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Entropy product measure for multipartite pure states

CAO Wancang[1,2], LIU Dan[2], PAN Feng[3] & LONG Guilu[2,4]

1. Department of Physics, Chifeng College, Chifeng 024001, China;
2. Key Laboratory for Quantum Information and Measurements, Department of Physics, Tsinghua University, Beijing 100084, China;
3. Department of Physics, Liaoning Normal University, Dalian 116029, China;
4. Key Laboratory for Atomic and Molecular Nano-Sciences, Tsinghua University, Beijing 100084, China

Correspondence should be addressed to Long Guilu (email: gllong@tsinghua.edu.cn)

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Abstract An entanglement measure for multipartite pure states is formulated using the product of the von Neumann entropy of the reduced density matrices of the constituents. Based on this new measure, all possible ways of the maximal entanglement of the triqubit pure states are studied in detail and all types of the maximal entanglement have been compared with the result of 'the average entropy'. The new measure can be used to calculate the degree of entanglement, and an improvement is given in the area near the zero entropy.

Keywords: multipartite pure state, von Neumann entropy, entanglement measure.

The concept of entanglement originated from the papers on 'Schrödinger cat'[1] and 'EPR paradox'[2]. Entanglement plays an important role in the theory of quantum information and quantum computation[3−5], so its measure is of significant importance. Entanglement of bipartite pure states can be measured by the von Neumann entropy of either of the two reduced density matrices[6]. The entropy of the two reduced density matrices is equal because Schmidt decomposition[7] exists in a bipartite system.

In general, Schmidt decomposition does not exist for a multipartite pure state[8,9], which had been proved by A. Peres, who further gave the necessary and sufficient condition of the existence of a Schmidt decomposition for some special tripartite pure states[10]. The minimal number of orthogonal product states can be found by representation transformation, and a general pure state can be characterized by a superposition of this minimal set of orthogonal products states. Acin *et al*. found that there need five terms for a triqubit system in the worst case. They called the representation in terms of this minimal set of products states Schmidt representation or canonical representation[11], and classified the entanglement of the triqubit system by this representation.

An 'averaged entropy' measure was given for a multipartite system in ref. [12] in which the conditions of triqubit pure states[13] and bipartite qutrit pure states[14] were discussed and a significant result was found. Some of us found that this measure is equivalent to the scheme of the distance from the maximal entangled state1). For the measure in ref. [12] is the average of entropy, some details cannot be described well, especially in the area of zero entropy of the reduced density matrices. Another method is proposed in this paper using the product of the entropies of the reduced density matrices of constituent particles.

For an N-qubit pure state $|\psi\rangle$, the average entropy[10] was defined by

$$S_2 = \begin{cases} \frac{1}{N}\sum_{i=1}^{N} E_i, & \text{if for all } i \text{ there is } E_i \neq 0, \\ 0, & \text{if for any } i \text{ there is } E_i = 0, \end{cases} \tag{1}$$

where

$$E_i = -Tr[(\rho_\psi)_i \log_2(\rho_\psi)_i] \qquad (i = 1, 2, \cdots, N) \tag{2}$$

is the reduced von Neumann entropy for the i-th particle with other $N-1$ particles traced out.

We propose a measure using the product of entropies of the reduced density matrices for an N-particle system:

$$S_1 = \prod_{i=1}^{N} E_i, \tag{3}$$

where the definition of E_i is just given in eq. (2).

1 Analysis and comparison for the measure of multipartite pure states

We calculate all types of the triqubit system to prove the feasibility of the entropy product measure and show that the discontinuation when $E=0$ in ref. [12] can be avoided in this new measure.

