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Linear-quadratic optimal control for partially observed forward-backward stochastic systems with random jumps

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Abstract In this paper, we investigate a linear-quadratic (LQ) optimal control problem for partially observed forward-backward stochastic systems with random jumps, where the observation's drift term is linear with respect to the state x and control variable v. In our model, the observation process is no longer a Brownian motion but a controlled stochastic process driven by Brownian motions and Poisson random measures, which also have correlated noises with the state equation. Applying a backward separation approach to decompose the state and observation, we overcome the problem of cyclic dependence of control and observation. Then, the necessary and sufficient conditions for optimal control are derived. We also obtain the feedback representation of optimal control and provide two special cases to illustrate the significance of our results. Moreover, we also provide a financial application to demonstrate the practical significance of our results.

Keywords LQ optimal control, partially observed stochastic system, random jumps, backward separation principle, optimal filtering

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1 Introduction

The partially observed optimal control problem for forward-backward stochastic differential equations with Poisson jump (FBSDEP) is considered in this study. Our main motivation for this study is an asset-liability management problem. Consider a company's liability process l_t^v governed by

$$-\mathrm{d}l_t^v = (b_t v_t - \bar{b}_t)\mathrm{d}t + c_t \mathrm{d}W_t + \bar{c}_t \mathrm{d}\bar{W}_t + \int_E f_{(t,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e),$$

where v is the decision maker's strategy that means to inject or withdraw funds to achieve a specific target. Let $\bar{b}_t > 0$ denote the expected liability rate. $c_t > 0$ and $\bar{c}_t > 0$ are the volatility of liability; $f_{(t,\cdot)}$ and $\bar{f}_{(t,\cdot)}$ represent the jump amplitude of liability. Suppose that the initial investment of this company is x_0 and only invests in a riskless bond with an interest rate of $r_t > 0$. The company's cash balance process x_t^v is

$$x_t^v = \mathrm{e}^{\int_0^t r_s \mathrm{d}s} \left(x_0 - \int_0^t \mathrm{e}^{-\int_0^s r_u \mathrm{d}u} \mathrm{d}l_t^v \right).$$

It follows from Itô's formula that

$$\begin{cases} \mathrm{d}x_t^v = (r_t x_t^v + b_t v_t - \bar{b}_t)\mathrm{d}t + c_t \mathrm{d}W_t + \bar{c}_t \mathrm{d}\bar{W}_t + \displaystyle\int_E f_{(t,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ x_0^v = x_0. \end{cases} \tag{1}$$

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Because of the company strategy and asymmetry account statement information, the decision maker can only observe the cash balance by the stock price,

$$\mathrm{d}S_t^v = S_t^v\bigg\{\Big(h_t x_t^v + \bar{g}_t + \frac{1}{2}\sigma_t^2 + \int_E \vartheta_{(t,e)}\lambda_2(\mathrm{d}e)\Big)\mathrm{d}t + \sigma_t \mathrm{d}\bar{W}_t + \int_E \kappa_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e)\bigg\}$$

with $S_0^v = 1$, where $\vartheta_{(t,e)} = \tilde{\kappa}_{(t,e)} - \kappa_{(t,e)} = \ln(1+\kappa_{(t,e)}) - \kappa_{(t,e)}$. Then the decision maker's available information is $\sigma\{S_s^v; 0 \leqslant s \leqslant t\}$ rather than $\mathcal{F}_t$ at time t. Setting $Y_t^v = \ln S_t^v$, we have

$$Y_t^v = Y_0^v + \int_0^t (h_s x_s^v + \bar{g}_s)\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}\bar{W}_s + \int_0^t\int_E \tilde{\kappa}_{(s,e)}\tilde{N}_2(\mathrm{d}s,\mathrm{d}e). \tag{2}$$

Obviously, $\mathcal{F}_t^{Y^v} = \sigma\{Y_s^v; 0 \leqslant s \leqslant t\} = \sigma\{S_s^v; 0 \leqslant s \leqslant t\}$. The generalized stochastic recursive utility problem under consideration is as follows.

