DOI: 10.1007/s11433-006-0291-0

The refined theory of deep rectangular beams based on general solutions of elasticity

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Received August 9, 2005; accepted April 6, 2006

Abstract The problem of deducing one-dimensional theory from two-dimensional theory for a homogeneous isotropic beam is investigated. Based on elasticity theory, the refined theory of rectangular beams is derived by using Papkovich-Neuber solution and Lur'e method without *ad hoc* assumptions. It is shown that the displacements and stresses of the beam can be represented by the angle of rotation and the deflection of the neutral surface. Based on the refined beam theory, the exact equations for the beam without transverse surface loadings are derived and consist of two governing differential equations: the fourth-order equation and the transcendental equation. The approximate equations for the beam under transverse loadings are derived directly from the refined beam theory and are almost the same as the governing equations of Timoshenko beam theory. In two examples, it is shown that the new theory provides better results than Levinson's beam theory when compared with those obtained from the linear theory of elasticity.

Keywords: deep rectangular beams, the refined theory, Papkovich-Neuber solution, Lur'e method, governing equation.

The beam theory has been studied for many years. In the early eighteenth century, Bernoulli and Euler presented the classical beam theory, and Timoshenko[1] introduced the shear theory of beams, then Cowper[2] gave the shear coefficients. Since then, more and more work on the subject has been done by the following researchers, i.e., Levinson[3], Bickford[4], Tutek and Aganović[5], Lewiński[6], Fan and Widera[7], Tullini and Savoia[8].

Cheng[9] gave a refined plate theory from Boussinesq-Galerkin elasticity solution and Lur'e method[10] without *ad hoc* assumptions. The refined plate theory consists of three parts: the biharmonic equation, the shear equation and the transcendental equation. People may doubt the legitimacy of manipulations performed on differential operators in the derivation; however, the final results obtained by his method can be justified by the satisfaction of all equations in the three-dimensional theory of elasticity. The only approximation in Cheng's plate theory is due to the approximate specification of boundary conditions at the edges of plates; therefore, regarding Saint-Venant's principle, Cheng's theory is a very accurate one.

A parallel development of Cheng's theory by Barrett and Ellis[11] has been obtained for the isotropic plates under transverse surface loadings (only homogeneous cases are considered in the previous works). The paper also presents a detailed discussion on the specification of boundary conditions in light of the work of Gregory and Wan[12,13]. Their work actually indicates that various approximate theories for plates subjected to surface loadings can be developed directly from the three-dimensional theory of elasticity.

Wang and Shi[14] derived a new thick plate theory by using Papkovich-Neuber solution and Lur'e method[10] without *ad hoc* assumptions, and derived the shear theory of plates from the refined plate theory. Moreover, from the nonuniqueness of Papkovich-Neuber solution, a rigorous proof was given: the deformation of bending plates may be described by three generalized displacements on the neutral surface of the plate without loss in generality. Yin and Wang[15] extended it for the transversely isotropic plates by using Elliott-Lodge solution. Xu and Wang[16] applied the results[14] to the problem of a transversely isotropic piezoelectric plate, and derived approximate equations for the plate under transverse loadings. Recently, several extensions have been found in the rectangular beam problems of magnetoelastic beams[17], thermoelastic beams[18] and piezoelectric beams[1)], and the refined theory of beams in the coupling fields has been obtained. Moreover, the exact equations for the beam without transverse surface loadings and the approximate equations for the beam under transverse loadings are derived from the refined beam theory, respectively.

This paper presents the theory for a deep beam of rectangular cross-section by using the plate method developed by Wang and Shi[14]. In the next section, based on elasticity theory, the refined theory of rectangular beams is derived by using Papkovich-Neuber solution and Lur'e method without *ad hoc* assumptions. In sec. 2, based on the refined beam theory, the exact equations for the beam without transverse surface loadings consist of two governing differential equations: the fourth-order equation and the transcendental equation. The approximate equations for the beam under transverse loadings are derived directly from the refined beam theory in sec. 3, and they are almost the same as the governing equations of Timoshenko beam theory[19]. In the end, two examples are examined to illustrate the application of the new theory and compare the results with the known exact and approximate beam theories.

The method used in this paper is obtained by extending our previous work: Wang and Xu[20], Wang and Wang[21,22].