When $N=3$, the basis vectors of a triqubit system are denoted by

$$\{|W_1\rangle = |000\rangle, \quad |W_2\rangle = |110\rangle, \quad |W_3\rangle = |101\rangle, \quad |W_4\rangle = |011\rangle,$$
$$|\overline{W}_1\rangle = |111\rangle, \quad |\overline{W}_2\rangle = |001\rangle, \quad |\overline{W}_3\rangle = |010\rangle, \quad |\overline{W}_4\rangle = |100\rangle\}. \tag{4}$$

Any entangled triqubit state can be classified according to the number of terms in the expressions of basis vectors given in eq. (4). Now we proceed to discuss separately the various types of states according to the classification given in ref. [13].

1.1 Triqubit states with two terms

When $|\Psi\rangle$ is a linear combination of two terms in eq. (4), there are 28 such combinations. When the combination is $|W_i\rangle$ and $|\overline{W}_i\rangle$ with $i \in P$ ($P = \{1,2,3,4\}$), $|\Psi\rangle$ is an entangled state. There are four such combinations. Taking

1) Cao W, Liu D. A compare of the entanglement representation for multipartite pure states (submitted)

$$|\psi\rangle = a|W_1\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|\overline{W}_1\rangle = a|000\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|111\rangle \tag{5}$$

as an example, where a and b are real numbers satisfying $a^2 + b^2 = 1$, and α is the relative phase. Using formula (3), we can get the expression for the entropy product entanglement measure as

$$S_1(a,b) = \prod_{i=1}^{3} E_i = -(a^2 \log_2 a^2 + b^2 \log_2 b^2)^3. \tag{6}$$

The corresponding average entropy is marked by $S_2(a,b)$ (hereafter we use S_2 to denote the corresponding average entropy). S_1 and S_2 are plotted in Fig. 1. It is seen that the trends of the two curves are the same, so is the position of the extremes of the two curves.

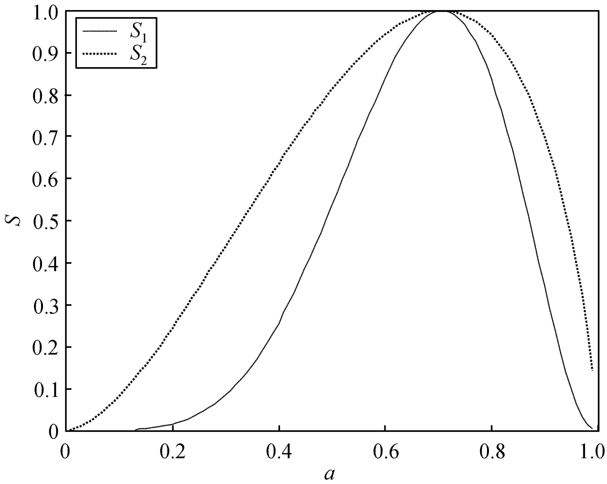

Fig. 1. The entanglement measure of state (5).

1.2 Triqubit states with three terms

When $|\Psi\rangle$ is a linear combination of three terms in eq. (4), there are 56 such combinations, of which 32 are genuine entangled triqubit pure states. Now we discuss two cases of them.

(i) When the state is a linear combination of three terms $(|W_i\rangle, |\overline{W}_i\rangle, |W_j\rangle)$ or $(|W_i\rangle, |\overline{W}_i\rangle, |\overline{W}_j\rangle)$ with $i, j \in P$ $(i \neq j,\ P = \{1,2,3,4\})$, these are the first type of three-term entanglement. There are 24 such states. We take

$$|\psi\rangle = a|W_2\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|\overline{W}_2\rangle + c\mathrm{e}^{\mathrm{i}\beta}|\overline{W}_3\rangle = a|110\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|001\rangle + c\mathrm{e}^{\mathrm{i}\beta}|010\rangle \tag{7}$$

as an example, where a, b and c are real numbers satisfying the normalizing condition; α and β are the relative phases. According to the measure defined in eq. (3), we have the expression of the entanglement measure as

$$S_1(a,b)=\prod_{i=1}^{3} E_i = -(b^2\log_2 b^2+(a^2+c^2)\log_2(a^2+c^2))^2 \times\left(\frac{1}{2}(1-u)\log_2\left[\frac{1}{2}(1-u)\right]+\frac{1}{2}(1+u)\log_2\left[\frac{1}{2}(1+u)\right]\right), \tag{8}$$

where $c=\sqrt{1-a^2-b^2}$ and $u=\sqrt{1-4a^2b^2}$. When a, b and c are all not equal to zero, S_1 and S_2 have no extremes. They have the maximal entanglement when $c = 0$ and $a=b=1/\sqrt{2}$, and this is the case of triqubit form with two terms as given in Fig. 1.