Problem 1 (SRU)**.** Find an $\mathcal{F}^{Y^v}$-adapted control v to minimize

$$\bar{J}[v] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T Q_t(v_t - \bar{q}_t)^2\mathrm{d}t + R(x_T^v - \bar{r})^2 - 2\bar{s}y_0^v\bigg],$$

subject to (1), (2) and

$$\begin{aligned} y_t^v = x_T^v &+ \int_t^T \tilde{g}\Big(s, x_s^v, y_s^v, z_s^v, \bar{z}_s^v, \int_E k_{(s,e)}^v\lambda_1(\mathrm{d}e), \int_E \bar{k}_{(s,e)}^v\lambda_2(\mathrm{d}e), v_s\Big)\mathrm{d}s \\ &- \int_t^T z_s^v\mathrm{d}W_s - \int_t^T \bar{z}_s^v\mathrm{d}\bar{W}_s - \int_t^T\int_E k_{(s,e)}^v\tilde{N}_1(\mathrm{d}s,\mathrm{d}e) - \int_t^T\int_E \bar{k}_{(s,e)}^v\tilde{N}_2(\mathrm{d}s,\mathrm{d}e). \end{aligned} \tag{3}$$

The first term in the performance function calculates the difference between the control variable v and benchmark $\bar{q}$, the middle term estimates the risk of terminal wealth, and the last term denotes a stochastic recursive utility. On the basis of this example, we study the partially observed LQ problem as follows:

$$\begin{cases} \mathrm{d}x_t^v = \big\{a_t x_t^v + b_t v_t + \bar{b}_t\big\}\mathrm{d}t + c_t\mathrm{d}W_t + \bar{c}_t\mathrm{d}\bar{W}_t + \displaystyle\int_E f_{(t,e)}\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \bar{f}_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \mathrm{d}y_t^v = -\bigg\{A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t\bar{z}_t^v + \displaystyle\int_E D_t k_{(t,e)}^v\lambda_1(\mathrm{d}e) + \int_E \bar{D}_t\bar{k}_{(t,e)}^v\lambda_2(\mathrm{d}e) + F_t v_t + G_t\bigg\}\mathrm{d}t \\ \qquad + z_t^v\mathrm{d}W_t + \bar{z}_t^v\mathrm{d}\bar{W}_t + \displaystyle\int_E k_{(t,e)}^v\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_e \bar{k}_{(t,e)}^v\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ x_0^v = x_0, \quad y_T^v = Lx_T^v + M, \end{cases}$$

where $(W,\bar{W})$ is the standard Brownian motion, and $\tilde{N}_1$ and $\tilde{N}_2$ are the compensated martingale measures. Suppose that the state is partially observed through

$$Y_t^v = Y_0^v + \int_0^t \big\{h_s x_s^v + g_s v_s + \bar{g}_s\big\}\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}\bar{W}_s + \int_0^t\int_E \kappa_{(s,e)}\tilde{N}_2(\mathrm{d}s,\mathrm{d}e).$$

The problem is to find an $\mathcal{F}_t^{Y^v}$-adapted control v to minimize the cost functional,

$$\begin{aligned} J[v] = \frac{1}{2}\mathbb{E}\bigg[&\int_0^T \Big\{O_t(x_t^v)^2 + P_t(y_t^v)^2 + Q_t v_t^2 + 2o_t x_t^v + 2p_t y_t^v + 2q_t v_t\Big\}\mathrm{d}t \\ &+ R(x_T^v)^2 + 2r x_T^v + S(y_0^v)^2 + 2s y_0^v\bigg]. \end{aligned}$$

The optimal control of a partially observed stochastic system is usually encountered in finance studies, such as recursive utility or mean-variance problems. A substantial body of literature is available on partial information control systems, such as [1–3]. Li and Tang [4] obtained the general maximum principle using a purely probabilistic approach for a partially observed system, whose diffusion term contains control, and the observation also depends on control. Framstad et al. [5] obtained the sufficient condition for the optimal control of SDEP and studied the financial applications. Meng [6] obtained one sufficient condition and one necessary condition for partial information optimal control of a stochastic control system governed by fully coupled FBSDE with a convex control domain. Wang and Wu [7] obtained a maximum principle for the partially observed optimal control of FBSDE with nonconvex control domains and an uncontrolled diffusion term. Øksendal and Sulem [8] obtained a sufficient condition for the partial information optimal control of FBSDE with Lévy processes by using Malliavin calculus to handle control systems with random coefficients. Wang et al. [9] obtained the sufficient and necessary conditions for optimal control of FBSDEP and observation noises. Li et al. [10] studied the stabilization problem for discrete-time Markov jump linear systems involving multiplicative noise with an infinite horizon. Mu and Hu [11] studied the exponential stability analysis for semi-Markovian switched stochastic systems with asynchronously impulsive jumps. The above literature on partial information solved the problem using a change of probability measure; thus, the Girsanov transformation plays a vital role. The partially observed stochastic system also has important applications in engineering, such as the filtering problem in the field of wireless communication. In most practical problems, the observation process is just an ordinary stochastic process, which cannot be assumed to be a Brownian motion. However, if the observation is no longer a Brownian motion and depends on the control variable, then a circular dependency relationship is formed. In this case, the Wonham separation principle is critical in decoupling state estimation and optimal control (see [12,13]). In general, because the mean square error of state estimate relies on control, the separation principle is usually invalid when handling partially observed stochastic control problems. To address this flaw, Wang and Wu [14] proposed a backward separation technique that can be used to solve some partially observed stochastic control problems, such as the stochastic LQ problem. Xiao and Wang [15] studied the filtering equations of FBSDE with random jumps. Wang et al. [16] studied an LQ optimal control problem of FBSDE with partial information using the backward separation approach. Wang et al. [17] studied an optimal control problem derived using mean-field FBSDE with correlated noises, whose drift term depends on the state and its expectation. Li et al. [18] studied an LQ control problem for FBSDE with delay under full and partial information. They also proved the unique solvability of a class of FBSDEs with delay.