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1) Gao Y, Wang M Z. The refined theory of transversely isotropic piezoelectric rectangular beams. Sci China Ser G-Phys, Mech & Astro (accepted)

1 Lur'e method

We consider a straight beam of narrow rectangular cross-section as a plane stress problem. In a fixed rectangular coordinate system, z is the coordinate normal to the neutral surface (x-y plane) of the beam. We assume that the beam length in x-direction is l, the beam width in y-direction is 1, the beam height in z-direction is h, and $l \gg h \gg 1$. In the absence of body force, the equilibrium equations of elasticity plane stress problem expressed by displacement u_x and u_z are

$$\nabla^2 u_x + \frac{1+\nu}{1-\nu}\frac{\partial \theta}{\partial x} = 0, \quad \nabla^2 u_z + \frac{1+\nu}{1-\nu}\frac{\partial \theta}{\partial z} = 0, \tag{1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is a two-dimensional Laplacian operator; $\theta = \partial u_x/\partial x + \partial u_z/\partial z$; ν is Poisson's ratio.

Papkovich-Neuber solution of the governing eq. (1) can be obtained as

$$u_x = P_1 - \frac{1+\nu}{4}\frac{\partial}{\partial x}\left(P_0 + xP_1 + zP_3\right), \quad u_z = P_3 - \frac{1+\nu}{4}\frac{\partial}{\partial z}\left(P_0 + xP_1 + zP_3\right), \tag{2}$$

where the displacement function $P_i \ (i = 0,1,3)$ is a two-dimensional harmonic function and satisfies

$$\nabla^2 P_i = \frac{\partial^2 P_i}{\partial z^2} + \partial_x^2 P_i = 0 \quad \left(i = 0,1,3;\ \partial_x = \frac{\partial}{\partial x}\right). \tag{3}$$

The problem of beam may be decomposed into two fundamental problems: the extension of a beam and the bending of a beam. In the case of bending of a beam, the beam is subjected only to anti-symmetrical loadings and edge conditions, thus only odd functions of z are required for u_x and even functions of z for u_z. From the Lur'e method and with these requirements satisfied, treating eq. (3) as an ordinary differential equation in z with constant coefficients, one obtains the following symbolic solution of eq. (3):

$$P_1(x,z) = \frac{\sin(z\partial_x)}{\partial_x} g_1(x), \ P_3(x,z) = \cos(z\partial_x) g_3(x), \tag{4}$$

where g_1 and g_3 are unknown functions of x yet to be determined, in which the trigonometric functions $\sin(z\partial_x)/\partial_x$ and $\cos(z\partial_x)$ have the following symbolic expressions:

$$\begin{aligned} \frac{\sin(z\partial_x)}{\partial_x} &= z\left(1 - \frac{1}{3!}z^2\partial_x^2 + \frac{1}{5!}z^4\partial_x^4 - \cdots\right), \\ \cos(z\partial_x) &= \left(1 - \frac{1}{2!}z^2\partial_x^2 + \frac{1}{4!}z^4\partial_x^4 - \cdots\right). \end{aligned} \tag{5}$$

From Appendix A of ref. [14], we can know that harmonic function P_0 always can satisfy the following expression without loss in generality:

$$P_0 + xP_1 + zP_3 = -z\cos(z\partial_x) f(x), \tag{6}$$

where

$$f(x) = \int_0^x g_1(t)\mathrm{d}t - g_3(x). \tag{7}$$

Substituting eqs. (4) and (6) into eq. (2), one obtains

$$u_x = \frac{\sin(z\partial_x)}{\partial_x} g_1 + \frac{1+\nu}{4} z\cos(z\partial_x) f',$$
$$u_z = \cos(z\partial_x) g_3 + \frac{1+\nu}{4}\left[\cos(z\partial_x) - z\partial_x \sin(z\partial_x)\right] f, \tag{8}$$

where the differential symbol "′" denotes differentiation with respect to x. The angle of rotation and the deflection of the neutral surface can be found to be

$$\psi = -\left.\frac{\partial u_x}{\partial z}\right|_{z=0} = -\left(g_1 + \frac{1+\nu}{4} f'\right), \quad w = u_z\big|_{z=0} = g_3 + \frac{1+\nu}{4} f. \tag{9}$$

