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Convergence of adaptive MPC for linear stochastic systems

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Abstract The convergence of an adaptive model predictive control (MPC) algorithm for discrete-time linear stochastic systems with unknown parameters is investigated in this paper. The proposed adaptive MPC is designed by solving a finite horizon constrained linear-quadratic optimal control problem of online estimated models, which are built on a recursive weighted least-squares (WLS) algorithm together with a random regularization method. By incorporating an attenuating excitation signal into adaptive MPC, the proposed adaptive MPC is shown to converge asymptotically to the ergodic MPC performance with known parameters by using the Markov chain ergodic theory.

Keywords adaptive control, linear-quadratic optimal control, model predictive control, uncertain stochastic system, weighted least-squares

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1 Introduction

Model predictive control (MPC) has become one of the most prospective and successful control approaches in handling control problems for systems with multiple constraints. Survey data show that MPC is considered, and likely to be, more impactful than other control technologies [1]. The provenance of MPC is not academic research but industry implementation [2], and many theoretical studies are needed, particularly on adaptive MPC. At each successive step, the MPC signal is obtained by solving a finite horizon optimal control problem in which the initial state is the current state of the system [3]. A vast amount of literature has been devoted to MPC design and analyses, and much progress has been made over the past several decades [4]. However, most of the existing studies on MPC, such as dynamic matrix control [5] and generalized predictive control [6], depend on the structure information of the control systems. When system model uncertainties and/or external disturbances emerge, robust and stochastic MPC algorithms have been proposed to handle such situations by assuming a known nominal model [7–9].

Of course, adaptive approaches or learning-based approaches may also be used in MPC design when model uncertainties exist. A major difficulty in the design and analysis of MPC based on online estimated models has been how to make the closed-loop control systems stable while keeping the constraints satisfied in the presence of uncertainties. Most of the research in the literature is parameter estimation-based MPC in which the system model has a parametric structure, and here we only mention a few examples as follows: recursive estimation algorithms are used to estimate parameters in a special uncertain structure for unconstrained [10] and constrained [11,12] state space models, iterative set-membership identification algorithms are presented for constrained linear input-output models [13–15], an adaptive MPC is proposed for a class of unknown non-affine systems based on estimating parameters in their linearized models [16], an adaptive MPC is designed assuming a fixed, robustly stabilizing feedback law for all model parameters in a given prior bounded set for constrained linear uncertain state space models [17,18], an adaptive dual MPC is proposed to handle the output uncertainties for output tracking problems [19], and an adaptive

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MPC with exponentially converging tracking errors with sufficiently rich reference trajectories is proposed for linearly parameterized nonlinear stochastic systems [20]. In another type of research line, the controller design is not based on parameter estimation; for example, a robust data-driven MPC is proposed for linear time-invariant systems by using an implicit model description based on the behavioral approach and the measured trajectories [21]. Overall, extensive research on adaptive MPC has been conducted with guaranteed stability of the closed-loop systems under various assumptions, and the convergence of the adaptive MPC has rarely been explored, even for the standard linear-quadratic (LQ) control problems.

In this paper, an adaptive MPC with LQ performance is proposed and proved for constrained linear stochastic systems with unknown system matrices. First, the unknown parameters are estimated using a weighted least-squares (WLS) algorithm, which possesses the celebrated self-convergence property (i.e., convergence regardless of the excitation property of the system signals). Then, the random regularization procedure introduced in [22, 23] is used to obtain a family of modified WLS estimates that lie in a given bounded set while keeping the self-convergence property. With this modified WLS estimate at each time instant, the adaptive MPC is solved by a finite horizon constrained LQ optimization problem for the estimated model. On the basis of some assumptions, the convergence of the adaptive control is proven. To the best of the authors' knowledge, this result is the first to give a complete proof of the convergence of an adaptive MPC with the standard LQ performance and the first to use the Markov chain ergodic theory in the convergence study of MPC.

The remainder of this paper is organized as follows. Section 2 presents the problem statement and the required assumptions for the general formulation. In Section 3, we introduce the WLS algorithm and the random regularization method that will be used to obtain the estimated model. Section 4 provides the design procedure of the adaptive MPC and presents the main theoretical results of the paper. The proofs of the main results are provided in Section 5. Finally, Section 6 concludes the paper.

2 Problem statement

This paper considers a discrete-time linear time-invariant stochastic system with unknown parameter matrices $A^* \in \mathbb{R}^{n\times n}, B^* \in \mathbb{R}^{n\times m}$:

$$x_{k+1} = A^* x_k + B^* u_k + w_{k+1}, \tag{1}$$

where $x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m$ and $w_k \in \mathbb{R}^n$ are the output, input and random disturbances at time $k \in \mathbb{N}$. The random variables are defined on a fixed complete probability space $(\Omega, \mathcal{F}, P)$, and the filtration $(\mathcal{F}_k, k \geqslant 0)$ is defined on this space. In addition, we suppose that the disturbance is bounded.

In this paper, we define $\|x\|_H$ as the weighted norm of the vector $x \in \mathbb{R}^n$ for a given positive definite matrix $H \in \mathbb{R}^{n\times n}$, i.e., $\|x\|_H = \sqrt{x^\tau H x}$, and define $\|A\|_H$ as the operator norm of a matrix $A \in \mathbb{R}^{n\times n}$ induced by the weighted norm, i.e., $\|A\|_H = \sup_{x\in\mathbb{R}^n, \|x\|_H=1} \|Ax\|_H$. When H is taken as the unit matrix I, the norm $\|\cdot\|_I$ is the Euclidean norm, which is usually defined as $\|\cdot\|$. We make the following assumption about the boundedness of the noise, which is widely assumed in the investigation of MPC, e.g., [3, 17].

Assumption 1. $\{w_k, \mathcal{F}_k, k \geqslant 1\}$ is a bounded independent stochastic sequence defined on the basic probability space $(\Omega, \mathcal{F}, P)$ with

$$w_k \in \mathbb{W} \triangleq \{w \big| \|w\| < \bar{w}\}, \quad \forall k \geqslant 0, \tag{2}$$

and

$$\sup_{k\geqslant 0} \|w_k\| < \bar{w}, \tag{3}$$

where $\bar{w}$ is a known positive number. Moreover, the noise sequence $\{w_k, k \geqslant 1\}$ is identically distributed and possesses a density ρ_w that is continuous and satisfies $\rho_w(x) > 0$ for all $x \in \mathbb{W}$.

Now, consider the following bounded sets:

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid \|x\| \leqslant \bar{x}\}, \tag{4}$$

$$\mathbb{U} = \{u \in \mathbb{R}^m \mid \|u\| \leqslant \bar{u}\}, \tag{5}$$

where $\bar{x}$ and $\bar{u}$ are known positive numbers. These two sets are usually used as the constraint sets of the input and the output. However, because of uncertainty, the input and the output of the system (1) are difficult to satisfy the hard constraints, and constraints must be loosened in the adaptive process. Letting $a \oplus B = \{a + b| \; \forall b \in B\}$ denote the Minkowski set addition, we define the soft constraint set as follows.

Definition 1 (Soft constraint sets). If there exists an adapted sequence of vectors $\{\delta_k, \mathcal{F}_k, k \geqslant 0\}$ with proper dimension such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \|\delta_k\|^2 = 0,$$

then, for any given set $\mathbb{Z}$, its corresponding soft constraint sets $\{\mathbb{Z}_k, \; k \geqslant 0\}$ are defined as

$$\mathbb{Z}_k = \mathbb{Z} \oplus \delta_k, \; k \geqslant 0. \tag{6}$$

In this paper, we study adaptive MPC problems in which the parameter estimates in the transient phase may not be good enough, and so we allow the MPC and the corresponding states to satisfy $\{x_k \in \mathbb{X}_k, \, u_k \in \mathbb{U}_k, \, k \geqslant 0\}$, where $\{\mathbb{X}_k, \, k \geqslant 0\}$ and $\{\mathbb{U}_k, \, k \geqslant 0\}$ are the soft constraint sets corresponding to $\mathbb{X}$ and $\mathbb{U}$, respectively. Actually, we will construct an adaptive MPC that satisfies not only the soft constraints but also the hard constraints $\mathbb{X}$ and $\mathbb{U}$ after a finite number of steps.

Now, we introduce the ergodic performance. Let $\theta^\tau = (A, B)$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. A constrained infinite horizon optimization problem $\mathrm{OP}(x_0; \theta)$ with initial state x_0 is defined as follows:

$$\begin{aligned} \mathrm{OP}(x_0; \theta): \; & \min_{\boldsymbol{u}} J(\boldsymbol{u}) \\ & x_{k+1} = Ax_k + Bu_k + w_{k+1}, \; w_{k+1} \in \mathbb{W}, \\ & x_k \in \mathbb{X}, \; u_k \in \mathbb{U}, \; k \geqslant 0, \end{aligned} \tag{7}$$

where $\boldsymbol{u} = (u_0, u_1, \ldots)$ is the sequence of optimizers, $\mathbb{W}$ is defined in Assumption 1, $\mathbb{X}$ and $\mathbb{U}$ are defined in (4) and (5), and the ergodic cost function is

$$J(\boldsymbol{u}) = \limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \left(x_k^\tau Q x_k + u_k^\tau R u_k\right), \tag{8}$$

where $Q \geqslant 0$ and $R > 0$ are known matrices with proper dimensions. The notation that $A > (\geqslant) \; 0$ means that A is a positive (semi) definite matrix. If the above problem $\mathrm{OP}(x_0; \theta)$ is solvable, let $J(\boldsymbol{u}^*; \theta)$ denote the optimal value and $\boldsymbol{u}^*$ denote the optimal control sequence.

Next, we consider the finite horizon constrained optimization problem $\mathrm{OP}_N(x; \theta)$ defined as follows:

$$\mathrm{OP}_N(x; \theta): \; \min_{\boldsymbol{u}_0^{N-1}} J_N(\boldsymbol{u}_0^{N-1}; \theta) \tag{9a}$$

$$x_{k+1} = Ax_k + Bu_k + w_{k+1}, \; w_{k+1} \in \mathbb{W}, \tag{9b}$$

$$u_k \in \mathbb{U}, \, x_k \in \mathbb{X}, \, 0 \leqslant k \leqslant N-1; \tag{9c}$$

$$x_N \in \mathbb{X}_f, \; x_0 = x, \tag{9d}$$

where $\boldsymbol{u}_0^{N-1} = (u_0, u_1, \ldots, u_{N-1})$ is the finite sequence of optimizers, $\mathbb{X}_f$ is a given terminal set, and the function $J_N(\boldsymbol{u}_0^{N-1}; \theta)$ to be minimized is defined as follows:

$$J_N(\boldsymbol{u}_0^{N-1}; \theta) = \sum_{k=0}^{N-1} \left(x_k^\tau Q x_k + u_k^\tau R u_k\right) + V_f(x_N; \theta),$$

where $V_f(x_N; \theta)$ is a given terminal cost.