In this paper, we study an LQ optimal control problem for partially observed FBSDEP, whose observation is linear with respect to the state x and control variable v. Our model assumes that the observation Y^v is a stochastic process depending on the control variable v. This assumption makes the problem more natural and consistent with the actual situation, because the observation process is not necessarily continuous in practice. For example, the stock price process in the financial market is usually discontinuous because of the impact of emergencies such as macro policies or because the signal reception process in wireless communication is interrupted for some reason. Therefore, we assume that the observation equation is driven by Brownian motions and Poisson random measures, which also have correlated noises with the state equation. In the literature, little research is available on the stochastic control problem and filtering problem when the observation process is discontinuous. This research is one of the main contributions of this paper. On the basis of the above assumptions about the observation process, the Girsanov transformation is invalid. Therefore, we apply the backward separation approach to solve our problem. Compared with Wang et al. [16], we extend the backward separation approach from the continuous stochastic system to the discontinuous system. We obtain the necessary and sufficient conditions for the optimal control. Inspired by [19], we also provide the optimal filtering of the state equation and adjoint equation. We also have the feedback representation of optimal control. Finally, we provide a financial application to illustrate the practical significance of our results.

The remaining paper is organized as follows. In Section 2, we formulate the LQ optimal control problem of FBSDEP with partial information. In Section 3, we provide the necessary and sufficient conditions for optimal control. We also give its feedback representation. In Section 4, we give two cases to illustrate the method for solving the optimal control. In Section 5, we solve the financial problem raised at the beginning of this paper. In Section 6, we give the conclusion.

2 Problem formulation and preliminary

2.1 Notation

Given $T > 0$, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, \mathbb{P})$ be a given filtered complete probability space. On this space, there is a two-dimensional standard Brownian motion $(W,\bar W)$ valued in $\mathbb{R}^2$ with $W_0=\bar W_0=0$. And there are two independent $\mathcal{F}_t$-adapted Poisson random measures N_1 and N_2 on $[0,T]\times E$, where E is a standard measure space with a σ-field $\mathcal{E}$. We also assume that Brownian motions and Poisson random measures are mutually independent. For $i=1,2$, the mean measure of N_i is a measure on $([0,T]\times E,\mathcal{B}([0,T])\otimes\mathcal{E})$ which has the form Leb $\times\,\lambda_i$, where Leb denotes the Lebesgue measure on $[0,T]$ and λ_i is a finite measure on E, respectively. For any $D\in\mathcal{E}$ and $t\in[0,T]$, since $\lambda_i(D)<\infty$, we set $\tilde N_i(\omega,[0,t]\times D):=N_i(\omega,[0,t]\times D)-t\lambda_i(D)$. It is well known that $\tilde N_i(\omega,[0,t]\times D)$ is a martingale for every D. We assume that $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$ is generated by $W,\bar W,N_1,N_2$, which satisfies usual condition. Now we introduce some spaces of stochastic processes and random variables:

- $L^\infty(0,T;\mathbb{R}):=\{\psi|\psi$ is a deterministic uniformly-bounded function$\}$;
- $L^2_{\mathcal{F}_T}(\mathbb{R}):=\{\psi|\psi$ is $\mathcal{F}_T$-measurable and $\mathbb{E}[|\psi|^2]<\infty\}$;
- $L^2_{\mathcal{F}}(0,T;\mathbb{R}):=\{\Phi|\Phi_t$ is $\mathcal{F}_t$-adapted and $\mathbb{E}\int_0^T|\Phi_t|^2\mathrm{d}t<\infty\}$;
- $S^2_{\mathcal{F}}(0,T;\mathbb{R}):=\{\Phi|\Phi_t$ is $\mathcal{F}_t$-adapted and $\mathbb{E}[\sup_{0\leqslant t\leqslant T}|\Phi_t|^2]<\infty\}$;
- $M^2_{\mathcal{F}}(0,T;\mathbb{R}):=\{\Phi|\Phi_t$ is $\mathcal{F}_t$-predictable and $\mathbb{E}\int_0^T|\Phi_t|^2\mathrm{d}t<\infty\}$;
- $F^2_{\mathcal{F}}(0,T;\mathbb{R}):=\{\Phi|\Phi_t$ is $\mathcal{F}_t$-predictable and $\mathbb{E}\int_0^T\int_E|\Phi_t|^2\lambda(\mathrm{d}e)\mathrm{d}t<\infty\}$.

2.2 Problem formulation

Define the processes $(x^0,y^0,z^0,\bar z^0,k^0,\bar k^0)$ and Y^0 by

$$\begin{cases}\mathrm{d}x_t^0=a_tx_t^0\mathrm{d}t+c_t\mathrm{d}W_t+\bar c_t\mathrm{d}\bar W_t+\displaystyle\int_E f_{(t,e)}\tilde N_1(\mathrm{d}t,\mathrm{d}e)+\int_E\bar f_{(t,e)}\tilde N_2(\mathrm{d}t,\mathrm{d}e),\\ \mathrm{d}y_t^0=-\left\{A_tx_t^0+B_ty_t^0+C_tz_t^0+\bar C_t\bar z_t^0+\displaystyle\int_E D_tk^0_{(t,e)}\lambda_1(\mathrm{d}e)+\int_E\bar D_t\bar k^0_{(t,e)}\lambda_2(\mathrm{d}e)\right\}\mathrm{d}t\\ \qquad\quad+z_t^0\mathrm{d}W_t+\bar z_t^0\mathrm{d}\bar W_t+\displaystyle\int_E k^0_{(t,e)}\tilde N_1(\mathrm{d}t,\mathrm{d}e)+\int_E\bar k^0_{(t,e)}\tilde N_2(\mathrm{d}t,\mathrm{d}e),\\ x_0^0=x_0,\quad y_T^0=Lx_T^0,\end{cases}\tag{4}$$

and

$$\begin{cases}\mathrm{d}Y_t^0=h_tx_t^0\mathrm{d}t+\sigma_t\mathrm{d}\bar W_t+\displaystyle\int_E\kappa_{(t,e)}\tilde N_2(\mathrm{d}t,\mathrm{d}e),\\ Y_0^0=0.\end{cases}\tag{5}$$

Define $(x^1,y^1,z^1,\bar z^1,k^1,\bar k^1)$ and Y^1 with the control process $v\in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ by

$$\begin{cases}\dot x_t^1=a_tx_t^1+b_tv_t+\bar b_t,\\ \mathrm{d}y_t^1=-\Bigg\{A_tx_t^1+B_ty_t^1+C_tz_t^1+\bar C_t\bar z_t^1+\displaystyle\int_E D_tk^1_{(t,e)}\lambda_1(\mathrm{d}e)+\int_E\bar D_t\bar k^1_{(t,e)}\lambda_2(\mathrm{d}e)\\ \qquad\quad+F_tv_t+G_t\Bigg\}\mathrm{d}t+z_t^1\mathrm{d}W_t+\bar z_t^1\mathrm{d}\bar W_t+\displaystyle\int_E k^1_{(t,e)}\tilde N_1(\mathrm{d}t,\mathrm{d}e)+\int_E\bar k^1_{(t,e)}\tilde N_2(\mathrm{d}t,\mathrm{d}e),\\ x_0^1=0,\quad y_T^1=Lx_T^1+M,\end{cases}\tag{6}$$

and

$$\begin{cases}\dot Y_t^1=h_tx_t^1+g_tv_t+\bar g_t,\\ Y_0^1=0.\end{cases}\tag{7}$$

Assumption 1. The coefficients a_t, b_t, $\bar b_t$, c_t, $\bar c_t$, A_t, B_t, C_t, $\bar C_t$, D_t, $\bar D_t$, $f_{(t,\cdot)}$, $\bar f_{(t,\cdot)}$, h_t, g_t, $\bar g_t$, σ_t, $1/\sigma_t$, $\kappa_{(t,\cdot)}$, $1/\kappa_{(t,\cdot)}$ belong to $L^\infty(0,T;\mathbb{R})$. x_0 and L are constants, $M\in L^2_{\mathcal{F}_T}(\mathbb{R})$.