From eqs. (9) and (8) , the final expressions for the displacements are

$$u_x = -\frac{\sin(z\partial_x)}{\partial_x}\psi + \frac{1+\nu}{4}\left[z\cos(z\partial_x) - \frac{\sin(z\partial_x)}{\partial_x}\right] f',$$
$$u_z = \cos(z\partial_x) w - \frac{1+\nu}{4} z\partial_x \sin(z\partial_x) f, \tag{10}$$

with the expression

$$f(x) = -\left[\int_0^x \psi(t)\mathrm{d}t + w(x)\right]. \tag{11}$$

Eq. (10) is the displacement expressions by the angle of rotation of the neutral surface ψ and the deflection of the neutral surface w.

2 Exact beam equations: No transverse surface loadings

In order to satisfy the homogeneous boundary conditions on the upper and lower surfaces of the beam, we set

$$\sigma_z = 0, \ \tau_{xz} = 0 \ \ (z = \pm h/2). \tag{12}$$

Using Hooke's law, from eq. (10) the stress components σ_x, τ_{xz} and σ_z can be indicated as

$$\sigma_x = -\frac{E}{4}\left\{\left[\frac{1-\nu}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x} - z\cos(z\partial_x)\right] f'' + \frac{4}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x}\psi'\right\},$$
$$\tau_{xz} = -\mu\left[\cos(z\partial_x)(\psi - w') + \frac{1+\nu}{2} z\partial_x \sin(z\partial_x) f'\right], \tag{13}$$
$$\sigma_z = -\frac{E}{4}\left\{\left[\frac{1-\nu}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x} + z\cos(z\partial_x)\right] f'' + \frac{4}{1+\nu}\frac{\sin(z\partial_x)}{\partial_x} w''\right\},$$

where E and $\mu = E/2(1+\nu)$ are the Young's modulus and the shear modulus of the beam, respectively. Substituting the stress expressions in eq. (13) into the boundary conditions (12) of the beam, we get the following matrix equation expressed by ψ and w:

$$\begin{bmatrix} D_1 - D_2\partial_x^2 & -(D_1 + D_2\partial_x^2)\partial_x \\ D_3\partial_x & -\frac{4}{(1+\nu)h} D_2\partial_x^2 + D_3\partial_x^2 \end{bmatrix}\begin{bmatrix} \psi \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{14}$$

The three differential operators D_1, D_2 and D_3 are defined by

$$D_1 = \frac{4}{1+\nu}\cos\left(\frac{h\partial_x}{2}\right),\ \ D_2 = \frac{h}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right),$$
$$D_3 = \frac{h}{2}\cos\left(\frac{h\partial_x}{2}\right) + \frac{1-\nu}{1+\nu}\frac{1}{\partial_x}\sin\left(\frac{h\partial_x}{2}\right). \tag{15}$$

Let L_0 be the determinant of the 2×2 matrix equation (14),

$$L_0 = \frac{4h}{1+\nu}\left\{\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\right\}\partial_x^4, \tag{16}$$

and $L_{ij}\ (i,j=1,2)$ be the elements of the matrix equation (14). The solutions of eq. (14) are

$$\begin{bmatrix}\psi \\ w\end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11}\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix}, \tag{17}$$

and ξ_i satisfies

$$L_0\xi_i = 0\ \ (i=1,2). \tag{18}$$

In Appendix A, it is proved that the solutions of eq. (18) can be decomposed into two parts, so there are two functions $\xi_i^{(1)}$ and $\xi_i^{(2)}$,

$$\xi_i = \xi_i^{(1)} + \xi_i^{(2)} \quad (i=1,2), \tag{19}$$

where the superscripts "(1)" and "(2)" indicate the fourth-order part and the transcendental part respectively, and $\xi_i^{(1)}$ and $\xi_i^{(2)}$ satisfy the following two governing differential equations of the beam problem, respectively:

$$\partial_x^4\xi_i^{(1)} = 0,\ \ \frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\xi_i^{(2)} = 0, \tag{20}$$

then the angle of rotation and the deflection of the beam can be decomposed into two parts:

$$\psi = \psi^{(1)} + \psi^{(2)},\ \ w = w^{(1)} + w^{(2)}. \tag{21}$$

The solutions of eq. (21) will be investigated in the following two sections.