Let $u_0(x, \theta)$ be the first vector component of $\boldsymbol{u}_0^{N-1}$ obtained by solving the above optimization problem $\mathrm{OP}_N(x; \theta)$. That is, if the optimal solution of $\mathrm{OP}_N(x; \theta)$ is $(\boldsymbol{u}_0^{N-1})^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$, then we take

$$u_0(x, \theta) = u_0^*, \tag{10}$$

which is called the MPC controller. Some usually used forms of the above terminal set and terminal cost may be found in [7, 17].

Now, for any $k \geqslant 0$, let $u_0(x_k, \theta)$ be the MPC with initial value x_k, where x_k is the state of (9b) under the previous MPC $u_0(x_{k-1}, \theta)$ and the noise $w_k \in \mathbb{W}$. We can define the MPC performance as follows.

Definition 2 (Ergodic MPC performance). Consider the constrained control system (1) under the MPC sequence $\{u_0(x_k, \theta^*),\ k \geqslant 1\}$. The following long-run average system performance is called ergodic MPC performance:

$$J_N^{\mathrm{MPC}}(\boldsymbol{u}_0; \theta^*) \triangleq \limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \left[x_k^\tau Q x_k + u_0^\tau(x_k, \theta^*) R u_0(x_k, \theta^*)\right].$$

In the unconstrained case, it is easily proven that $J_N^{\mathrm{MPC}}(x_0, \theta^*)$ will approach the optimal quadratic performance $J(\boldsymbol{u})$ defined by (8) when $N \to \infty$ (e.g., [24]) because in this case, the MPC has an explicit form $u = Kx$ for some gain matrix K. In the current constrained case, there is no explicit solution in general, but one may still expect that this property also holds under some proper conditions. In this paper, we will not need this property because we are only interested in constructing an adaptive MPC that achieves the above ergodic performance.

To conduct our theoretical investigation, we need to make the following basic assumptions.

Assumption 2. There exists a positive number h_θ such that the set $\Theta \triangleq \{\theta \in \mathbb{R}^{(n+m)\times n} | \|\theta\| \leqslant h_\theta\}$ contains the true parameter $\theta^{*\tau} = (A^*, B^*)$ as an interior point. In addition, the true parameter θ^* satisfies that (A^*, B^*) is controllable and $(A^*, Q^{1/2})$ is observable.

In Assumption 2, because constraints are placed on the input and output, the optimization problem $\mathrm{OP}_N(x; \theta)$ may not be solvable for all $\theta \in \mathbb{R}^{(n+m)\times n}$. We require that the parameter set Θ is bounded and define a neighborhood of the origin for simplicity.

Assumption 3. For any controllable $\theta \triangleq (A, B) \in \Theta$ and any observable $(A, Q^{1/2})$, the MPC controller $u_0(x, \theta)$ obtained by solving the MPC problem $\mathrm{OP}_N(x; \theta)$ defined by (9) with any $x \in \mathbb{X}$ exists and depends on θ and x continuously.

Under Assumptions 2 and 3, we can ensure that the MPC sequence exists for the true system. Furthermore, to prove the convergence of the adaptive MPC, the following assumption is also required.

Assumption 4. There exists a nonnegative continuous function $V(x)$ such that

$$V(0) = 0;\ V(x) > 0,\ \forall \|x\| > 0,$$

$$V(A^*x + B^*u_0(x, \theta^*)) \leqslant \rho V(x),\ \forall x \in \mathbb{X}, \tag{11}$$

where $\rho < 1$ is a positive value. Moreover, there exists a $\mathcal{K}$-function $\alpha(\cdot)$ such that for any $x, y \in \mathbb{X}$,

$$|V(x) - V(y)| \leqslant \alpha(\|x - y\|), \tag{12}$$

and that for some nonzero point $z \in \mathbb{X}$,

$$(1 - \rho)V(z) \geqslant E[\alpha(\|w\|)], \tag{13}$$

where w possesses a density ρ_w defined in Assumption 1.

Remark 1. If the MPC takes the form of $u_0(x, \theta^*) = K^*x$ such that $A^* + B^*K^*$ is stable, then there exists an induced matrix norm $\|\cdot\|_*$ such that $\rho = \|A^* + B^*K^*\|_* < 1$ [25]. In this case, we can choose $V(x) = \alpha(x) = \|x\|_*$, and as long as $(1 - \rho)\max_{z\in\mathbb{X}} \|z\|_* \geqslant \max_{w\in\mathbb{W}} \|w\|_*$, Eq. (13) will be satisfied.

The objective of this paper is to design an adaptive MPC for the uncertain system (1) with any initial value $x_0 \in \mathbb{X}$ such that the ergodic cost function $J(\boldsymbol{u})$ defined by (8) is identical to that for $u_0(x_k, \theta^*)$ when the parameter θ^* is known with some relaxed constraints closed to $\{x_k \in \mathbb{X},\ u_k \in \mathbb{U},\ k \geqslant 0\}$.

3 WLS estimation and random regularization

3.1 WLS estimation

We first introduce the WLS algorithm, which has some nice properties, particularly the self-convergence of the estimates [22, 26].

Let

$$\varphi_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}. \tag{14}$$

Then, Eq. (1) can be rewritten as a linear regression:

$$x_{k+1} = \theta^{*\tau}\varphi_k + w_{k+1}, \tag{15}$$

and the recursive WLS algorithm has the following form:

$$\begin{aligned} \theta_{k+1} &= \theta_k + L_k(x_{k+1}^{\tau} - \varphi_k^{\tau}\theta_k), \\ L_k &= \frac{P_k\varphi_k}{\alpha_k^{-1} + \varphi_k^{\tau}P_k\varphi_k}, \\ P_{k+1} &= P_k - \frac{P_k\varphi_k\varphi_k^{\tau}P_k}{\alpha_k^{-1} + \varphi_k^{\tau}P_k\varphi_k}, \end{aligned} \tag{16}$$

where $\theta_0^{\tau} = (A_0, B_0)$ and $P_0 > 0$ are arbitrary proper deterministic matrices. $\{\alpha_k\}$ is the weighting sequence defined by

$$\alpha_k = \frac{1}{\log^{1+\delta}(r_k)}, \quad r_k = \|P_0^{-1}\| + \sum_{i=0}^{k}\|\varphi_i\|^2, \tag{17}$$

where δ is an arbitrary positive number.

Since the noise is a martingale difference sequence, some basic properties of the WLS algorithm can be found in [22] and are stated as Lemma 1 below.

Lemma 1 ([22]). Let Assumption 1 hold and $\{\theta_k, P_k; k \geqslant 0\}$ be given by (16) and (17). Then, the following properties are satisfied:

(1) $\|P_k^{-1/2}(\theta^* - \theta_k)\|^2 = O(1)$, $k \to \infty$, a.s.,

(2) $\sum_{i=1}^{k}\|\varphi_i^{\tau}(\theta^* - \theta_i)\|^2 = o(r_k) + O(1)$, $k \to \infty$, a.s.,

(3) $\lim_{k\to\infty}\theta_k = \overline{\theta}$, a.s.,

where θ^* is the true parameter matrix, and $\overline{\theta}$ is a random matrix that may not be equal to θ^*.

3.2 Random regularization

In this subsection, we use the random regularization method to modify the family of WLS estimates to ensure that this new family lies in the prior parameter set Θ defined in Assumption 2 and is uniformly controllable and observable.

First, let $\beta_k^* = P_k^{-1/2}(\theta^* - \theta_k)$, where θ^* is the true parameter. From Lemma 1 (1), we know that the sequence $\{\beta_k^*,\ k \geqslant 0\}$ is bounded a.s. and that $\theta^* = \theta_k + P_k^{1/2}\beta_k^*$. Then, we can consider the regularization parameter $\bar{\theta}(k,x)$ having the following form:

$$\bar{\theta}(k,x) = \theta_k + P_k^{1/2}x, \tag{18}$$

and let

$$\bar{\theta}^{\tau}(k,x) = (A(k,x), B(k,x)), \tag{19}$$

where $x \in \mathcal{M}(n+m,n)$ and $\mathcal{M}(n+m,n)$ denotes the family of $(n+m)\times n$ real matrices.

Using the similar method as introduced in [22, 23], we proceed to choose x as a bounded vector β_k in (18) at each time step k and let $\hat{\theta}_k$ denote the corresponding $\bar{\theta}^{\tau}(k,\beta_k)$, that is

$$\hat{\theta}_k = \theta_k + P_k^{1/2}\beta_k. \tag{20}$$

This modification has almost the same nice properties as those of the WLS estimate θ_k.

Since for any fixed $x \in \mathcal{M}(n+m,n)$, the matrix sequence $\{A(k,x), B(k,x),\ k \geqslant 0\}$ is bounded where Q is defined in (8), the estimate $\bar{\theta}(k,x)$ defined in (18) that lies in the set Θ defined in Assumption 2 is equivalent to the uniform positivity of the family $\{f_k(x), k \geqslant 0\}$, where

$$f_k(x) = h_\theta - \|\theta_k + P_k^{1/2}x\|.$$

From [23], its uniform controllability and observability are equivalent to the uniform positivity of the families $\{g_k(x), k \geqslant 0\}$ and $\{h_k(x), k \geqslant 0\}$, where

$$g_k(x) = \det\left(\sum_{i=0}^{n-1} A^i(k,x)B(k,x)B^{\tau}(k,x)A^{i\tau}(k,x)\right),$$

$$h_k(x) = \det\left(\sum_{i=0}^{n-1} A^{i\tau}(k,x)QA^i(k,x)\right),$$

and $\det(\cdot)$ denotes the determinant of a matrix.

To design a convergent β_k, we define a function as follows:

$$F_k(x) = \min\{f_k(x), g_k(x), h_k(x)\},$$

and define the positive-valued set A_k:

$$A_k = \{x \in \mathcal{M}(n+m,n) | F_k(x) > 0\}. \tag{21}$$

In addition, we introduce the following time instant $t(k)$ for any integer $k > 0$:

$$t(k) = \min\left\{t \in \mathbb{N} : t \leqslant k,\, \mathcal{L}\left(\bigcap_{i=t}^{k} A_i\right) \geqslant \mu > 0\right\}, \tag{22}$$

where $\mathcal{L}(\cdot)$ is the Lebesgue measure and μ is some positive number.