(ii) When the state is a linear combination of three terms $(|W_i\rangle,|W_j\rangle,|W_q\rangle)$ or $(|\bar{W}_i\rangle,|\bar{W}_j\rangle,|\bar{W}_q\rangle)$ with $i,j,q\in P$ $(i\neq j\neq q,\ P=\{1,2,3,4\})$, they are the second type of three-term entangled state. We take

$$|\psi\rangle = a|W_3\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_2\rangle + c\mathrm{e}^{\mathrm{i}\beta}|W_4\rangle = a|101\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|110\rangle + c\mathrm{e}^{\mathrm{i}\beta}|011\rangle \tag{9}$$

as an example, where a, b and c are real numbers satisfying the normalizing condition; α and β are the relative phases. According to the measure defined in eq. (3), we have the expression for the entanglement measure as

$$S_1(a,b)=\prod_{i=1}^{3} E_i = -(a^2\log_2 a^2+(b^2+c^2)\log_2(b^2+c^2)) \times(b^2\log_2 b^2+(a^2+c^2)\log_2(a^2+c^2))(c^2\log_2 c^2+(a^2+b^2)\log_2(a^2+b^2)), \tag{10}$$

where $c=\sqrt{1-a^2-b^2}$. It can be verified that the maximal entanglement occurs at $a=b=c=1/\sqrt{3}$. We take $c=1/\sqrt{3}$, and the curves of $S_1(a,b)$ and $S_2(a,b)$ are shown in Fig. 2. The tendency of the two measures is identical and the locations of extremes are the same. But there is difference between them. The sharp change of S_1 is visible from the plot. The change of S_2 is smooth relatively, and when $a=0$ there is obviously a discontinuous point where S_2 drops from 0.65 to 0 abruptly. This phenomenon is avoided by the measure S_1.

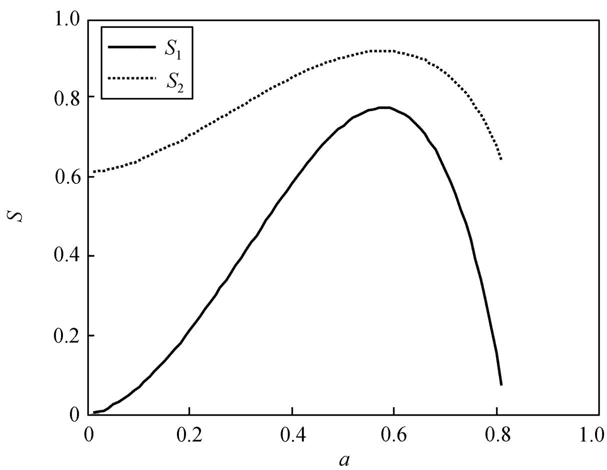

Fig. 2. The entanglement measure of state (9).

1.3 Triqubit states with four terms

When $|\Psi\rangle$ is a linear combination of four terms in eq. (4), there are 70 such combinations, of which 64 are genuine triqubit pure states. Now we discuss five cases of them in details.