2.1 The fourth-order equation and the fourth-order solution

$\xi_i^{(1)}$ satisfies the following fourth-order equation:

$$\partial_x^4\xi_i^{(1)} = 0, \tag{22}$$

and the solutions of the angle of rotation $\psi^{(1)}$ and the deflection $w^{(1)}$ become

$$\begin{bmatrix}\psi^{(1)} \\ w^{(1)}\end{bmatrix} = \begin{bmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11}\end{bmatrix}\begin{bmatrix}\xi_1^{(1)} \\ \xi_2^{(1)}\end{bmatrix}. \tag{23}$$

By using eqs. (22), (23) and Taylor series of the trigonometric functions (5), after tedious manipulation, the result turns out to be

$$\psi^{(1)}=\left(1+\frac{1+\nu}{4}h^2\partial_x^2\right)\partial_x w^{(1)},\tag{24}$$

where

$$\partial_x^4 w^{(1)}=0\,;\tag{25}$$

and from eq. (10), the total displacements can be found to be

$$u_x^{(1)}=-z\partial_x\left[1-\frac{1}{6}z^2\partial_x^2+\frac{1+\nu}{12}h^2\partial_x^2\left(3-2\frac{z^2}{h^2}\right)\right]w^{(1)},\quad u_z^{(1)}=\left(1+\frac{\nu}{2}z^2\partial_x^2\right)w^{(1)};\tag{26}$$

the normal stress and shear stress can be found to be

$$\sigma_x^{(1)}=-Ez\left(w^{(1)}\right)'',\quad \tau_{xz}^{(1)}=-\frac{E}{8}\left(h^2-4z^2\right)\left(w^{(1)}\right)''',\quad \sigma_z^{(1)}=0.\tag{27}$$

Calculating moment and shear force for the present case yields

$$M_x^{(1)}=-D\left(w^{(1)}\right)'',\quad Q_x^{(1)}=-D\left(w^{(1)}\right)''',\tag{28}$$

where $D=Eh^3/12$ is the flexural rigidity of beam.

By the same arguments made in Cheng[9], eqs. (26)—(28) constitute a first-order refined theory for the bending beams with the differential governing equation (25), which can satisfy two edge conditions along the boundary of beams and coincide with the corresponding expressions of the classical elasticity. Unlike the customary beam theory, all the fundamental equations of the refined beam theory are deduced directly.

2.2 The transcendental equation and the transcendental solution

$\xi_i^{(2)}$ satisfies the following transcendental equation:

$$\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\xi_i^{(2)}=0,\tag{29}$$

and the solutions of the angle of rotation $\psi^{(2)}$ and the deflection $w^{(2)}$ become

$$\begin{bmatrix}\psi^{(2)}\\ w^{(2)}\end{bmatrix}=\begin{bmatrix}L_{22} & -L_{12}\\ -L_{21} & L_{11}\end{bmatrix}\begin{bmatrix}\xi_1^{(2)}\\ \xi_2^{(2)}\end{bmatrix}.\tag{30}$$

Substituting eq. (30) into the displacement and stress expressions (10) and (13) respectively, one obtains the following expressions:

$$u_x^{(2)}=\frac{1}{E}\left[\frac{\partial^2 m}{\partial x^2}-(1+\nu)\frac{\partial^3\Phi}{\partial x^3}\right],\quad u_z^{(2)}=\frac{1}{E}\left[\frac{\partial^2 n}{\partial x^2}-(1+\nu)\frac{\partial^3\Phi}{\partial x^2\partial z}\right],\tag{31}$$

$$\sigma_x^{(2)}=\frac{\partial^4\Phi}{\partial x^2\partial z^2},\quad \tau_{xz}^{(2)}=-\frac{\partial^4\Phi}{\partial x^3\partial z},\quad \sigma_z^{(2)}=\frac{\partial^4\Phi}{\partial x^4},\tag{32}$$

where the functions $m(x,z)$ and $n(x,z)$ are conjugate harmonic function, and satisfy

$$\frac{\partial m}{\partial x}=\frac{\partial n}{\partial z}=\nabla^2\Phi.\tag{33}$$

