j}^{-}t)\right.\\&\left.-\left(\prod_{j=1}^{q}q(n_j+N_j)q(n_j)\right)\cos^2\left(\frac{\theta}{2}\right)\cos(\Omega_{n_j+N_j}^{+}t)\sin(\Omega_{n_j}^{+}t)\right]\end{aligned}$$

$$+\left(\prod_{j=1}^{q} q(n_j+2N_j)q(n_j)\right)\frac{1}{2}\mathrm{e}^{\mathrm{i}\left(\phi-2\sum_{j=1}^{q}N_j\varphi_j\right)}\sin\theta\sin(\Omega_{n_j+2N_j}^{-}t)\sin(\Omega_{n_j}^{+}t)\Bigg\}. \tag{17}$$

3 The field quantum entropy and quantum entanglement between the *q*-mode coherent fields and the two-level atom

Following the work by P-K in ref. [2], we can express the reduced quantum entropy $S_f(t)$ of the q-mode coherent fields in terms of the eigenvalue $\pi_f^{\pm}(t)$ of the field reduced density matrix $\rho_f(t)$, given by expression (13), that is

$$S_f(t)=-[\pi_f^{+}(t)\ln\pi_f^{+}(t)+\pi_f^{-}(t)\ln\pi_f^{-}(t)]. \tag{18}$$

If the minimum value of $S_f(t)$ is taken to be zero, the q-mode coherent fields and the atom are disentangled; if the maximum value of $S_f(t)$ is taken to be one, the q-mode coherent fields and the atom are in a maximal entangled state; if the value of $S_f(t)$ is between zero and one, the q-mode coherent fields and the atom are in usual entangled states. It seems to be impossible to express the sums in eqs. (15)—(17) in a closed form, but as for a not too large $\overline{n}_j$, the direct numerical temporal evolution of the entropy can take place based on the analytical solution presented by expression (18). In what follows, we shall consider the behaviors of the field quantum entropy and the quantum entanglement of the system. It should be emphasized that by computing all the infinite series for the eigenvalues of the reduced field density operator, we have mathematically invoked the sound truncation criteria.

Firstly, we examine the influence of the initial average photon number (namely, the intensity of the field) on the time evolution of the three-mode coherent field entropy $S_f(t)$. For simplicity, here we have set $\overline{n}_1=\overline{n}_2=\overline{n}_3$, and you can take each of them with different values according to your need.

According to the plots in Figure 1, one can see that the field and the atom are always in the maximum entangled state, and with the increase of $\overline{n}_1=\overline{n}_2=\overline{n}_3$, the value of the quantum field entropy decreases, especially when $\overline{n}_1=\overline{n}_2=\overline{n}_3$ are very large (the strong quantum field condition), the quantum field entropy $S_f(t)$ nearly tends to its zero value in the overall time evolution process (see Figure 1(c)), which indicates that the stronger the quantum field is, the weaker the entanglement between the quantum field and the atom is, and when the field is strong enough, the two subsystems may be in a disentangled state all the time.

Secondly, we investigate the evolution of the three-mode quantum field entropy with different atomic distribution angles θ under the condition that $q=3$, $\phi=0$, $\varphi_1=\varphi_2=\varphi_3=0$, $N_1=N_2=N_3=1$, $\overline{n}_1=\overline{n}_2=\overline{n}_3=5$. Figure 2 presents the numerical results.

Comparing these plots in Figure 2, one can find that the quantum field entropy is strongly dependent on the atomic initial state. The quantum field entropy reaches its maximum value after a very short time of oscillation, and then the field entropy almost remains in that sustained maximum value, and the larger the atomic distribution angle θ is, the shorter the time for the field entropy to evolve its maximum value is, which means that the quantum field and the two-level atom are always in an entangled state, and are nearly stable at maximum entanglement after a short time of vibration.