(i) The state is a linear combination of four terms $(|W_i\rangle, |W_j\rangle, |\bar{W}_j\rangle, |\bar{W}_j\rangle)$ with $i, j \in P$ $(i \neq j,\ P = \{1, 2, 3, 4\})$. This is called the first type of four-term entanglement, and there are 6 such combinations. For example we can write

$$|\psi\rangle = a|\bar{W}_1\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_1\rangle + c\mathrm{e}^{\mathrm{i}\beta}|\bar{W}_3\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|\bar{W}_3\rangle$$
$$= a|111\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|000\rangle + c\mathrm{e}^{\mathrm{i}\beta}|101\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|010\rangle, \quad (11)$$

where a, b, c and d are real numbers satisfying the normalizing condition; α, β and γ are the relative phases. According to the measure defined in eq. (3), we have the expression of the entanglement measure as

$$S_1(a,b) = \prod_{i=1}^{3} E_i = -((a^2+c^2)\log_2(a^2+c^2) + (b^2+d^2)\log_2(b^2+d^2))^2$$
$$\times\left(\frac{1}{2}(1-u)\log_2\left[\frac{1}{2}(1-u)\right] + \frac{1}{2}(1+u)\log_2\left[\frac{1}{2}(1+u)\right]\right), \quad (12)$$

where $a^2+b^2+c^2+d^2=1$,

$$u = \sqrt{1+4(-a^2b^2-c^2d^2+2abcd\cos(\theta)}, \qquad \theta = \alpha-\beta-\gamma. \quad (13)$$

It can be verified that the maximal entanglement occurs at $a=b=c=d=1/2$ and $\theta=(2n+1)\pi$, where $S_1=S_2=1$ when n is integer. We take $c=d=1/2$, $\theta=\pi$, and the plots of $S_1(a,b)$ and $S_2(a,b)$ versus a are shown in Fig. 3. The tendency of the two entropy measures is identical and the locations of extreme points are the same.

$S_1=S_2=0$ when $a=b=c=d=1/2$, $\theta=2n\pi$ (n is integer). We take $\theta=0$ and

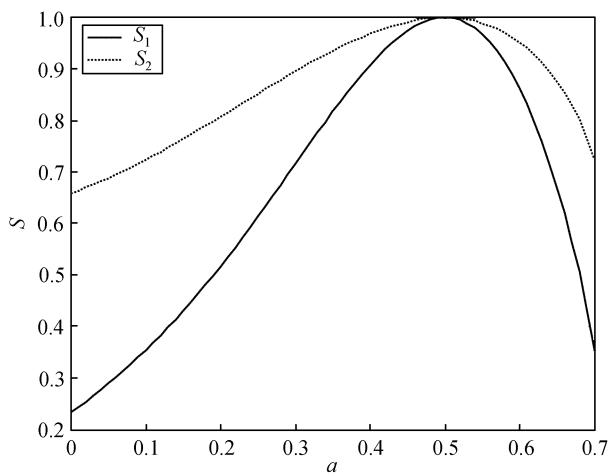

Fig. 3. Entanglement measure of state (11) with $\theta=\pi$ versus a.

$c = d = 1/2,$ and the change of $S_1(a,b)$ and $S_2(a,b)$ with a is shown in Fig. 4. The curve of S_2 is discontinuous at $a = 0.5$, because $E_2 = 0$ and consequently S_2 is also zero at this point. The values of S_2 at the vicinity are quite large; therefore it cannot reflect the smooth change of entanglement. Whereas, the smooth and continuous change feature can be reflected clearly by the product measure S_1.

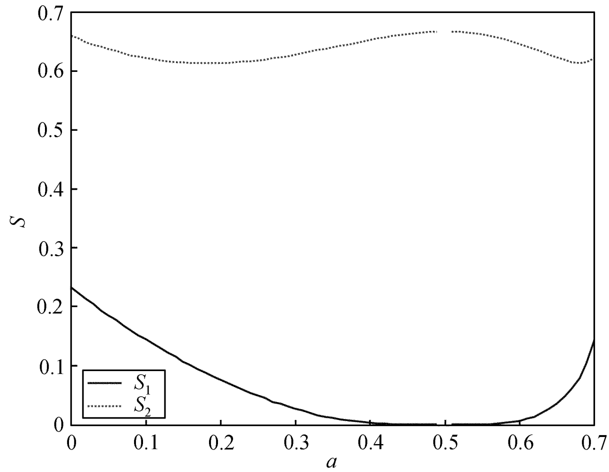

Fig. 4. Entanglement measure of state (11) with $\theta = 0$ versus a.