Furthermore, the function $\Phi(x,z)$ has the following expression:

$$
\begin{aligned}
-\frac{4\Phi}{E} = & \left[-\frac{2}{1+\nu} h\cos\left(\frac{h\partial_x}{2}\right)\sin\left(z\partial_x\right)+\frac{4}{1+\nu} z\sin\left(\frac{h\partial_x}{2}\right)\cos\left(z\partial_x\right)\right]\frac{1}{\partial_x^2}\xi_1^{(2)} \\
& +\left[2\cos\left(\frac{h\partial_x}{2}\right)\frac{\sin(z\partial_x)}{\partial_x} - h\sin\left(\frac{h\partial_x}{2}\right)\sin\left(z\partial_x\right)\right. \\
& \left. -2z\cos\left(\frac{h\partial_x}{2}\right)\cos\left(z\partial_x\right)\right]\frac{1}{\partial_x^2}\xi_2^{(2)},
\end{aligned}
\tag{34}
$$

and Φ satisfies the following equations:

$$\nabla^2\nabla^2\Phi = 0, \tag{35}$$

$$\Phi = 0,\ \ \partial\Phi/\partial z = 0 \quad \left(z = \pm h/2\right). \tag{36}$$

Therefore, the moment and shear force are

$$M_x^{(2)} = 0,\ \ Q_x^{(2)} = 0. \tag{37}$$

Eq. (37) shows that the transcendental solution does not yield moment and shear force which are yielded only from the fourth-order solution. Eqs. (31), (32) and (37) satisfy two edge conditions along the boundary of beams, and yet satisfy exactly all the fundamental equations in the theory of elasticity.

Combining the fourth-order solution of eqs. (26)—(28) and the transcendental solution of eqs. (31), (32) and (37), we arrive at a second-order refined theory for the bending beams with the two differential governing equations (25) and (35). It is important to note that the equilibrium equation (1) is satisfied by any solution of the refined beam theory, and the only approximation in the theory is introduced by the approximate specification of the boundary conditions at the edges of the beam (i.e. the boundary conditions are specified in terms of the stress resultants or some combination of the angle of rotation and the deflection of the neutral surface, instead of the stress or displacement distribution over the thickness $-h/2 \leqslant z \leqslant h/2$). Therefore, in the cases where Saint-Venant's principle holds, the refined beam theory should be a very accurate one.

3 Approximate beam equations: Transverse surface loadings

Now let us consider the case that the beam is subjected only to the transverse surface loadings, i.e.

$$\tau_{xz} = 0,\ \ \sigma_z = \pm q/2 \quad (z = \pm h/2). \tag{38}$$

There are various beam theories. Bernoulli-Euler beam theory and Timoshenko beam theory are the most famous two among them. The governing equation of Bernoulli-Euler beam theory[19] is

$$Dw'''' = q. \tag{39}$$

The governing equations of Timoshenko beam theory[19] are

$$Dw'''' = \left[1-\left(\frac{1}{5}+\frac{11}{60}\nu\right)h^2\partial_x^2\right]q, \tag{40}$$

$$D\psi''' = q. \tag{41}$$

Now the governing equations of the refined beam theory will be derived. Substituting the stress expressions in eq. (13) into the boundary conditions (38) of the beam, we get the following equations expressed by ψ and w:

$$
\begin{aligned}
&(D_1 - D_2\partial_x^2)\psi - (D_1 + D_2\partial_x^2)\partial_x w = 0, \\
&D_3\partial_x\psi - \left[\frac{4}{(1+\nu)h}D_2 - D_3\right]\partial_x^2 w = \frac{2}{E}q.
\end{aligned}
\tag{42}
$$

Taking the operator $D_1 - D_2\partial_x^2$ on both sides of the second expression of eq. (42) and then using the first expression of eq. (42), one obtains

$$
\left[2D_1D_3 - \frac{4}{(1+\nu)h}D_1D_2 + \frac{4}{(1+\nu)h}D_2^2\partial_x^2\right]w'' = \frac{2}{E}(D_1 - D_2\partial_x^2)q. \tag{43}
$$