Thirdly, we see the time evolution of the three-mode coherent quantum field entropy under the condition that $\theta=\pi/3$, $\varphi_1=\varphi_2=\varphi_3=0$, $N_1=N_2=N_3=1$, $\overline{n}_1=\overline{n}_2=\overline{n}_3=2$ and different values

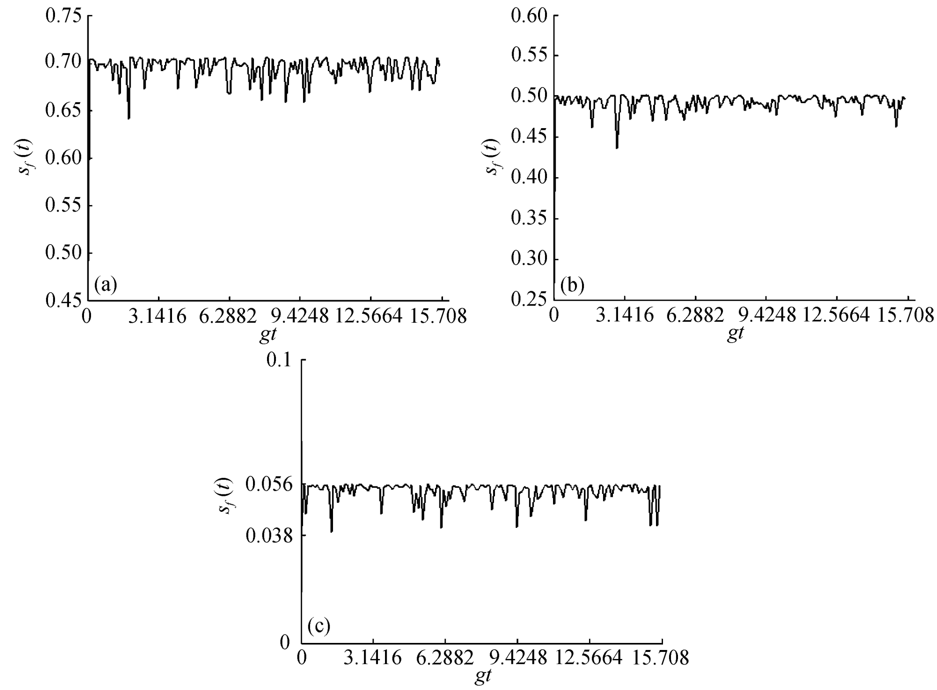

Figure 1 The evolution of the quantum field entropy $S_f(t)$ as a function of the scaled time gt with $q = 3$, $\theta = 0$, $\phi = 0$, $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $N_1 = N_2 = N_3 = 2$. (a) $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 2$; (b) $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 5$; (c) $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 10$.