Remark 2. In fact, the above μ can be chosen by the following method. Let $\{\theta_k,\, P_k,\, k \geqslant 0\}$ be the sequence generated by (16). Then, for a given positive number $\eta < h_\theta$, we define a set $B_k = \{x \in \mathcal{M}(n+m,n) | x = (\frac{\lambda}{\|\theta_k\|} - 1)P_k^{-1/2}\theta_k,\, |\lambda| \leqslant h_\theta - \eta\} \oplus \{x \in \mathcal{M}(n+m,n) |\ x^\tau \cdot P_k^{-1/2}\theta_k = 0,\, \|x\| < \|P_k^{1/2}\|^{-1}\eta\}$. It is easy to verify that $B_k - N_k \subseteq A_k$, where $N_k \triangleq \{x \in A_k | g_k(x) = 0,\, h_k(x) = 0\}$ is a set of measure zero [22], and that $\mathcal{L}(B_k - N_k) = \mathcal{L}(B_k) \geqslant 2\|P_k^{1/2}\|^{-1}(h_\theta - \eta)\eta^{2n+m-1}$. Moreover, by the definition of P_k, we know that $\{\|P_k^{1/2}\|^{-1},\, k \geqslant 0\}$ is non-decreasing. Hence, if we choose $0 < \mu \leqslant 2\|P_0^{1/2}\|^{-1}(h_\theta - \eta)\eta^{2n+m-1}$, then we have $\mathcal{L}(A_k) \geqslant \mathcal{L}(B_k) \geqslant \mu$; thus, $t(k)$ is well-defined because $k \in \{t \in \mathbb{N} : t \leqslant k,\, \mathcal{L}(\bigcap_{i=t}^{k} A_i) \geqslant \mu > 0\}$ for any $k \geqslant 0$.

Then, we can define a non-empty set:

$$D_k = \bigcap_{i=t(k)}^{k} A_i.$$

We now proceed to show that there are two integers $t_1 \geqslant t_0 > 0$ such that $t(k) = t_0$ for all $k \geqslant t_1$, i.e., $D_k = \bigcap_{i=t_0}^{k} A_i$, $\forall k \geqslant t_1$. In this case, it is obvious that $D_{k+1} \subseteq D_k$ for all $k \geqslant t_1$. In fact, by the convergence of the parameter estimate θ_k as shown in Lemma 1, we can indeed prove that t_0 and t_1 exist in the following lemma.

Lemma 2. There exists a positive number μ together with two integers $t_1 \geqslant t_0 > 0$ such that $t(k) = t_0$ for all $k \geqslant t_1$, i.e., $D_k = \bigcap_{i=t_0}^{k} A_i$, $\forall k \geqslant t_1$. In addition, D_∞ exists and $L(D_\infty) \geqslant \mu$.

The proof is provided in Appendix A.

Let $\{\eta_k,\, k \geqslant 0\}$ be an independent sequence in $\mathbb{R}^{(n+m)\times n}$ that is independent of the system noise $\{w_k,\, k \geqslant 0\}$, and for each $k \geqslant 0$, η_k is uniformly distributed on D_k. Then, the sequence $\{\beta_k, k \geqslant 0\}$ to be used in (20) can be defined recursively as follows:

$$\beta_k = \begin{cases} \eta_k, & F_k(\eta_k) \geqslant (1+\gamma)F_k(\beta_{k-1}), \\ \beta_{k-1}, & F_k(\eta_k) < (1+\gamma)F_k(\beta_{k-1}), \end{cases} \tag{23}$$

where γ is small enough so that $2\gamma + \gamma^2 \leqslant 1$. One can show that the sequence $\{\beta_k,\, k \geqslant 0\}$ defined above can ensure that the estimates $\{\hat{\theta}_k,\, k \geqslant 0\}$ have the required properties as described in the following lemma whose proof is given in Appendix B.

Lemma 3. Under Assumptions 1 and 2, the regularized WLS estimate $\hat{\theta}_k$ defined by (20) with β_k chosen as in (23) has the following properties:

(1) $\hat{\theta}_k \in \Theta$ converges to a random matrix $\hat{\theta}$, a.s.,

(2) the family $\{\hat{A}_k,\, \hat{B}_k,\, Q^{1/2},\, k \geqslant 0\}$ is uniformly controllable and uniformly observable, a.s.,

(3) $\|P_k^{-1/2}\tilde{\theta}_k\|^2 = O(1)$, $k \to \infty$, a.s.,

(4) $\sum_{i=1}^{k} \|\varphi_i^\tau \tilde{\theta}_i\|^2 = o(r_k) + O(1)$, $k \to \infty$, a.s.,

where $\tilde{\theta}_k = \theta^* - \hat{\theta}_k$ and r_k is defined in (17).

4 Optimal adaptive MPC algorithm and main results

In this section, we introduce the design of the adaptive MPC with the estimate $\hat{\theta}_k$ defined by (20) and (23) such that the state sequence and the control sequence lie in the soft constraint sets of $\mathbb{X}$ and $\mathbb{U}$ defined in (4) and (5).

From Assumption 2 and Lemma 3, if $x_k \in \mathbb{X}$ at the k-th step, we can choose the control $u_k = u_0(x_k, \hat{\theta}_k)$ as introduced in Assumption 3. Note that

$$x_{k+1} = A^* x_k + B^* u_0(x_k, \hat{\theta}_k) + w_{k+1},$$

where $\hat{\theta}_k$ is defined in Lemma 3. Because an error may exist between the estimate and the true parameter, the state x_{k+1} may not be in $\mathbb{X}$, and thus the control $u_0(x_{k+1}, \hat{\theta}_{k+1})$ may not be well-defined. To address this case, we try to project the state x_{k+1} onto $\mathbb{X}$ using the following method:

$$\bar{x}_{k+1} = \arg\min_{x \in \mathbb{X}} \|x - x_{k+1}\|^2_{Q_k}, \tag{24}$$

where Q_k is a positive matrix to be defined as follows.

First, as is well known, for any controllable and observable triple $(A, B, Q^{1/2})$, the following Riccati equation has a positive definite matrix Φ as its solution:

$$\Phi = Q + A^\tau \Phi A - A^\tau \Phi B(B^\tau \Phi B + R)^{-1} B^\tau \Phi A, \tag{25}$$

where Q and R are defined in (8), and the matrix $A + BK$ is stable with the following matrix K:

$$K \triangleq -(B^\tau \Phi B + R)^{-1} B^\tau \Phi A. \tag{26}$$

Under this motivation, let $\{\hat{\theta}_k^\tau = (\hat{A}_k, \hat{B}_k),\ k \geqslant 0\}$ be the family of the regularized WLS estimates defined in Lemma 3. Knowing that $\{\hat{A}_k,\ \hat{B}_k,\ Q^{1/2},\ k \geqslant 0\}$ is uniformly controllable and uniformly observable, we can define $\hat{K}_k$ following the similar approach to the definition of K in (25) and (26) but with (A, B) replaced by $(\hat{A}_k,\ \hat{B}_k)$. Let us recursively define $\{Q_k,\ k \geqslant 0\}$ used in (24) by the following formula with any given initial condition $Q_0 > 0$:

$$Q_{k+1} = (\hat{A}_k + \hat{B}_k \hat{K}_k)^\tau Q_k (\hat{A}_k + \hat{B}_k \hat{K}_k) + I, \tag{27}$$

where I is the identity matrix with dimension n. Since $\hat{A}_k + \hat{B}_k \hat{K}_k$ is stable for any $k \geqslant 1$ and converges to a stable matrix, we know that $\{Q_k > 0,\ k \geqslant 0\}$ is bounded and has a limit, which is defined as $\bar{Q}$, i.e., $\bar{Q} = \lim_{k\to\infty} Q_k$ [16].

Using the projection (24), we know that $\bar{x}_k$ must be in $\mathbb{X}$ and that $u_0(\bar{x}_k, \hat{\theta}_k)$ is well-defined. In addition, to counteract the influence of the residue $x_k - \bar{x}_k$, we introduce the following adaptive MPC:

$$\bar{u}_k = u_0(\bar{x}_k, \hat{\theta}_k) + \hat{K}_k (x_k - \bar{x}_k), \tag{28}$$

where $\hat{K}_k$ is defined above.

Clearly, u_k is meaningful for any $x_k \in \mathbb{R}^n$. The next theorem will further show that this defined control sequence, together with the corresponding state sequence, satisfies the soft constraints defined in Definition 1.

Theorem 1. Let Assumptions 1–3 hold. Then, the state x_k of the system (1) starting at any $x_0 \in \mathbb{X}$ and the control $\bar{u}_k$ defined by (28) lie in the sets $\mathbb{X}_k$ and $\mathbb{U}_k$, respectively, for all $k \geqslant 0$, where $\{\mathbb{X}_k,\ k \geqslant 0\}$ and $\{\mathbb{U}_k,\ k \geqslant 0\}$ are the soft constraint sets of $\mathbb{X}$ and $\mathbb{U}$, respectively, defined in Definition 1.

The proof is given in Section 5.

In Theorem 1, the performance of the closed-loop control system under adaptive MPC is not easily analyzed because the parameter estimates may not be strongly consistent. To obtain a strong consistency for the family of estimates, we add an attenuating excitation to the adaptive MPC (28) by using the method introduced in [22].

Let $\{\epsilon_k,\ k \geqslant 0\}$ be a sequence of m-dimensional i.i.d random vectors independent of $\{\omega_k, \eta_k,\ k \geqslant 0\}$ with

$$E[\epsilon_k] = 0,\ E[\epsilon_k \epsilon_k^\tau] = I_m,\ \|\epsilon_k\| \leqslant h_\epsilon, \tag{29}$$

where h_ϵ is a known number, and let the attenuating excitation ξ_k be defined by

$$\xi_k = \frac{\epsilon_k}{k^{\epsilon/2}},\ \epsilon \in (0, 1/4n). \tag{30}$$

Then, the new adaptive MPC is taken as

$$u_k = \bar{u}_k + \xi_k, \tag{31}$$

where u_k is defined in (28).

By using a method similar to the one introduced in [27], we can prove that the above control ensures that the estimate $\hat{\theta}_k$ converges to the true parameter.

Lemma 4 ([27])**.** Assume that Assumptions 1–4 hold and that $\{\hat{\theta}_k,\ k \geqslant 0\}$ is the family of estimates given by (20) and (23) using the control (31) in the system (1). Then

$$\lim_{k\to\infty} \hat{\theta}_k = \theta^*,$$

where θ^* is the true parameter.

Because the estimate converges to the true parameter, we can prove that the ergodic performance under the adaptive MPC converges to that of the non-adaptive one.