Obviously, Eqs. (4)–(7) admit unique solutions under Assumption 1, respectively (see [20,21]). Let

$$\begin{aligned}&x_t^v=x_t^0+x_t^1,\qquad y_t^v=y_t^0+y_t^1,\qquad z_t^v=z_t^0+z_t^1,\qquad \bar z_t^v=\bar z_t^0+\bar z_t^1,\\ &k^v_{(t,\cdot)}=k^0_{(t,\cdot)}+k^1_{(t,\cdot)},\qquad \bar k^v_{(t,\cdot)}=\bar k^0_{(t,\cdot)}+\bar k^1_{(t,\cdot)},\qquad Y_t^v=Y_t^0+Y_t^1.\end{aligned}\tag{8}$$

According to Itô's formula and (4)–(8), $(x^v, y^v, z^v, \bar{z}^v, k^v, \bar{k}^v)$ and Y^v are the unique solutions of

$$\begin{cases} \mathrm{d}x_t^v = \{a_t x_t^v + b_t v_t + \bar{b}_t\}\mathrm{d}t + c_t \mathrm{d}W_t + \bar{c}_t \mathrm{d}\bar{W}_t + \int_E f_{(t,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}y_t^v = -\Big\{ A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + \int_E D_t k_{(t,e)}^v \lambda_1(\mathrm{d}e) + \int_E \bar{D}_t \bar{k}_{(t,e)}^v \lambda_2(\mathrm{d}e) + F_t v_t \\ \qquad + G_t \Big\}\mathrm{d}t + z_t^v \mathrm{d}W_t + \bar{z}_t^v \mathrm{d}\bar{W}_t + \int_E k_{(t,e)}^v \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{k}_{(t,e)}^v \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ x_0^v = x_0, \quad y_T^v = L x_T^v + M, \end{cases} \tag{9}$$

and

$$\begin{cases} \mathrm{d}Y_t^v = \{h_t x_t^v + g_t v_t + \bar{g}_t\}\mathrm{d}t + \sigma_t \mathrm{d}\bar{W}_t + \int_E \kappa_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ Y_0^v = 0, \end{cases} \tag{10}$$

respectively. The superscript of every process emphasizes that they rely on the control variable v. $(x^v, y^v, z^v, \bar{z}^v, k^v, \bar{k}^v)$ and Y^v are the state and observation corresponding to the control v, respectively.

Set $\mathcal{F}_t^{Y^v} = \sigma\{Y_s^v; 0 \leqslant s \leqslant t\}$ and $\mathcal{F}_t^{Y^0} = \sigma\{Y_s^0; 0 \leqslant s \leqslant t\}$. Let U be a nonempty convex set of $\mathbb{R}$, and let $\mathcal{U}_{ad}^0$ be the set of all $\mathcal{F}_t^{Y^0}$-adapted processes with values in U such that $\mathbb{E}\sup_{0\leqslant t\leqslant T}|v_t|^2 < +\infty$.

Remark 1. In general, v is called admissible if $v \in L^2_{\mathcal{F}^{Y^v}}(0,T;\mathbb{R})$. It means that the control variable needs to be given through the observed results. But the circular dependence leads to the essential difficulty in finding the optimal control. This is the essential reason that Eqs. (9) and (10) are split into two parts. The Girsanov's transformation cannot be used to solve this problem in our model.

Definition 1. Let $\mathcal{U}_{ad}$ denote the set of admissible controls, which is in the form of $\mathcal{U}_{ad} = \{v | v \in \mathcal{U}_{ad}^0,\ v \text{ is } \mathcal{F}_t^{Y^v}\text{-adapted}\}$.

The cost function has the following form:

$$\begin{aligned} J[v] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\{ O_t (x_t^v)^2 + P_t (y_t^v)^2 + Q_t v_t^2 + 2o_t x_t^v + 2p_t y_t^v + 2q_t v_t \Big\}\mathrm{d}t \\ + R(x_T^v)^2 + 2r x_T^v + S(y_0^v)^2 + 2s y_0^v \bigg]. \end{aligned} \tag{11}$$

Assumption 2. The coefficients $O_t \geqslant 0$, $P_t \geqslant 0$, $Q_t \geqslant 0$, o_t, p_t and q_t belong to $L^\infty(0,T;\mathbb{R})$. $R \geqslant 0$, $S \geqslant 0$, r and s are constants.