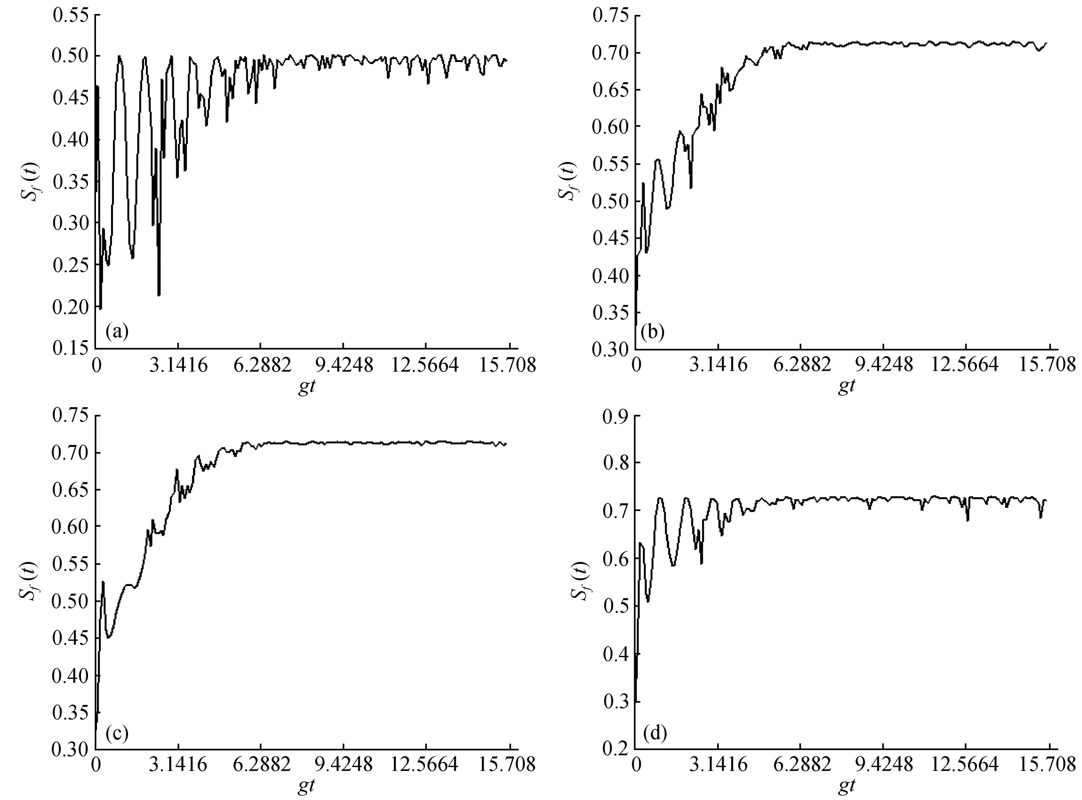

Figure 2 The evolution of the quantum field entropy $S_f(t)$ as a function of the scaled time gt with $q = 3$, $\phi = 0$, $\varphi_1 = \varphi_2 = \varphi_3 = 0$, $N_1 = N_2 = N_3 = 1$, $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = 5$. (a) $\theta = 0$; (b) $\theta = \pi/3$; (c) $\theta = \pi/2$; (d) $\theta = \pi$.

of the phase angle of the atom dipole ϕ.

It is shown that the changes of the phase angle of the atom dipole have no obvious effects on the quantum field entropy, except for $\phi = \pi$ (see Figure 3(c)), which means the phases of the atom dipole almost have no influence on the entanglement between the quantum field and the two-level atom.