Theorem 2. Let Assumptions 1–4 hold and the adaptive MPC (31) be applied to the uncertain system (1) with any initial value $x_0 \in \mathbb{X}$. Then, the state x_k and the control u_k lie in the soft constraint sets $\bar{\mathbb{X}}_k$ and $\bar{\mathbb{U}}_k$, respectively, for all $k \geqslant 0$ and will further lie in the constraints $\mathbb{X}$ and $\mathbb{U}$, respectively, for all $k \geqslant k_0$, where k_0 is a positive constant. Moreover, the ergodic performance under the adaptive MPC will converge to that of the non-adaptive one, i.e.,

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} (x_k^\tau Q x_k + u_k^\tau R u_k) = J_N^{\mathrm{MPC}}(\boldsymbol{u}_0; \theta^*), \tag{32}$$

where $J_N^{\mathrm{MPC}}(\boldsymbol{u}_0; \theta^*)$ is the non-adaptive ergodic performance defined in Definition 2.

The proof will be provided in Subsection 5.2.

Remark 3. Note that the system (1) under the adaptive MPC (31) is nonlinear, so the convergence of the ergodic performance to the non-adaptive one is not easily proven using the traditional approaches. Inspired by [28], we show that the state sequence of the closed-loop system is an ergodic random process under Assumption 4, and thus, the Markov chain ergodic theory can be applied in the convergence study of adaptive MPC.

Remark 4. We note that the adaptive continuous-time linear-quadratic Gaussian control for the unconstrained stochastic system has been studied and solved in [23]. Similar results as in Theorems 1 and 2 may be established for continuous-time adaptive MPC of linear time-invariant stochastic systems by using similar methods.

5 Proofs of the main results

5.1 The proof of Theorem 1

Proof. First, let x_k be the state of the system (1) starting at any $x_0 \in \mathbb{X}$, and let $\bar{u}_k$ be the adaptive control defined by (28). It suffices to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \|x_k - \bar{x}_k\|^2 = 0, \tag{33}$$

where $\bar{x}_k$ is defined in (24).

Let $\{\hat{\theta}_k^\tau = (\hat{A}_k, \hat{B}_k),\ k \geqslant 0\}$ be the family of estimates given by (20) and (23). For any state x_k, by the definition of $\bar{u}_k$, we have

$$\begin{aligned} x_{k+1} &= A^* x_k + B^* \bar{u}_k + \omega_{k+1} \\ &= \hat{A}_k \bar{x}_k + \hat{B}_k u_0(\bar{x}_k, \hat{\theta}_k) + \omega_{k+1} + (\hat{A}_k + \hat{B}_k \hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^\tau \varphi_k, \end{aligned} \tag{34}$$

where $\tilde{\theta}_k = \theta^* - \hat{\theta}_k$, φ_k is defined in (14), and $\hat{K}_k$ is defined following a similar approach to the definition of K in (25) and (26) but with (A, B) replaced by $(\hat{A}_k, \hat{B}_k)$.

From (2) of Lemma 3 and Assumption 3, we know that $A_k\bar{x}_k + B_k u_0(\bar{x}_k, \hat{\theta}_k) + \omega_{k+1} \in \mathbb{X}$, and combining this fact with the definition of $\bar{x}_k$ introduced in (24), we know that

$$\begin{aligned}\|x_{k+1} - \bar{x}_{k+1}\|_{Q_k}^2 &\leqslant \|(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 \\ &= (x_k - \bar{x}_k)^\tau (\hat{A}_k + \hat{B}_k\hat{K}_k)^\tau Q_k (\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \|\tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 + 2\zeta_k \\ &= \|x_k - \bar{x}_k\|_{Q_{k+1}}^2 - \|x_k - \bar{x}_k\|^2 + \|\tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 + 2\zeta_k,\end{aligned} \tag{35}$$

where $\zeta_k = (x_k - \bar{x}_k)^\tau (\hat{A}_k + \hat{B}_k\hat{K}_k)^\tau Q_k \tilde{\theta}_k^\tau \varphi_k$, and Q_k is defined by (27).

Summing both sides of (35), we have

$$\begin{aligned}\bar{r}_N \triangleq \sum_{k=0}^{N} \|x_k - \bar{x}_k\|^2 = \|x_0 - \bar{x}_0\|_{Q_1}^2 - \|x_{N+1} - \bar{x}_{N+1}\|_{Q_N}^2 + \sum_{k=0}^{N} \|\tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 + 2\sum_{k=0}^{N} \zeta_k \\ + \sum_{k=1}^{N} (x_k - \bar{x}_k)^\tau \left(Q_{k+1} - Q_{k-1}\right)(x_k - \bar{x}_k).\end{aligned} \tag{36}$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{k=0}^{N} \zeta_k \leqslant \left(\sum_{k=0}^{N} \|(\hat{A}_k + \hat{B}_k\hat{K}_k) Q_k (x_k - \bar{x}_k)\|^2\right)^{1/2} \left(\sum_{k=0}^{N} \|\tilde{\theta}_k^\tau \varphi_k\|^2\right)^{1/2}. \tag{37}$$

Because $\{\hat{A}_k, \hat{B}_k, \hat{K}_k, Q_k, k \geqslant 0\}$ is bounded and convergent, we can introduce the following finite-valued notations:

$$\begin{aligned}&\hat{k}_m = \max_{0\leqslant k\leqslant\infty} \|\hat{K}_k\|^2, \quad \hat{a}_m = \max_{0\leqslant k\leqslant\infty} \|\hat{A}_k + \hat{B}_k\hat{K}_k\|^2, \\ &\hat{a}_{qm} = \max_{0\leqslant k\leqslant\infty} \|(\hat{A}_k + \hat{B}_k\hat{K}_k) Q_k\|^2.\end{aligned} \tag{38}$$

Then, we have

$$\sum_{k=0}^{N} \zeta_k \leqslant (\hat{a}_{qm}\bar{r}_N)^{1/2} \left(\sum_{k=0}^{N} \|\tilde{\theta}_k^\tau \varphi_k\|^2\right)^{1/2}. \tag{39}$$

In addition, let $\lambda_{\max}(Q_k)$ be the maximum eigenvalue of the matrix $Q_k \in \mathbb{R}^{n\times n}$, and let $\bar{\lambda}_{\max} \triangleq \max_{0\leqslant k\leqslant\infty} \lambda_{\max}(Q_k) < \infty$. Then, we have

$$\|\tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 \leqslant \bar{\lambda}_{\max} \cdot \|\tilde{\theta}_k^\tau \varphi_k\|^2, \quad \forall k \geqslant 0.$$

By (4) of Lemma 3, we have

$$\sum_{k=0}^{N} \|\tilde{\theta}_k^\tau \varphi_k\|_{Q_k}^2 = o(r_N), \quad N \to \infty, \tag{40}$$

where r_N is defined by (17).

From the convergence of $\{Q_k, k \geqslant 0\}$, we have $\lim_{k\to\infty} \|Q_{k+1} - Q_{k-1}\| = 0$. It is easy to see that

$$\sum_{k=1}^{N} (x_k - \bar{x}_k)^\tau (Q_{k+1} - Q_{k-1})(x_k - \bar{x}_k) = o(\bar{r}_N), \ N \to \infty. \tag{41}$$

Hence, using (36), (39)–(41), we have

$$\bar{r}_N = O(1) + o(r_N) + o(r_N^{1/2}\bar{r}_N^{1/2}) + o(\bar{r}_N), \quad N \to \infty, \tag{42}$$

from which it is not difficult to see that

$$\bar{r}_N = O(1) + o(r_N), \quad N \to \infty. \tag{43}$$

Moreover, from the definition of r_N, we have

$$r_N = \|P_0^{-1}\| + \sum_{i=0}^{N} \|\varphi_i\|^2 = \|P_0^{-1}\| + \sum_{i=0}^{N} (\|x_i\|^2 + \|\bar{u}_i\|^2)$$
$$\leqslant \|P_0^{-1}\| + 2\sum_{i=0}^{N} \left[(1 + \|\hat{K}_i\|^2)\|x_i - \bar{x}_i\|^2 + \|\bar{x}_i\|^2 + \|u_0(\bar{x}_i, \hat{\theta}_i)\|^2\right]$$
$$\leqslant \|P_0^{-1}\| + 2\sum_{i=0}^{N} \left[(1 + \hat{k}_m)\|x_i - \bar{x}_k\|^2 + \|\bar{x}_i\|^2 + \|u_0(\bar{x}_i, \hat{\theta}_i)\|^2\right]. \tag{44}$$

From (24) and Assumption 3, we have $\|\bar{x}_i\| \leqslant \bar{x}$ and $\|u_0(\bar{x}_i, \hat{\theta}_i)\| \leqslant \bar{u}$ for all $i \geqslant 0$. Then we can obtain

$$r_N = O(\bar{r}_N) + O(N),\ N \to \infty. \tag{45}$$

Combining (45) and (43), we have $r_N = O(N)$ when $N \to \infty$. Furthermore, we can obtain $\bar{r}_N = \sum_{k=0}^{N} \|x_k - \bar{x}_k\|^2 = o(N)$ when $N \to \infty$. Hence, Eq. (33) is proven.

Now, letting

$$\delta_{1,k} = (\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^{\tau}\varphi_k,$$
$$\delta_{2,k} = \hat{K}_k(x_k - \bar{x}_k), \tag{46}$$

from (34) and the definition of $\bar{u}_k$, we have

$$x_{k+1} = \hat{A}_k\bar{x}_k + \hat{B}_k u_0(\bar{x}_k, \hat{\theta}_k) + \omega_{k+1} + \delta_{1,k} \in \mathbb{X} \oplus \delta_{1,k},$$

and

$$\bar{u}_k = u_0(\bar{x}_k, \hat{\theta}_k) + \delta_{2,k} \in \mathbb{U} \oplus \delta_{2,k}.$$

Let $\mathbb{X}_k = \mathbb{X} \oplus \delta_{1,k}$ and $\mathbb{U}_k = \mathbb{U} \oplus \delta_{2,k}$. To verify that $\{\mathbb{X}_k,\ k \geqslant 0\}$ and $\{\mathbb{U}_k,\ k \geqslant 0\}$ are the soft constraint sets of $\mathbb{X}$ and $\mathbb{U}$, respectively, we only need to prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|\delta_{1,k}\|^2 = 0,\quad \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \|\delta_{2,k}\|^2 = 0. \tag{47}$$

From (38), we have $\|\delta_{1,k}\|^2 \leqslant 2(\hat{a}_m\|x_k - \bar{x}_k\|^2 + \|\tilde{\theta}_k^{\tau}\varphi_k\|^2)$. Then the first equation in (47) is easily verified by using (33). Similarly, we can prove that $\|\delta_{2,k}\|^2 \leqslant \hat{k}_m\|x_k - \bar{x}_k\|^2$, and the second equation is also proven.

5.2 The proof of Theorem 2

Proof. Without ambiguity, we also use $\{\hat{\theta}_k^{\tau} = (\hat{A}_k, \hat{B}_k),\ k \geqslant 0\}$ to denote the family of estimates given by (20) and (23) with the data generated from the system (1) under the control (31). In addition, the definitions of $\hat{K}_k$ and Q_k are similar to those introduced in Theorem 1.