Substitutions of eq. (15) into (43) and the first expression of eq. (42) give

$$
\frac{Eh}{2}\left[1 - \frac{\sin(h\partial_x)}{h\partial_x}\right]w'' = \left[\cos\left(\frac{h\partial_x}{2}\right) - \frac{1+\nu}{4}h\partial_x\sin\left(\frac{h\partial_x}{2}\right)\right]q, \tag{44}
$$

$$
\left[\cos\left(\frac{h\partial_x}{2}\right) - \frac{1+\nu}{4}h\partial_x\sin\left(\frac{h\partial_x}{2}\right)\right]\psi = \left[\cos\left(\frac{h\partial_x}{2}\right) + \frac{1+\nu}{4}h\partial_x\sin\left(\frac{h\partial_x}{2}\right)\right]\partial_x w. \tag{45}
$$

Eq. (44) is the exact governing equation for the deflection w at the neutral surface of the beam subjected to the transverse surface loadings. Since this equation is of infinite order, however, it is not applicable in most cases. Using Taylor series of the trigonometric functions in eq. (5) and then dropping all the terms associated with h^4 or the higher orders, we arrive at the following equations:

$$
D\left(1 - \frac{1}{20}h^2\partial_x^2\right)w'''' = \left(1 - \frac{2+\nu}{8}h^2\partial_x^2\right)q, \tag{46}
$$

$$
\left(1 - \frac{2+\nu}{8}h^2\partial_x^2\right)\psi = \left(1 + \frac{\nu}{8}h^2\partial_x^2\right)\partial_x w. \tag{47}
$$

Taking the operators $1 + h^2\partial_x^2/20$ and $1 + h^2\partial_x^2(2+\nu)/8$ on both sides of eqs. (46) and (47) respectively, and then omitting the h^4 order terms, one obtains

$$
Dw'''' = \left[1 - \left(\frac{1}{5} + \frac{1}{8}\nu\right)h^2\partial_x^2\right]q, \tag{48}
$$

$$
D\psi''' = \left[1 + \left(\frac{1}{20} + \frac{1}{8}\nu\right)h^2\partial_x^2\right]q. \tag{49}
$$

Eqs. (48) and (49) form the basic equations for an approximate first-order theory for the bending of the beam. Eqs. (48) and (49) are almost the same as the governing equations of Timoshenko beam theory (40) and (41). With the h^2 order term omitted, the governing equation (48) has the same structure as that of Bernoulli-Euler beam theory (39).

From eqs. (48) and (49), the expressions about the displacements, stresses and stress resultants become

$$
\begin{aligned}
u_x &= -\frac{z}{D\partial_x^3}\left(1 + \frac{2+5\nu}{40}h^2\partial_x^2 - \frac{2+\nu}{6}z^2\partial_x^2\right)q, \\
u_z &= \frac{1}{D\partial_x^4}\left(1 - \frac{8+5\nu}{40}h^2\partial_x^2 + \frac{\nu}{2}z^2\partial_x^2\right)q,
\end{aligned}
\tag{50}
$$

$$\sigma_x = -\frac{12}{h^2\partial_x^2}\frac{z}{h}\left(1+\frac{1}{20}h^2\partial_x^2-\frac{1}{3}z^2\partial_x^2\right)q,$$
$$\tau_{xz} = -\frac{3}{2h\partial_x}\left(1-4\frac{z^2}{h^2}\right)q, \quad \sigma_z = \frac{z}{h}\left(\frac{3}{2}-2\frac{z^2}{h^2}\right)q, \tag{51}$$

$$M_x'' = -q, \quad Q_x' = -q. \tag{52}$$

Clearly, even if people doubt the legitimacy of the manipulation performed on differential operators, the beam equations obtained above can be justified by comparing their forms with the forms of the corresponding equations in other well-known beam theories.