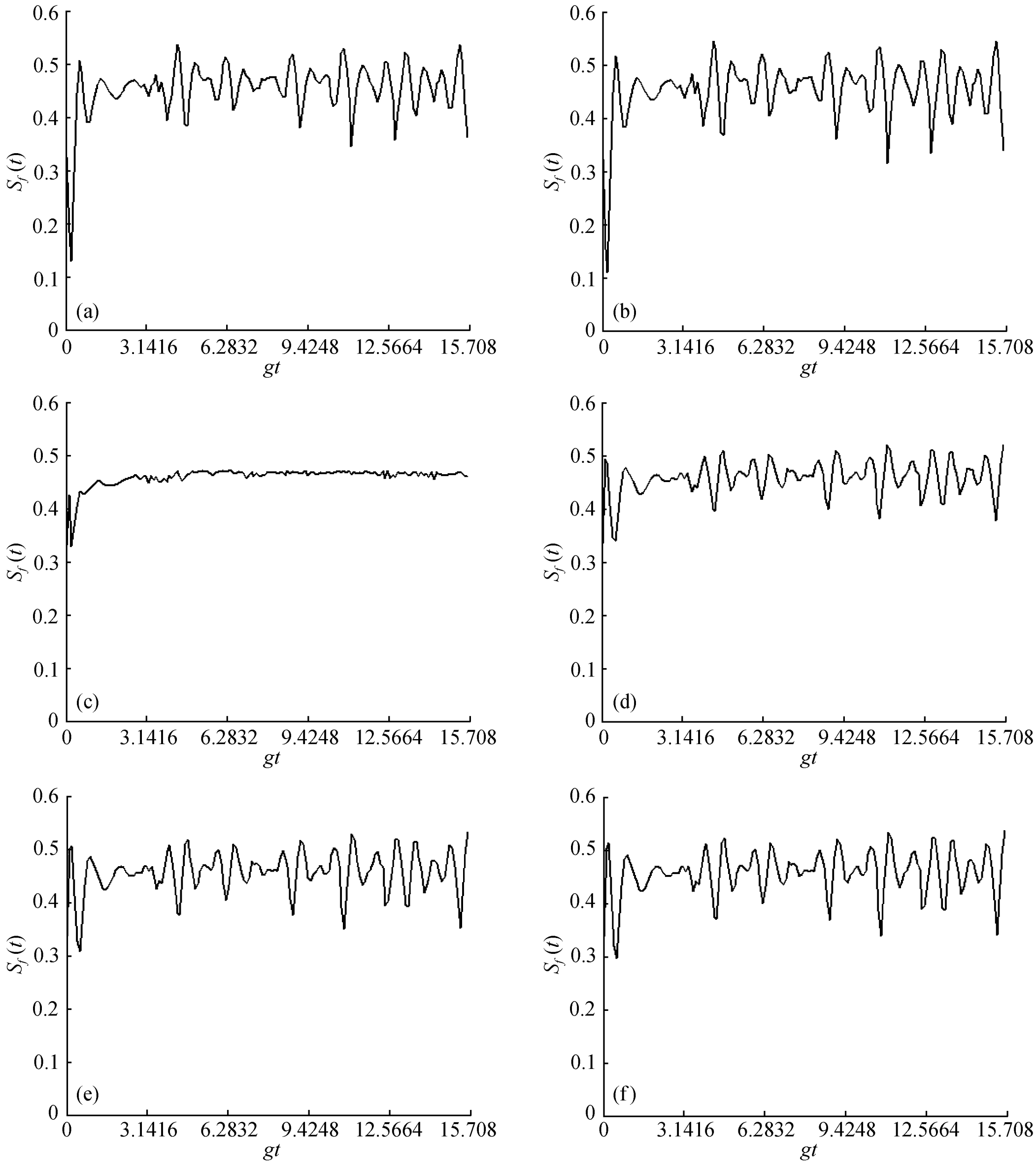

Figure 3 The evolution of the quantum field entropy $S_f(t)$ as a function of the scaled time gt with (a) $\phi = \pi/3$; (b) $\phi = \pi/2$; (c) $\phi = \pi$; (d) $\phi = 5\pi/4$; (e) $\phi = 4\pi/3$; (f) $\phi = 13\pi/8$.

4 Conclusions

We have not only examined the time evolution of the quantum field entropy and the entanglement of the multi-mode field resonantly interacting with a two-level atom via any N_j-degree degenerate N^{Σ}-photon process and obtained the analytical expression of the quantum field entropy $S_f(t)$ for the

q-mode coherent field. Besides, we have presented the numerical results for three modes coherent field ($q = 3$) and discussed in detail the influence of the different parameters $\overline{n}_j$, θ and ϕ on the evolution of the quantum field entropy. Our numerical calculation results show that: (1) With the increase of the initial average photon number $\overline{n}_1 = \overline{n}_2 = \overline{n}_3$, the value of the quantum field entropy $S_f(t)$ decreases, especially when $\overline{n}_1 = \overline{n}_2 = \overline{n}_3$ are large enough, the quantum field entropy $S_f(t)$ nearly tends to its zero value in the overall time evolution process, which indicates that the stronger the quantum field is, the weaker the entanglement between the quantum field and the atom is, and when the field is strong enough, the two subsystems may be in a disentangled state all the time. (2) The quantum field entropy is strongly dependent on the atomic initial state. The quantum field entropy reaches its maximum value after a very short time of oscillation, and then the field entropy almost remains in that sustained maximum value, and the larger the atomic distribution angle θ is, the shorter the time for the field entropy to evolve its maximum value is, which means that the quantum field and the two-level atom are always in an entangled state, and are nearly stable at the maximum entanglement after a short time of vibration. (3) The changes of the phase angle of the atom dipole have no obvious effects on the quantum field entropy, except for $\phi = \pi$, which means the phases of the atom dipole almost have no influence on the entanglement between the quantum field and the two-level atom.