Since the attenuating excitation ξ_k defined in (30) converges to 0, Theorem 1 also holds under the control (31). In fact, we only need to replace $\delta_{1,k+1}$ and $\delta_{2,k}$ defined in (46) by

$$\bar{\delta}_{1,k+1} = (\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^{\tau}\varphi_k + \hat{B}_k\xi_k,$$
$$\bar{\delta}_{2,k} = \hat{K}_k(x_k - \bar{x}_k) + \xi_k, \tag{48}$$

where $\hat{K}_k$ is the solution of (25) and (26) with the estimate $\hat{\theta}_k$, and ξ_k is defined in (30). The corresponding soft constraint sets are $\bar{\mathbb{X}}_k = \mathbb{X} \oplus \bar{\delta}_{1,k+1}$ and $\bar{\mathbb{U}}_k = \mathbb{U} \oplus \bar{\delta}_{2,k}$.

We will prove the other parts of this theorem as follows.

Step 1: We will prove that $\limsup_{k\to\infty} \|x_k - \bar{x}_k\| = 0$.

Since $\{\hat{A}_k, \hat{B}_k,\ k \geqslant 0\}$ converges to the true parameter (A^*, B^*) by Lemma 4, we know that $\{\hat{K}_k,\ Q_k,\ k \geqslant 0\}$ is convergent, and we can define

$$K^* = \lim_{k\to\infty} \hat{K}_k,\quad \bar{Q} = \lim_{k\to\infty} Q_k.$$

Hence, by (27), it is easy to see that

$$\bar{Q} = (A^* + B^*K^*)^\tau \bar{Q}(A^* + B^*K^*) + I,$$

and $\bar{Q} \geqslant I$. Then, by using the definition of the weighted norm, it is not difficult to prove that $\|A^* + B^*K^*\|_{\bar{Q}} = \lambda_1 < 1$, where $\lambda_1 = (1 - \min_{\{x\in\mathbb{R}^n, x^\tau \bar{Q}x=1\}} \|x\|)^{1/2}$. Since $\bar{\theta}_k$, Q_k, and ξ_k are convergent, for any given $\varepsilon > 0$, there exists a number $k_1 > 0$ such that

$$\|Q_k - \bar{Q}\| \leqslant \lambda_{\min}(\bar{Q})\varepsilon,\ \|\tilde{\theta}_k\| \leqslant \varepsilon,\ \|\xi_k\| \leqslant \varepsilon,\ \|\hat{A}_k + \hat{B}_k\hat{K}_k\|_{\bar{Q}} \leqslant \lambda_1 + \varepsilon,\ \forall k \geqslant k_1. \tag{49}$$

Now, by the definition of φ_k in (14), we have

$$\begin{aligned}
\|\varphi_k\| &\leqslant \|\bar{x}_k\| + \|x_k - \bar{x}_k\| + \|u_0(\bar{x}_k, \hat{\theta}_k)\| + \|\hat{K}_k\|\|x_k - \bar{x}_k\| + \|\xi_k\| \\
&\leqslant \bar{x} + \lambda_{\min}^{-1/2}(\bar{Q})\|x_k - \bar{x}_k\|_{\bar{Q}} + \bar{u} + \hat{k}_{1m}\lambda_{\min}^{-1/2}(\bar{Q})\|x_k - \bar{x}_k\|_{\bar{Q}} + \varepsilon \\
&= \phi_1\|x_k - \bar{x}_k\|_{\bar{Q}} + \phi_2,
\end{aligned} \tag{50}$$

where $\phi_1 = (1+\hat{k}_{1m})\lambda_{\min}^{-1/2}(\bar{Q})$, $\phi_2 = \bar{x}+\bar{u}+\varepsilon$, $\bar{x}$ and $\bar{u}$ are defined in (4) and (5), $\hat{k}_{1m} = \max_{0\leqslant k\leqslant\infty} \|\hat{K}_k\|$, and $\lambda_{\min}(\bar{Q})$ is the minimum eigenvalue of $\bar{Q}$.

Now, we calculate the upper bound of $\|x_k - \bar{x}_k\|^2_{\bar{Q}}$. Similar to (35), we have

$$\|x_{k+1} - \bar{x}_{k+1}\|^2_{Q_k} \leqslant \|(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^\tau\varphi_k + \hat{B}_k\xi_k\|^2_{Q_k},$$

and when $k \geqslant k_1$, for any $x \in \mathbb{R}^n$, we have

$$\|x\|^2_{Q_k} = \|x\|^2_{\bar{Q}} + x^\tau(Q_k - \bar{Q})x \leqslant \|x\|^2_{\bar{Q}} + \lambda_{\min}^{-1}(\bar{Q})\|\bar{Q} - Q_k\|\|x\|^2_{\bar{Q}} \leqslant (1+\varepsilon)\|x\|^2_{\bar{Q}}. \tag{51}$$

Then, for $k \geqslant k_1$, we have

$$\begin{aligned}
\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}} &= \|x_{k+1} - \bar{x}_{k+1}\|^2_{Q_k} + (x_{k+1} - \bar{x}_{k+1})^\tau(\bar{Q} - Q_k)(x_{k+1} - \bar{x}_{k+1}) \\
&\leqslant \|x_{k+1} - \bar{x}_{k+1}\|^2_{Q_k} + \lambda_{\min}^{-1}(\bar{Q})\|\bar{Q} - Q_k\|\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}} \\
&\leqslant \|(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^\tau\varphi_k + \hat{B}_k\xi_k\|^2_{Q_k} + \varepsilon\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}} \\
&\leqslant (1+\varepsilon)\|(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + \tilde{\theta}_k^\tau\varphi_k + \hat{B}_k\xi_k\|^2_{\bar{Q}} + \varepsilon\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}}, \\
&\leqslant (1+\varepsilon)\left[\|(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k)\|^2_{\bar{Q}} + \|\tilde{\theta}_k^\tau\varphi_k\|^2_{\bar{Q}} + \|\hat{B}_k\xi_k\|^2_{\bar{Q}} + 2\bar{\zeta}_k\right] \\
&\quad + \varepsilon\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}},
\end{aligned} \tag{52}$$

where $\bar{\zeta}_k = (\tilde{\theta}_k^\tau\varphi_k)^\tau\bar{Q}(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + (\hat{B}_k\xi_k)^\tau\bar{Q}(\hat{A}_k + \hat{B}_k\hat{K}_k)(x_k - \bar{x}_k) + (\tilde{\theta}_k^\tau\varphi_k)^\tau\bar{Q}\hat{B}_k\xi_k$.

Moving the last term to the left, it is not difficult to see that

$$\begin{aligned}
\|x_{k+1} - \bar{x}_{k+1}\|^2_{\bar{Q}} \leqslant \frac{1+\varepsilon}{1-\varepsilon}\Big(&\|\hat{A}_k + \hat{B}_k\hat{K}_k\|^2_{\bar{Q}}\|x_k - \bar{x}_k\|^2_{\bar{Q}} + \|\bar{Q}\|\|\tilde{\theta}_k^\tau\|^2\|\varphi_k\|^2 + \|\bar{Q}\|\|\hat{B}_k\|^2\|\xi_k\|^2 \\
&+ 2\lambda_{\min}^{-1}(\bar{Q})\|\bar{Q}\|\|\tilde{\theta}_k^\tau\|\|\varphi_k\|\|\hat{A}_k + \hat{B}_k\hat{K}_k\|\|x_k - \bar{x}_k\|_{\bar{Q}} \\
&+ 2\lambda_{\min}^{-1}(\bar{Q})\|\bar{Q}\|\|\hat{B}_k\|\|\xi_k\|\|\hat{A}_k + \hat{B}_k\hat{K}_k\|\|x_k - \bar{x}_k\|_{\bar{Q}} + 2\|\bar{Q}\|\|\tilde{\theta}_k^\tau\|\|\varphi_k\|\|\hat{B}_k\|\|\xi_k\|\Big).
\end{aligned} \tag{53}$$

From (49) and (50), we have

$$\begin{aligned}
&\|\bar{Q}\|\|\tilde{\theta}_k^\tau\|^2\|\varphi_k\|^2 \leqslant \varepsilon^2\|\bar{Q}\|\left[\phi_1^2\|x_k - \bar{x}_k\|^2_{\bar{Q}} + 2\phi_1\phi_2\|x_k - \bar{x}_k\|_{\bar{Q}} + \phi_2^2\right], \\
&\|\bar{Q}\|\|\hat{B}_k\|^2\|\xi_k\|^2 \leqslant \|\bar{Q}\|\hat{b}_m^2\varepsilon^2, \\
&\lambda_{\min}^{-1}(\bar{Q})\|\bar{Q}\|\|\tilde{\theta}_k^\tau\|\|\varphi_k\|\|\hat{A}_k + \hat{B}_k\hat{K}_k\|\|x_k - \bar{x}_k\|_{\bar{Q}} \\
&\quad \leqslant \lambda_{\min}^{-1}(\bar{Q})\hat{a}_m\varepsilon\|\bar{Q}\|(\phi_1\|x_k - \bar{x}_k\|^2_{\bar{Q}} + \phi_2\|x_k - \bar{x}_k\|_{\bar{Q}}), \\
&\|\bar{Q}\|\|\tilde{\theta}_k^\tau\|\|\varphi_k\|\|\hat{B}_k\|\|\xi_k\| \leqslant \varepsilon^2\hat{b}_m\|\bar{Q}\|(\phi_1\|x_k - \bar{x}_k\|_{\bar{Q}} + \phi_2),
\end{aligned} \tag{54}$$

where $\hat{a}_m$ is defined in (38) and $\hat{b}_m = \max_{0\leqslant k\leqslant\infty} \|\hat{B}_k\|$.

Then, using (53), we have

$$\begin{aligned}\|x_{k+1}-\bar{x}_{k+1}\|_{\bar{Q}}^2 &\leqslant a_\varepsilon\|x_k-\bar{x}_k\|_{\bar{Q}}^2+O(\varepsilon\|x_k-\bar{x}_k\|_{\bar{Q}})+O(\varepsilon^2)\\ &\leqslant [a_\varepsilon+O(\varepsilon)]\,\|x_k-\bar{x}_k\|_{\bar{Q}}^2+O(\varepsilon),\ \forall k\geqslant k_1,\end{aligned}\tag{55}$$

where $a_\varepsilon=\frac{1+\varepsilon}{1-\varepsilon}\left[(\lambda_1+\varepsilon)^2+\phi_1^2\|\bar{Q}\|\varepsilon^2+2\lambda_{\min}^{-1}(\bar{Q})\hat{a}_m\phi_1\|\bar{Q}\|\varepsilon\right]$. As long as ε is small enough, it is not difficult to see that

$$\limsup_{k\to\infty}\|x_{k+1}-\bar{x}_{k+1}\|_{\bar{Q}}^2=O(\varepsilon),$$

and since ε can be arbitrarily small, we have

$$\limsup_{k\to\infty}\|x_k-\bar{x}_k\|_{\bar{Q}}^2=0.$$

Then, using the equivalency of norms, we can obtain

$$\limsup_{k\to\infty}\|x_k-\bar{x}_k\|=0,\ \text{a.s.}$$

Step 2: We will prove that there is a finite time k_0 such that $u_k=u_0(x_k,\hat{\theta}_k)+\xi_k,\ \forall\ k\geqslant k_0+1$, a.s.