(ii) When the combination is $\left(|W_i\rangle, |W_j\rangle, |W_t\rangle, |W_q\rangle\right)$ or $\left(|\bar{W}_i\rangle, |\bar{W}_j\rangle, |\bar{W}_t\rangle, |\bar{W}_q\rangle\right)$ with $i, j, t, q \in P$ ($P = \{1, 2, 3, 4\}$; i, j, q are all unequal). This is the second type of four-term entanglement, and there are 2 such combinations. We take

$$\begin{aligned} |\psi\rangle &= a|\bar{W}_2\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_3\rangle + c\mathrm{e}^{\mathrm{i}\beta}|W_4\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|W_1\rangle \\ &= a|110\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|101\rangle + c\mathrm{e}^{\mathrm{i}\beta}|011\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|000\rangle \end{aligned} \tag{14}$$

as an example, where a, b, c and d are real numbers satisfying the normalizing condition; α, β and γ are the relative phases. According to the measure defined in eq. (3), we have the expression of the entanglement measure as

$$\begin{aligned} S_1(a,b) = \prod_{i=1}^{3} E_i &= -((a^2+b^2)\log_2(a^2+b^2) + (c^2+d^2)\log_2(c^2+d^2)) \\ &\times((a^2+c^2)\log_2(a^2+c^2) + (b^2+d^2)\log_2(b^2+d^2)) \\ &\times((a^2+d^2)\log_2(a^2+d^2) + (b^2+c^2)\log_2(b^2+c^2)), \end{aligned} \tag{15}$$

where $a^2+b^2+c^2+d^2=1$. Through the calculation we get $S_1 = S_2 = 1,$ when $a = b = c = d = 1/2$ and α, β, γ are arbitrary values. We take $c = d = 1/2,$ and the curve of $S_1(a,b)$ is given in Fig. 5 by eq. (15). The tendency of S_1 and S_2 is identical and the locations of extreme points are the same. The change of S_2 is smoother than S_1.

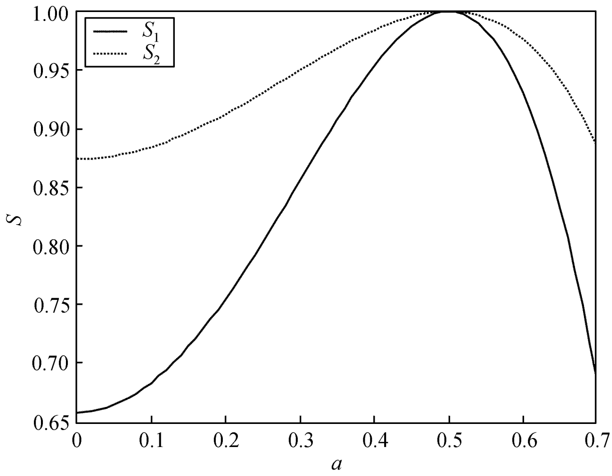

Fig. 5. The entanglement measure of state (14) versus *a*.