4 Examples and comparison

Dropping all the terms associated with h^4 or the higher orders in eq. (42), we arrive at the following equations:

$$\psi - w' - \frac{h^2}{8}\left[(2+\nu)\psi'' + \nu w'''\right] = 0, \quad \frac{2}{3}(\psi' - w'') = \frac{2(1+\nu)}{Eh}q. \tag{53}$$

According to the stress expressions in eq. (13), omitting all the terms associated with h^4 or the higher orders, one obtains the expressions of the moment and shear force as

$$M_x = -\frac{D(2+5\nu)}{10(1+\nu)}\left(\frac{8+5\nu}{2+5\nu}\psi' + w''\right), \quad Q_x = -\frac{2}{3}\mu h(\psi - w'). \tag{54}$$

Fan and Widera[23] employed the asymptotic expansion approach, and derived the proper new boundary conditions of a beam for the outer expansion without a consideration of the inner solution by adopting Gregory and Wan's technique[12,13]. It is interesting to note that the new boundary conditions in the stress data case are consistent with the conventional boundary conditions[24]. To illustrate the applications of the refined beam theory developed in the previous sections, we present the following two typical examples by using the boundary conditions given by Fan and Widera[23]. Results for the examples are given for both the new theory and Levinson's beam theory, and are compared with the well-known exact solutions and the solutions by Levinson[3].

4.1 The uniformly loaded cantilever beam

Considering a cantilever beam of uniform cross-section loaded by a uniformly distributed load of intensity $q = q_0$ and clamped at $x = l$, where q_0 is a constant. For the present theory, the boundary conditions are

$$\frac{8+5\nu}{2+5\nu}\psi'(0) + w''(0) = 0, \quad -\frac{2}{3}\mu h\left[\psi(0) - w'(0)\right] = 0, \quad w(l) = \psi(l) = 0. \tag{55}$$

From eqs. (53) and (55), the solution for the deflection curve of the neutral surface is

$$w = \frac{q_0 l^4}{24EI}\left(\frac{x^4}{l^4} - 4\frac{x}{l} + 3\right) + \frac{q_0 l^4}{20EI}\frac{h^2}{l^2}\left[\left(2+\frac{5}{4}\nu\right)\left(1-\frac{x^2}{l^2}\right) + \left(1+\frac{5}{2}\nu\right)\left(1-\frac{x}{l}\right)\right]. \tag{56}$$

Whereas the theory of Levinson solution[3] is

$$w=\frac{q_0 l^4}{24EI}\left(\frac{x^4}{l^4}-4\frac{x}{l}+3\right)+\frac{q_0 l^4}{20EI}\frac{h^2}{l^2}\left[(2+2\nu)\left(1-\frac{x^2}{l^2}\right)+(1+\nu)\left(1-\frac{x}{l}\right)\right]. \tag{57}$$

The solution of eq. (56) based on the refined theory is the exact solution of elasticity theory[24] and is the same as the solution by Levinson[3] if $\nu=0$.

4.2 The linear loaded and simply supported beam

The other example is a beam of uniform cross-section which carries a linear distributed load $q(x)=q_0 x$ and is simply supported at $x=\pm l$. For the new theory, the boundary conditions are

$$\frac{8+5\nu}{2+5\nu}\psi'(\pm l)+w''(\pm l)=0,\quad w(\pm l)=0. \tag{58}$$

From eqs. (53) and (58), the solution for the deflection curve of the neutral surface is

$$w(x)=\frac{q_0 l^5}{360EI}\left(3\frac{x^5}{l^5}-10\frac{x^3}{l^3}+7\frac{x}{l}\right)+\left(1+\frac{5}{8}\nu\right)\frac{q_0 l^5}{30EI}\frac{h^2}{l^2}\left(\frac{x}{l}-\frac{x^3}{l^3}\right). \tag{59}$$

Whereas the theory of Levinson solution[3] is

$$w(x)=\frac{q_0 l^5}{360EI}\left(3\frac{x^5}{l^5}-10\frac{x^3}{l^3}+7\frac{x}{l}\right)+(1+\nu)\frac{q_0 l^5}{30EI}\frac{h^2}{l^2}\left(\frac{x}{l}-\frac{x^3}{l^3}\right). \tag{60}$$

The solution of eq. (59) based on the refined theory is the exact solution of elasticity theory[24]. From eqs. (59) and (60), for this problem both of the beam theories equally overestimate the “shear correction” term at the center of the beam less than 14% if $0\leqslant\nu\leqslant 0.5$.

For the above-mentioned two cases, eqs. (56) and (59) that we give from the refined theory are the exact solutions for the neutral surface given for the case of plane stress by the linear theory of elasticity[24]. The refined theory provides the better results than Levinson’s beam theory when compared with those obtained from the linear theory of elasticity.