First, combining Assumption 3 with the fact that $\|\tilde{\theta}_k^\tau\|$, $\|x_k-\bar{x}_k\|$ and ξ_k converge to 0, for any given positive number $\epsilon_1<\bar{w}-\sup_{k\geqslant 0}\|w_k\|$, there is a time k_0 such that for all $k\geqslant k_0$,

$$\|(A^*+B^*\hat{K}_k)(x_k-\bar{x}_k)\|\leqslant\frac{1}{3}\epsilon_1,\ \|B^*(u_0(\bar{x}_k,\theta_k)-u_0(\bar{x}_k,\theta^*))\|\leqslant\frac{1}{3}\epsilon_1,\ \|B^*\xi_k\|\leqslant\frac{1}{3}\epsilon_1.$$

Letting $\tilde{\delta}_k=w_{k+1}+(A^*+B^*\hat{K}_k)(x_k-\bar{x}_k)+B^*(u_0(\bar{x}_k,\hat{\theta}_k)-u_0(\bar{x}_k,\theta^*))+B^*\xi_k$, it is easy to see that $\tilde{\delta}_k\in\mathbb{W}$ for all $k\geqslant k_0$. From the definition of $u_0(x,\theta^*)$ in Assumption 3, we know that for all $x\in\mathbb{X}$,

$$A^*x+B^*u_0(x,\theta^*)+w\in\mathbb{X},\quad\forall w\in\mathbb{W}.\tag{56}$$

Note that by (31), $u_k=u_0(\bar{x}_k,\hat{\theta}_k)+\hat{K}_k(x_k-\bar{x}_k)+\xi_k$, and it is not difficult to see that for all $k\geqslant k_0$,

$$\|x_{k+1}\|=\|A^*x_k+B^*u_k+w_{k+1}\|=\|A^*\bar{x}_k+B^*u_0(\bar{x}_k,\theta^*)+\tilde{\delta}_k\|\leqslant\bar{x}.$$

Thus, $x_{k+1}\in\mathbb{X}$ for all $k\geqslant k_0$, and therefore $x_{k+1}=\bar{x}_{k+1}$ and $u_{k+1}=u_0(x_{k+1},\hat{\theta}_{k+1})+\xi_{k+1},\forall k\geqslant k_0$, a.s.

Step 3: By Assumption 3 from (9b) and (9c), we consider the state sequence $\{\hat{x}_k\in\mathbb{X},k\geqslant 0\}$ generated by the MPC controller $u_0(\cdot,\theta^*)$ with the true parameter θ^*, i.e.,

$$\hat{x}_{k+1}=A^*\hat{x}_k+B^*u_0(\hat{x}_k,\theta^*)+w_{k+1},\quad\hat{x}_0\in\mathbb{X},\tag{57}$$

where w_{k+1} is the same noise as in the system (1). We will prove that the sequence $\{\hat{x}_k\in\mathbb{X},k\geqslant 0\}$ is ergodic; i.e., there exists an invariant probability measure $\bar{\pi}(\cdot)$ such that for any $n\geqslant 1$,

$$\lim_{n\to\infty}\|\bar{\mathcal{P}}_0^n-\bar{\pi}\|=0,\tag{58}$$

where $\|\bar{\mathcal{P}}\|\triangleq\sup_{x\in\mathbb{X}}|\int_{\mathbb{X}}\bar{\mathcal{P}}(x,y)\mathrm{d}y|$, $\bar{\mathcal{P}}_k^n(x,y)\triangleq P(\hat{x}_{k+n}=y|\hat{x}_k=x)$ is the n-step transition probability at time k, and $\bar{\pi}(\cdot)$ satisfies $\hat{x}_k$ having the distribution $\bar{\pi}$ for all $k\geqslant 1$ when $\hat{x}_0$ has the distribution $\bar{\pi}$ [29].

Letting $\mathbb{M}\triangleq\{x\in\mathbb{X}|V(x)\leqslant\max_{w\in\mathbb{W}}\alpha(\|w\|)\}$, by using (11) and choosing $w_k\equiv 0$, it is not difficult to prove that any invariant set in $\mathbb{X}$ contains $\mathbb{M}$ and that $0\in\mathbb{M}$ is attracting for all $x\in\mathbb{X}$. Then, by Proposition 7.2.5 (pp. 160) introduced in [29], there exists a minimal set $\mathbb{X}_\rho$ such that $\mathbb{M}\subseteq\mathbb{X}_\rho\subseteq\mathbb{X}$ (the definitions of attracting, invariant set, and minimal set can be found in [29]). Obviously, $\mathbb{M}$ is connected, and from the attracting of $\mathbb{M}$ and Assumption 1, we know that $\mathbb{X}_\rho$ is connected. From Theorem 7.2.4 (pp. 159) and Proposition 7.3.4 (pp. 163) introduced in [29], we can prove that $\{\hat{x}_k\in\mathbb{X},k\geqslant 0\}$ is φ-irreducible and aperiodic. Then, combining (13) provided in Assumption 4 with Theorem 8.4.3 (pp. 191) introduced in [29], it is not difficult to prove that the sequence is ergodic.

Step 4: We now prove that

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}(x_k^\tau Qx_k+u_k^\tau Ru_k)=J_N^{\mathrm{MPC}}(\boldsymbol{u}_0;\theta^*).$$

From Step 2, $u_k = u_0(x_k, \hat{\theta}_k) + \xi_k$ for all $k \geqslant k_0 + 1$; thus, the true state x_k generated by the adaptive MPC satisfies

$$x_{k+1} = A^* x_k + B^*(u_0(x_k, \hat{\theta}_k) + \xi_k) + w_{k+1}, \ \ k \geqslant k_0 + 1, \tag{59}$$

and $x_k \in \mathbb{X}$ for all $k \geqslant k_0 + 1$. Note that by the proof of Lemma 3, there exists a time $k_2 > k_0$ such that $\beta_k = \beta_{k_2}$ for all $k \geqslant k_2$, where β_k is defined in (23). Now, let $\Phi_k = (\varphi_k, \hat{\theta}_k, P_k)$ and $\bar{w}_{k+1} = w_{k+1} + B^*\xi_k$, where $\varphi_k^\tau = (x_k^\tau, u_0^\tau(x_k, \hat{\theta}_k)) \in \mathbb{Z} \triangleq \mathbb{X} \times \mathbb{U}$, $\hat{\theta}_k \in \Theta$, and $P_k \in \mathbb{M}^+ \triangleq \{P \in \mathbb{R}^{(n+m)^2} | P \geqslant 0\}$ defined in (14), (20), and (16), respectively. Similar to [28], it is not difficult to know that the stochastic process $\{\Phi_k, \, k \geqslant k_2\}$ evolving on $\mathbb{D} \triangleq \mathbb{Z} \times \Theta \times \mathbb{M}^+$ is a Feller-Markov chain; i.e., there exists a continuous function $G : \mathbb{D} \times \mathbb{R}^n \to \mathbb{D}$ such that $\Phi_{k+1} = G(\Phi_k, \bar{w}_{k+1})$. Considering $\hat{\Phi}_k = ((\hat{x}_k^\tau, u_0^\tau(\hat{x}_k, \theta^*))^\tau, \theta^*, 0)$, it is easy to know that $\hat{\Phi}_{k+1} = G(\hat{\Phi}_k, w_{k+1})$. Furthermore, $\{\hat{\Phi}_k, \, k \geqslant k_0\}$ has an invariant probability measure $\pi(\cdot)$ on $(\mathbb{D}_1, \mathcal{F}_1)$ from Step 3, where $\mathbb{D}_1 = \mathbb{Z} \times \{\theta^*\} \times \{0\}$ and $\mathcal{F}_1$ is a σ-algebra on this space.

Now, let $\mathcal{P}^{(k,k+n)}(x, y) \triangleq P(\Phi_{k+n} = y | \Phi_k = x)$ and $\hat{\mathcal{P}}^{(k,k+n)}(x, y) \triangleq P(\hat{\Phi}_{k+n} = y | \hat{\Phi}_k = x)$ be the n-step transition probability of $\{\Phi_k, \, k \geqslant k_2\}$ and $\{\hat{\Phi}_k, \, k \geqslant k_2\}$, respectively, at time k. Since ξ_k converges to 0, the function ρ_k, which is the density of $\bar{w}_{k+1}$, uniformly converges to ρ_w defined in Definition 1. Because $\lim_{k\to\infty} \hat{\theta}_k = \theta^*$ and $\lim_{k\to\infty} \xi_k = 0$, we know that for any given $x \in \mathbb{D}_1$ and any set $S \in \mathcal{F}_1$,

$$\begin{aligned}
&\limsup_{k\to\infty} \left| \mathcal{P}^{(k,k+1)}(x, S) - \hat{\mathcal{P}}^{(k,k+1)}(x, S) \right| \\
&\quad = \limsup_{k\to\infty} \left| \int_{\mathbb{W}} P\left(G(x, z) \in S\right) \rho_k(z) \mathrm{d}z - \int_{\mathbb{W}} P\left(G(x, z) \in S\right) \rho_w(z) \mathrm{d}z \right| \\
&\quad \leqslant \int_{\mathbb{W}} P\left(G(x, z) \in S\right) \limsup_{k\to\infty} \left|\rho_k(z) - \rho_w(z)\right| \mathrm{d}z = 0.
\end{aligned} \tag{60}$$