1 Jaynes E T, Cummings F W. Comparison of quantum and semiclassical radiation theories with application to the beam maser. Proc IEEE, 1963, 51(1): 89－109
2 Phoenix S J D, Knight P L. Fluctuations and entropy in models of quantum optical resonance. J Ann Phys (NY), 1988, 186: 381－407
3 Phoenix S J D, Knight P L. Establishment of an entangled atom-field state in the Jaynes-Cummings model. Phys Rev A, 1991, 44(9): 6023－6029
4 Phoenix S J D, Knight P L. Periodicity, phase, and entropy in models of two-photon resonance. J Opt Soc Am B, 1990, 7(1): 116－124
5 Bennett C H, Divincenzo D P. Quantum information and computation. Nature, 2000, 404: 247－255
6 Ekert A K. Quantum cryptography bassed on Bell's theorem. Phys Rev Lett, 1991, 67: 661－663
7 Qian C. Quantum entanglement and quantum computation. Comput Sci (in Chinese), 2006, 33(12): 230－234
8 Cao W C, Liu D, Pan F, et al. Entropy product measure for multipartite pure states. Sci China Ser G-Phys Mech Astron, 2006, 49(5): 606－615
9 Ding S C, Jin Z. Review on the study of entanglement in quantum computation speedup. Chin Sci Bull, 2007, 52(16): 2161－2166
10 Bennett C H, Brassard G, Grepeau C, et al. Teleporting an unknown quantum state via dual classical and Einstein-Podosky-Rosen channels. Phys Rev Lett, 1993, 70: 1895－1899
11 Zhai Z H, Li Y M, Wang S K, et al. Experimental study of continuous-variable quantum teleportation. Acta Phys Sin (in Chinese), 2005, 54(6): 2710－2716
12 Zhou X Q, Wu Y W. Discussion on building the net of quantum teleportation using three-particle entangled states. Acta Phys Sin (in Chinese), 2007, 56(4): 1881－1887
13 Gea-Banacloche J. Atom-and field-state evolution in the Jaynes-Cummings model for large initial fields. Phys Rev A, 1991, 44(9): 5913－5931
14 Li C X, Fang M F. Linear entropy dynamics of the field in the Jaynes-Cummings model with an intensity-dependent coupling in the dispersive approximation. Chin Phys, 2003, 12(8): 866－871
15 Chen F, Wang Z J, Wu J. Influence of entanglement degree on the light field in the nondegenerate two-photon Jaynes-Cummings model. J Anhui Norm Univ (Nat Sci) (in Chinese), 2006, 29(4): 345－347
16 Zhang C Q, Xie F S. The entangled properties of motorial atom and cavity field in two-photon Jaynes-Cummings model. Laser J (in Chinese), 2007, 28(2): 70－72
17 Fang M F, Zhou G H. Influence of atomic coherence on the evolution of field entropy in multiphoton processes. Phys Lett A, 1994, 184(1): 397－402
18 Lu D M. Evolution of field entropy in the multiphoton Jaynes-Cummings model. High Energy Phys Nucl Phys (in Chinese), 2005, 29(4): 435－438
19 Liu X J, Fang M F. Entangled states in the two-mode coherent fields interacting with a two-level atom. Chin Phys, 2003, 12(9): 971－976