(iii) The combination is $\left(|W_i\rangle,|W_j\rangle,|W_t\rangle,|\bar{W}_i\rangle\right)$ or $\left(|\bar{W}_i\rangle,|\bar{W}_j\rangle,|\bar{W}_t\rangle,|W_i\rangle\right)$ with $i,j,t\in P$ ($P=\{1,2,3,4\}$; $i,\ j,\ t$ are unequal). This is the third type of four-term entanglement, and there are 24 such combinations. We take

$$\begin{aligned}|\psi\rangle &= a|W_2\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_3\rangle + c\mathrm{e}^{\mathrm{i}\beta}|W_4\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|\bar{W}_4\rangle \\ &= a|110\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|101\rangle + c\mathrm{e}^{\mathrm{i}\beta}|011\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|100\rangle \end{aligned} \tag{16}$$

as an example, where *a*, *b*, *c* and *d* are real numbers satisfying the normalizing condition; α, β and γ are the relative phases. According to the measure defined in eq. (3), we have the expression of the entanglement measure as

$$\begin{aligned}S_1(a,b) &= \prod_{i=1}^{3} E_i \\ &= -(c^2\log_2 c^2 + (a^2+b^2+d^2)\log_2(a^2+b^2+d^2)) \\ &\quad \times\left(\frac{1}{2}(1-u)\log_2\left[\frac{1}{2}(1-u)\right]+\frac{1}{2}(1+u)\log_2\left[\frac{1}{2}(1+u)\right]\right) \\ &\quad \times\left(\frac{1}{2}(1-v)\log_2\left[\frac{1}{2}(1-v)\right]+\frac{1}{2}(1+v)\log_2\left[\frac{1}{2}(1+v)\right]\right),\end{aligned} \tag{17}$$

where $a^2+b^2+c^2+d^2=1$,

$$u=\sqrt{1-4(a^2b^2+a^2c^2+c^2d^2)}, \qquad v=\sqrt{1-4(a^2b^2+b^2c^2+c^2d^2)}. \tag{18}$$

Through calculation we get the maximal entanglement when $a = b = 0.4607$, $c = 0.6515$, $d = 0.3889$. We take $c=0.6515,\ d=0.3889$, and the curve of $S_1(a,b)$ is given in Fig. 6. The tendency of S_1 and S_2 is identical and the locations of the extreme points are the same. Again, the change of S_2 is smoother than S_1.

(iv) $|\Psi\rangle$ is a linear combination of four terms $(\left(|W_i\rangle,|W_j\rangle,|\bar{W}_i\rangle,|\bar{W}_q\rangle\right)$ with $i,j,q\in P$ ($P=\{1,2,3,4\}$; $i,\ j,\ q$ are all not equal). This is the fourth type of the

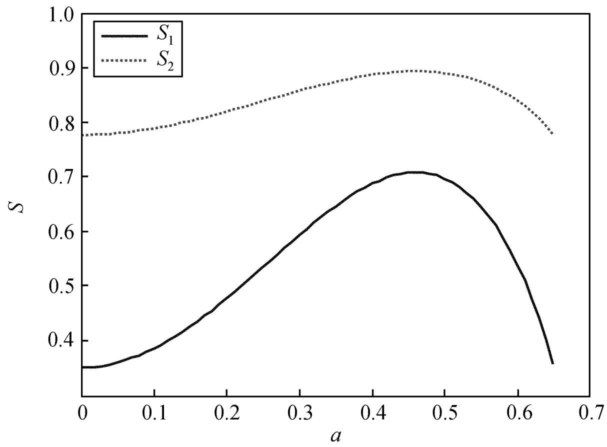

Fig. 6. The entanglement measure of state (17) versus a.

four-term entanglement, and there are 24 such combinations. We take

$$\begin{aligned}|\psi\rangle &= a|\overline{W}_1\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_2\rangle + c\mathrm{e}^{\mathrm{i}\beta}|W_3\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|\overline{W}_3\rangle \\ &= a|111\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|110\rangle + c\mathrm{e}^{\mathrm{i}\beta}|101\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|010\rangle \end{aligned} \tag{19}$$

as an example, where a, b, c and d are real numbers satisfying the normalizing condition; α, β and γ are the relative phases. According to the measure defined in eq. (3), we have the expression of the entanglement measure as