Moreover, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \left| \mathcal{P}^{(k,k+2)}(x, S) - \hat{\mathcal{P}}^{(k,k+2)}(x, S) \right| \\
&= \limsup_{k\to\infty} \left| \int_{\mathbb{D}_1} \mathcal{P}^{(k,k+1)}(x, z) \mathcal{P}^{(k+1,k+2)}(z, S) \mathrm{d}z - \int_{\mathbb{D}_1} \hat{\mathcal{P}}^{(k,k+1)}(x, z) \hat{\mathcal{P}}^{(k+1,k+2)}(z, S) \mathrm{d}z \right| \\
&\leqslant \int_{\mathbb{D}_1} \mathcal{P}^{(k,k+1)}(x, z) \limsup_{k\to\infty} \left| \mathcal{P}^{(k+1,k+2)}(z, S) - \hat{\mathcal{P}}^{(k+1,k+2)}(z, S) \right| \mathrm{d}z \\
&\quad + \int_{\mathbb{D}_1} \limsup_{k\to\infty} \left| \mathcal{P}^{(k,k+1)}(x, z) - \hat{\mathcal{P}}^{(k,k+1)}(x, z) \right| \hat{\mathcal{P}}^{(k+1,k+2)}(z, S) \mathrm{d}z \\
&= 0.
\end{aligned} \tag{61}$$

Similarly, for any $v \geqslant 1$, we can prove that

$$\lim_{k\to\infty} \left| \mathcal{P}^{(k,k+v)}(x, S) - \hat{\mathcal{P}}^{(k,k+v)}(x, S) \right| = 0, \ \ \forall x \in \mathbb{D}_1, \ \ \forall S \in \mathcal{F}_1. \tag{62}$$

In addition, using the property of the invariant probability measure $\pi(\cdot)$, for any distribution function $\lambda(x)$ on $(\mathbb{D}_1, \mathcal{F}_1)$, we know that

$$\lim_{n\to\infty} \left| \int_{x\in\mathbb{D}_1} \lambda(x) \hat{\mathcal{P}}^{(k,k+n)}(x, S) \mathrm{d}x - \pi(S) \right| = 0, \ \ \forall k \geqslant k_2. \tag{63}$$

From (62) and (63), for any $\epsilon > 0$, there exists $N_1 > k_2$ such that for all $N_2 > 2N_1$,

$$\left| \mathcal{P}^{(k_2+N_1,k_2+N_2)}(x, S) - \hat{\mathcal{P}}^{(k_2+N_1,k_2+N_2)}(x, S) \right| \leqslant \frac{\epsilon}{2},$$

and

$$\left| \int_{\mathbb{D}_1} \mathcal{P}^{(k_2,k_2+N_1)}(x, z) \hat{\mathcal{P}}^{(k_2+N_1,k_2+N_2)}(x, S) \mathrm{d}x - \pi(S) \right| \leqslant \frac{\epsilon}{2}.$$

Then, we have

$$\left| \mathcal{P}^{(k_2,k_2+N_2)}(x, S) - \pi(S) \right|$$

$$\begin{aligned}
&\leqslant \left|\int_{\mathbb{D}_1} \mathcal{P}^{(k_2,k_2+N_1)}(x,z)\left[\mathcal{P}^{(k_2+N_1,k_2+N_2)}(z,S)-\hat{\mathcal{P}}^{(k_2+N_1,k_2+N_2)}(z,S)\right]\mathrm{d}z\right| \\
&\quad+\left|\int_{\mathbb{D}_1} \mathcal{P}^{(k_2,k_2+N_1)}(x,z)\hat{\mathcal{P}}^{(k_2+N_1,k_2+N_2)}(z,S)\mathrm{d}z-\pi(S)\right| \\
&\leqslant \int_{\mathbb{D}_1} \mathcal{P}^{(k_2,k_2+N_1)}(x,z)\left|\mathcal{P}^{(k_2+N_1,k_2+N_2)}(z,S)-\hat{\mathcal{P}}^{(k_2+N_1,k_2+N_2)}(z,S)\right|\mathrm{d}z+\frac{\epsilon}{2} \\
&\leqslant \frac{\epsilon}{2}\int_{\mathbb{D}_1} \mathcal{P}^{(k_2,k_2+N_1)}(x,z)\mathrm{d}z+\frac{\epsilon}{2}\leqslant \epsilon. \qquad (64)
\end{aligned}$$

Hence, we can prove that

$$\lim_{n\to\infty}\left|\mathcal{P}^{(k_2,k_2+n)}(x,S)-\pi(S)\right|=0.$$

Thus $\{\Phi_k,\ k \geqslant k_2\}$ is ergodic (Harris recurrent and aperiodic), and $\pi(\cdot)$ is its invariant probability measure on the invariant space $\mathbb{D}_1$.

Let $E_1=[\mathbf{1}_{n\times n},\mathbf{0}_{n\times m}]\in\mathbb{R}^{n\times(n+m)}$ and $E_2^\tau=[1,\mathbf{0}_{1\times(2n+m)}]$, where $\mathbf{1}_{n\times n}\in\mathbb{R}^{n\times n}$ has all elements with 1. From the definition of $\hat{\Phi}_k$, we have $\hat{x}_k=E_1\hat{\Phi}_kE_2$, and similarly, $x_k=E_1\Phi_kE_2$. Now, let $L(x)\triangleq x^\tau Qx+u^\tau 0(x,\theta^*)Ru_0(x,\theta^*)$. Using the strong law of large numbers introduced in [29] (Theorem 17.1.7, pp. 416), we have

$$J_N^{\mathrm{MPC}}(\boldsymbol{u}_0;\theta^*)=\lim_{n\to\infty}\frac{1}{n-k_2}\sum_{k=k_2+1}^{n}L(\hat{x}_k)=\int_{\mathbb{D}_1}L(E_1\Phi E_2)\pi(\Phi)\mathrm{d}\Phi=\lim_{n\to\infty}\frac{1}{n-k_2}\sum_{k=k_2+1}^{n}L(x_k). \qquad (65)$$

From Assumption 3, the controller $u_0(x_k,\theta_k)$ continuously depends on x_k and θ_k. Hence, because θ_k converges to θ^* and ξ_k converges to 0, we have

$$\limsup_{n\to\infty}\frac{1}{n-k_2}\sum_{k=k_2+1}^{n}(x_k^\tau Qx_k+u_k^\tau Ru_k)=\limsup_{n\to\infty}\frac{1}{n-k_2}\sum_{k=k_2+1}^{n}L(x_k)=J_N^{\mathrm{MPC}}(\boldsymbol{u}_0;\theta^*),\ \text{a.s.} \qquad (66)$$

Because k_2 is finite, we have

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}(x_k^\tau Qx_k+u_k^\tau Ru_k)=J_N^{\mathrm{MPC}}(\boldsymbol{u}_0;\theta^*),\ \text{a.s.}$$

6 Conclusion

In this paper, we proposed an adaptive MPC for discrete-time-constrained linear stochastic systems with unknown parameters. This technique relies on the design of the finite horizon constrained linear-quadratic optimization problem associated with the estimate at each step. Under some reasonable assumptions, we give a WLS estimation-based MPC algorithm in the general formulation and prove the convergence of this adaptive MPC by using the Markov chain ergodic theory together with powerful methods in stochastic adaptive control. It is worth mentioning that the similar method may be used to design adaptive MPC in the continuous-time case.

However, one of the desirable goals of adaptive MPC is to cope with possibly time-varying unknown parameters in uncertain dynamic systems, which belongs to further investigation. Moreover, combining machine learning with adaptive MPC is also an attractive research direction, since machine learning may help us to find some stabilizing MPC, which can then be used in the design of the adaptive MPC to further improve the adaptivity and performance of the closed-loop control systems.

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Appendix A The proof of Lemma 2

Proof. From (3) of Lemma 1, there exists a random matrix $\overline{\theta}$ such that $\lim_{k\to\infty}\theta_k=\bar{\theta}$. In addition, from the definition of P_k, it is easy to know that $\{P_k,\ k\geqslant 0\}$ is a non-increasing positive definite matrix sequence, so there exists P_∞ such that $\lim_{k\to\infty}P_k=P_\infty$. We now prove the lemma by separately considering three cases.

Case 1: $P_\infty=0$. From Lemma 1 (1), we know that $\overline{\theta}=\theta^*$. Now, considering the set $B_0(\mu^{1/(2n+m)})\triangleq\{x\in\mathcal{M}(n+m,n)|\|x\|\leqslant\mu^{1/(2n+m)}\}$, we know that $\mathcal{L}(B_0(\mu^{1/(2n+m)}))>\mu$. Since $\theta^*\in\Theta^o$, there exists a positive number ε_1 such that $\varepsilon_1+\varepsilon_1\mu^{1/(2n+m)}<h_\theta-\|\theta^*\|$. Then, for the given ε_1, there exists a time t_1 such that for all $k\geqslant t_1$,

$$\|\theta_k-\theta^*\|\leqslant\varepsilon_1,\quad \|P_k^{1/2}\|\leqslant\varepsilon_1. \tag{A1}$$

Hence, for any $x\in B_0(\mu^{1/(2n+m)})$ and $k\geqslant t_1$, we have

$$\|\theta_k+P_k^{1/2}x\|\leqslant\|\theta^*\|+\|\theta_k-\theta^*\|+\|P_k^{1/2}\|\|x\|\leqslant\|\theta^*\|+\varepsilon_1+\varepsilon_1\mu^{1/(2n+m)}<h_\theta. \tag{A2}$$

Thus, $f_k(x)>0,\ \forall k\geqslant t_1$. Now, letting $N_k\triangleq\{x\in A_k|g_k(x)=0,\ h_k(x)=0\}$, we know that $\mathcal{L}(N_k)=0$ [22]. Then, it is easy to know that

$$B_0\left(\mu^{1/(2n+m)}\right)-N_k\subseteq A_k,\ \ \forall k\geqslant t_1.$$

Hence, we have

$$B_0\left(\mu^{1/(2n+m)}\right)-\bigcup_{i=t_1}^{\infty}N_i\subseteq\bigcap_{i=t_1}^{k}A_i,\ \ \forall\, k\geqslant t_1.$$

Thus, $\mathcal{L}(\bigcap_{i=t_1}^{k}A_i)\geqslant\mu,\ \forall\, k\geqslant t_1$; hence, there exists a time $t_0\leqslant t_1$ such that $D_k=\bigcap_{i=t_0}^{k}A_i$ and $\mathcal{L}(D_k)\geqslant\mu,\forall k\geqslant t_1$.