5 Conclusions

In the above sections, a refined theory for rectangular beam has been deduced systematically and directly from the elasticity theory by using Papkovich-Neuber solution and Lur’e method without *ad hoc* assumptions. For the homogenous beam, the refined beam theory is exact in the sense that a solution of the refined beam theory satisfies all the equations in the elasticity theory and consists of two parts: the fourth-order equation and the transcendental equation. Especially, the distribution of stresses described by the fourth-order equation is the same as that of stresses in the classical elasticity. For the beam under a transverse loading, the approximate governing equations and solutions are accurate up to the second-order terms with respect to beam thickness, and they are almost the same as the governing equations of Timoshenko beam theory. Furthermore, the two examples studied also indicate that the refined beam theory for the loaded beams can still be justified by comparing its form with that of other well-known beam theories. Therefore, in these cases where Saint-Venant's principle holds, the refined beam theory should be a very accurate one.

Acknowledgements The authors are very grateful to the anonymous reviewers for their helpful suggestions. Support from the National Natural Science Foundation of China (Grant No. 10372003) is acknowledged.

Appendix A

The method used in this appendix is obtained by extending the previous work[25]. Next, we will give and prove a lemma and a theorem.

A.1 The lemma

Supposing that $H(x)$ satisfies

$$\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]H=0, \tag{A1}$$

then there exists function $B(x)$ which satisfies the following two equations:

$$\partial_x^4 B=H,\quad \frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B=0. \tag{A2}$$

Proof: Assume function $C(x)$, which satisfies the following equation, can be found:

$$\partial_x^2 C=H. \tag{A3}$$

We can obtain the following equation:

$$\partial_x^2\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]C=\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]H=0. \tag{A4}$$

Set

$$B_1=C-\frac{6}{h^2}\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]C. \tag{A5}$$

Using eqs. (A3) and (A4), we can get

$$\partial_x^2 B_1=\partial_x^2 C=H. \tag{A6}$$

After tedious manipulation by using eqs. (A4) and (A5), the result turns out to be

$$\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B_1=0. \tag{A7}$$

Because $B_1(x)$ and $H(x)$ satisfy the same equation, $B_1(x)$ can be used instead of $H(x)$. Repeating eqs. (A3)—(A7), we can obtain $B(x)$ as

$$\partial_x^2 B=B_1,\quad \frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]B=0. \tag{A8}$$

From eqs. (A6) and (A8), it is not difficult to verify that $B(x)$ satisfies eq. (A2). So the proof of the lemma is finished.

A.2 The theorem

Supposing that ξ satisfies the following equation:

$$\left\{\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\right\}\partial_x^4\xi = 0, \tag{A9}$$

then there exist $\xi^{(1)}$ and $\xi^{(2)}$ such that

$$\xi = \xi^{(1)} + \xi^{(2)}, \tag{A10}$$

satisfying the following two equations:

$$\partial_x^4\xi^{(1)} = 0, \quad \frac{1}{\partial_x^2}\left[1-\frac{\sin\left(h\partial_x\right)}{h\partial_x}\right]\xi^{(2)} = 0. \tag{A11}$$

Proof: Let

$$F = \partial_x^4\xi, \tag{A12}$$

then

$$\frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]F = 0. \tag{A13}$$

According to the lemma, there exists $\xi^{(2)}$ such that

$$\partial_x^4\xi^{(2)} = F, \quad \frac{1}{\partial_x^2}\left[1-\frac{\sin(h\partial_x)}{h\partial_x}\right]\xi^{(2)} = 0. \tag{A14}$$

From eq. (A12) and the first equation of eq. (A14), we get

$$\partial_x^4\xi^{(2)} = F = \partial_x^4\xi, \tag{A15}$$

namely,

$$\partial_x^4\left(\xi - \xi^{(2)}\right) = 0. \tag{A16}$$

Let

$$\xi^{(1)} = \xi - \xi^{(2)}. \tag{A17}$$

Eq. (A16) becomes the first equation of eq. (A11), so there are functions $\xi^{(1)}$ and $\xi^{(2)}$ which satisfy eq. (A11). This completes the proof of the theorem.

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