$$\begin{aligned} S_1(a,b) &= \prod_{i=1}^{3} E_i \\ &= -\left(\frac{1}{2}(1-u)\log_2\left[\frac{1}{2}(1-u)\right] + \frac{1}{2}(1+u)\log_2\left[\frac{1}{2}(1+u)\right]\right) \\ &\times\left(\frac{1}{2}(1-v)\log_2\left[\frac{1}{2}(1-v)\right] + \frac{1}{2}(1+v)\log_2\left[\frac{1}{2}(1+v)\right]\right) \\ &\times\left(\frac{1}{2}(1-k)\log_2\left[\frac{1}{2}(1-k)\right] + \frac{1}{2}(1+k)\log_2\left[\frac{1}{2}(1+k)\right]\right), \end{aligned} \tag{20}$$

where $a^2+b^2+c^2+d^2=1$,

$$u=\sqrt{1-4b^2c^2-4c^2d^2}, \qquad v=\sqrt{1-4a^2d^2-4c^2d^2},$$
$$k=\sqrt{1-4[b^2c^2+(a^2+c^2)d^2]}\ . \tag{21}$$

Through calculation we found that S_1 and S_2 have no extreme points when none of a, b, c and d is equal to zero. The maximal entanglement occurs when $a=b=0$, $c=d=1/\sqrt{2}$ with arbitrary relative phases. In this case, it reduces to the triqubit entangled state with two terms.

(v) When the state is a linear combination of four terms $(|W_i\rangle,|W_j\rangle,|W_t\rangle,|\overline{W}_q\rangle)$ or $(|\overline{W}_i\rangle,|\overline{W}_j\rangle,|\overline{W}_t\rangle,|W_q\rangle)$ with $i,j,t,q\in P$ ($P=\{1,2,3,4\}$; i, j, t, q are not equal), this is the fifth type of four-term entanglement. There are 8 such combinations. We take

$$\begin{aligned}|\psi\rangle &= a|W_2\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|W_3\rangle + c\mathrm{e}^{\mathrm{i}\beta}|W_4\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|\bar{W}_1\rangle \\ &= a|110\rangle + b\mathrm{e}^{\mathrm{i}\alpha}|101\rangle + c\mathrm{e}^{\mathrm{i}\beta}|011\rangle + d\mathrm{e}^{\mathrm{i}\gamma}|111\rangle \end{aligned} \quad (22)$$

as an example, where a, b, c and d are real numbers satisfying the normalizing condition; α, β and γ are the relative phases. According to the measure defined in eq. (3), we have the expression for the entanglement measure same as eq. (20) with different u, v and k:

$$u = \sqrt{1-4a^2b^2-4b^2c^2}, \quad v = \sqrt{1-4a^2c^2-4b^2c^2}, \quad k = \sqrt{1-4a^2(b^2+c^2)}. \quad (23)$$

Through calculation, it is seen that S_1 and S_2 have no extreme points when none of a, b, c and d is equal to zero. The system will degenerate to three-term entangled state when $d = 0$.

We also calculate such linear combinations up to eight terms and find no maximal entanglement other than the three different classes of entangled states discussed above. We obtain the same conclusion as ref. [12].

2 Conclusion

Triqubit pure states have three inequivalent classes of maximal entanglement through calculation and analysis of the triqubit pure states above. Through this study we can get the following conclusions: (1) The entropy product measurement S_1 has a simple form, and we do not need to single out the special case of zero entropy of constituent subsystems as in the average entropy measure; (2) the properties of the states with zero entanglement and the states nearby can be well described; (3) the conclusion is the same as that from the average entropy measure for the case of the maximal entanglement being 1; (4) the location of the maximal entanglement is the same as that from the average entropy measure.

In summary, we have formulated a simple entanglement measure for multipartite pure states based on the product of partial entropy of a series of reduced density matrices. The new definition is simpler.

Viewed from the property of mathematics, the change of product entropy is faster than the average entropy. The location of the maximal entanglement is the same with the two definitions. This is a remarkable phenomenon, and more work remains to be done on it.

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