Case 2: $P_\infty\neq 0,\det(P_\infty)=0$. From (1) of Lemma 1, there exists a bounded sequence $\{\beta_k^*\}$ such that

$$\theta^*=\theta_k+P_k^{1/2}\beta_k^*.$$

Hence, there exists a convergent subsequence $\{\beta^*_{k_t}, t \geqslant 0\}$ whose limit is β^*. Since $\theta^* \in \Theta^o$ and $P_\infty \neq 0$, there exists a positive number $\varepsilon_2 \in (0,1)$ small enough such that

$$\varepsilon_2\left[1+\mu^{1/(2n+m)}+\varepsilon_2\|\beta^*\|+\varepsilon_2(\mu\varepsilon_2)^{\frac{1}{2n+m}}\right]+\|P_\infty^{1/2}\|(\mu\varepsilon_2)^{1/(2n+m)}<h_\theta-\|\theta^*\|,$$

and for the given ε_2, there exists a time t_2 such that for all $k \geqslant t_2$,

$$\|\theta_k-\bar{\theta}\| \leqslant \varepsilon_2,\quad \|P_k^{1/2}-P_\infty^{1/2}\| \leqslant \varepsilon_2^2. \tag{A3}$$

Now, consider a set $S=\beta^*+S_1\oplus S_2$, where

$$\begin{aligned} S_1&=\left\{y\in\mathcal{M}(n+m,n)\middle|P_\infty^{1/2}y\neq 0,\ \|y\|\leqslant(\mu\varepsilon_2)^{1/(2n+m)}\right\},\\ S_2&=\left\{z\in\mathcal{M}(n+m,n)\middle|P_\infty^{1/2}z=0,\ \|z\|\leqslant\frac{\mu^{1/(2n+m)}}{\varepsilon_2}\right\}, \end{aligned} \tag{A4}$$

and it is not difficult to see that $\mathcal{L}(S) \geqslant \mu$. For any $x \in S$, there exist $y \in S_1$ and $z \in S_2$ such that $x = \beta^* + y + z$ and $\|x\| \leqslant \|\beta^*\| + (\mu\varepsilon_2)^{1/(2n+m)} + \frac{\mu^{1/(2n+m)}}{\varepsilon_2}$. Then, for any $k \geqslant t_2$, we have

$$\begin{aligned} &\|\theta_k+P_k^{1/2}x\|\leqslant\|\bar{\theta}+P_\infty^{1/2}\beta^*\|+\|\theta_k-\bar{\theta}\|+\|P_k^{1/2}x-P_\infty^{1/2}\beta^*\|\\ &\leqslant\|\theta^*\|+\varepsilon_2+\|P_t^{1/2}-P_\infty^{1/2}\|\|x\|+\|P_\infty^{1/2}(x-\beta^*)\|\leqslant\|\theta^*\|+\varepsilon_2+\|P_t^{1/2}-P_\infty^{1/2}\|\|x\|+\|P_\infty^{1/2}\|\|y\|\\ &\leqslant\|\theta^*\|+\varepsilon_2\left[1+\mu^{1/(2n+m)}+\varepsilon_2\|\beta^*\|+\varepsilon_2(\mu\varepsilon_2)^{\frac{1}{2n+m}}\right]+\|P_\infty^{1/2}\|(\mu\varepsilon_2)^{1/(2n+m)}<h_\theta. \end{aligned} \tag{A5}$$

Hence, similar to the analysis in Case 1, we have

$$S-\bigcup_{i=t_2}^{\infty}N_i\subseteq\bigcap_{i=t_2}^{k}A_i,\ \ \forall\, k\geqslant t_2.$$

Thus, $\mathcal{L}(\bigcap_{i=t_2}^k A_i) \geqslant \mu, \forall\, k \geqslant t_2$; hence, there exists a time $t_0 \leqslant t_2$ such that $t(k) = t_0$, i.e., $D_k = \bigcap_{i=t_0}^k A_i$ and $\mathcal{L}(D_k) \geqslant \mu, \forall k \geqslant t_2$.

Case 3: $P_\infty > 0$. From (1) of Lemma 1, we know that $\bar{\theta}_1$ exists such that $\|\bar{\theta}_1\| < h_\theta - \eta_1$ and $\|P_k^{-1/2}(\theta_k - \bar{\theta}_1)\| = O(1)$, where η_1 is a given number such that $\|P_\infty^{1/2}\|\mu^{1/(2n+m)} < \eta_1 < h_\theta$.

Similarly, there is a bounded convergent sequence $\{\beta_k^1\}$ such that

$$\bar{\theta}_1=\theta_k+P_k^{1/2}\beta_k^1,$$

and let $\beta^1 = \lim_{k\to\infty}\beta_k^1$.

Now, consider the set $B_{\beta^1}(\mu^{1/(2n+m)}) \triangleq \{x \in \mathcal{M}(n+m,n) | \|x-\beta^1\| \leqslant \mu^{1/(2n+m)}\}$ and take a small enough number ε_3 such that $\varepsilon_3(\varepsilon_3 + \mu^{1/(2n+m)} + \|P_\infty^{1/2}\|) < \eta_1 - \|P_\infty^{1/2}\|\mu^{1/(2n+m)}$. Then, for the given ε_3, there exists a time t_3 such that for all $k \geqslant t_3$,

$$\|\beta_k^1-\beta^1\|\leqslant\varepsilon_3,\quad\|P_k^{1/2}-P_\infty^{1/2}\|\leqslant\varepsilon_3. \tag{A6}$$

For any $x \in B_{\beta^1}(\mu^{1/(2n+m)})$ and $k \geqslant t_3$, we have

$$\begin{aligned} \|\theta_k+P_k^{1/2}x\|&\leqslant\|\bar{\theta}_1\|+\|P_k^{1/2}\|(\|x-\beta^1\|+\|\beta_k^1-\beta^1\|)\\ &\leqslant h_\theta-\eta_1+(\varepsilon_3+\|P_\infty^{1/2}\|)(\varepsilon_3+\mu^{1/(2n+m)})<h_\theta. \end{aligned} \tag{A7}$$

Hence, similar to the analysis in Case 1, we have

$$B_{\beta^1}\left(\mu^{1/(2n+m)}\right)-\bigcup_{i=t_3}^{\infty}N_i\subseteq\bigcap_{i=t_3}^{k}A_i,\ \ \forall\, k\geqslant t_3.$$

Thus, $\mathcal{L}(\bigcap_{i=t_1}^k A_i) \geqslant \mu, \forall\, k \geqslant t_3$; hence, there exists a time $t_0 \leqslant t_3$ such that $t(k) = t_0$, i.e., $D_k = \bigcap_{i=t_0}^k A_i$ and $\mathcal{L}(D_k) \geqslant \mu, \forall k \geqslant t_3$.

Finally, note that $D_k = \bigcap_{i=t_0}^k A_i$ for all $k \geqslant t_1$ and that $D_k \supseteq D_{k+1}$ and $\mathcal{L}(D_k) \geqslant \mu$; hence, D_∞ exists and $\mathcal{L}(D_\infty) \geqslant \mu$.

Appendix B The proof of Lemma 3

First, by the definition of η_k, it easy to know that $F_k(\eta_k) > 0$; hence,

$$F_k(\beta_k)\geqslant\frac{F_k(\eta_k)}{1+\gamma}>0. \tag{B1}$$

Thus, $\hat{\theta}_k \in \Theta$ for all $0 \leqslant k < \infty$.

Next, we prove that there exists a positive random variable $\delta_\infty > 0$ such that

$$\limsup_{k\to\infty} F_k(\eta_k) \geqslant \delta_\infty, \text{ a.s.} \tag{B2}$$

Note that for any adaptive input $\{u_k\}$, the random process $F_k(\cdot)$ is measurable with respect to the σ-algebra $\sigma\{\omega_i, \eta_{i-1}, i \leqslant k\} \triangleq \mathfrak{G}_{k-1}$. Let $I(\cdot)$ denote the indicator function of a set and $\mu_k(\cdot)$ denote the probability density measure of uniform distribution on D_k. Without loss of generality, we only consider the case of $k \geqslant t_1$. It is easy to find that $D_{k+1} \subseteq D_k$.

Let

$$\delta_k \triangleq \max_{x\in D_k} F_k(x),$$

$$E_k \triangleq \left\{ x \in D_k : F_k(x) \geqslant \frac{\delta_k}{2} \right\}.$$

Because $\theta_k, P_k^{1/2}$, and δ_k are $\mathfrak{G}_{k-1}$-measurable, and η_k is independent of $\mathfrak{G}_{k-1}$, we have

$$\begin{aligned} P\left(F_k(\eta_k) \geqslant \frac{\delta_k}{2} | \mathfrak{G}_{k-1}\right) &= \int_{x\in D_k} I\left(F_k(x) \geqslant \frac{\delta_k}{2}\right) \mu_k \mathrm{d}x \\ &= \mu_k(E_k) \geqslant \mu_{t_1}(E_k). \end{aligned} \tag{B3}$$

We now proceed to show that $\mu_{t_1}(E_k) \not\to 0$ a.s. as $k \to \infty$. Let $F(x) \triangleq \lim_{k\to\infty} F_k(x)$. From the proof of Lemma 2, we have that $\mu \leqslant \mathcal{L}(D_\infty) < \infty$, which implies that $\max_{x\in D_\infty} F(x) > 0$. Furthermore, it is easy to see that $F_k(x)$ converges to $F(x)$ uniformly on a bounded set. Consequently, $\delta_k \to \delta_\infty$ with $\delta_\infty = \max_{x\in D_\infty} F(x) > 0$, a.s.

Since $F(x)$ is a continuous function, we have $\mathcal{L}(E_\infty) > 0$ a.s., where E_∞ is defined by

$$E_\infty = \{x \in D_\infty : F(x) \geqslant \lambda\delta_\infty\}, \quad \frac{1}{2} < \lambda < 1.$$

Hence, it is easy to see from the convergence of $\{F_k(x), \delta_k\}$ to $\{F(x), \delta_\infty\}$ that for a sufficiently large k, $\mathcal{L}(E_k) \geqslant \mathcal{L}(E_\infty)$, which implies that $\mu_{t_1}(E_k) \not\to 0$ a.s. since $\mu_{t_1}(E_k) = \frac{\mathcal{L}(E_k)}{\mathcal{L}(D_{t_1})}$.

Hence, by (B3), we have that

$$\sum_{k=1}^{\infty} P\left(F_k(\eta_k) \geqslant \frac{\delta_k}{2} | \mathfrak{G}_{k-1}\right) = \infty, \text{ a.s.}$$

Consequently, by the Borel-Cantelli-Lévy Lemma, we have

$$\sum_{k=1}^{\infty} P\left(F_k(\eta_k) \geqslant \frac{\delta_k}{2}\right) = \infty, \text{ a.s.,}$$

which implies that

$$\limsup_{k\to\infty} F_k(\eta_k) \geqslant \frac{1}{2} \lim_{k\to\infty} \delta_k = \frac{\delta_\infty}{2} > 0, \text{ a.s.}$$

Hence, Eq. (B2) is proven.

Using the method introduced in [22], we can prove that there are positive variables δ_1 and k_m such that

$$F(\beta_k) \geqslant \delta_1, \text{ a.s., } \forall k \geqslant k_m, \tag{B4}$$

and that the limit $\lim_{k\to\infty} F(\beta_k) = F$ exists and $F > 0$ a.s.

To summarize, the proof of (2) of Lemma 3 is completed.

The proof for the other part of Lemma 3 is similar to the proof in [22].