Then we give the following LQ problem.

Problem 2 (LQC). Find an admissible control $u \in \mathcal{U}_{ad}$ that satisfies (9) and (10) such that $J[u] = \inf_{v\in\mathcal{U}_{ad}} J[v]$. u is called an optimal control if it makes the above equation hold. $(x^u, y^u, z^u, \bar{z}^u, k^u, \bar{k}^u)$ and $J[u]$ are the corresponding state and cost functional, respectively.

2.3 Preliminary result

Lemma 1. For any $v \in \mathcal{U}_{ad}$, $\mathcal{F}_t^{Y^v} = \mathcal{F}_t^{Y^0}$.

Proof. For any $v \in \mathcal{U}_{ad}$, v_t is $\mathcal{F}_t^{Y^0}$-adapted. Then we know that x_t^1 is $\mathcal{F}_t^{Y^0}$-adapted by (6), so Y_t^1 is $\mathcal{F}_t^{Y^0}$-adapted by (7). Because $Y_t^v = Y_t^0 + Y_t^1$, we know that Y_t^v is $\mathcal{F}_t^{Y^0}$-adapted. It implies $\mathcal{F}_t^{Y^v} \subseteq \mathcal{F}_t^{Y^0}$. Similar discussions show that $\mathcal{F}_t^{Y^0} \subseteq \mathcal{F}_t^{Y^v}$ via $Y_t^0 = Y_t^v - Y_t^1$. Then the proof is completed.

Then we give the L^2 estimates of (9) without proof (see [20–22] for details).

Lemma 2. Let Assumption 1 hold. For any $v_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$, let $(x^{v_i}, y^{v_i}, z^{v_i}, \bar{z}^{v_i}, k^{v_i}, \bar{k}^{v_i})$ be the solution of (9) corresponding to $v_i, i = 1,2$. Then there exists a positive constant $\bar{C}$ such that

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \mathbb{E}|x_t^{v_1} - x_t^{v_2}|^2 \leqslant \bar{C}\mathbb{E}\bigg[\int_0^T |v_{1,t} - v_{2,t}|^2 \mathrm{d}t \bigg], \\ &\sup_{0\leqslant t\leqslant T} \mathbb{E}|y_t^{v_1} - y_t^{v_2}|^2 \leqslant \bar{C}\bigg[\sup_{0\leqslant t\leqslant T} \mathbb{E}|x_t^{v_1} - x_t^{v_2}|^2 + \mathbb{E}\int_0^T |v_{1,t} - v_{2,t}|^2 \mathrm{d}t \bigg]. \end{aligned}$$

Similar to Lemma 2.3 in Wang et al. [16], we have the main lemma without proof as follows.

Lemma 3. If Assumptions 1 and 2 hold, we have

$$\inf_{\bar{v}\in\mathcal{U}_{ad}} J[\bar{v}] = \inf_{v\in\mathcal{U}_{ad}^0} J[v].$$

Remark 2. Lemma 3 plays a key role in solving Problem (LQC). This lemma implies that we can find an optimal control $v \in \mathcal{U}_{ad}^0$ instead of the optimal control $v \in \mathcal{U}_{ad}$ to minimize J.

3 Optimal solution of LQ problem

3.1 Optimality condition

We establish the necessary and sufficient conditions for optimal control of Problem (LQC).

Theorem 1. Let Assumptions 1 and 2 hold. Suppose that u is an optimal control of Problem (LQC) and $(x, y, z, \bar{z}, k, \bar{k})$ is the corresponding optimal state. Then the FBSDEP

$$\begin{cases} \mathrm{d}\varphi_t = \big(B_t\varphi_t - P_t y_t - p_t\big)\mathrm{d}t + C_t\varphi_t\mathrm{d}W_t + \bar{C}_t\varphi_t\mathrm{d}\bar{W}_t + \int_E \varphi_t D_t\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \varphi_t\bar{D}_t\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \mathrm{d}\xi_t = -\Big\{a_t\xi_t + O_t x_t + o_t - A_t\varphi_t\Big\}\mathrm{d}t + \eta_t\mathrm{d}W_t + \bar{\eta}_t\mathrm{d}\bar{W}_t + \int_E \vartheta_{(t,e)}\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \bar{\vartheta}_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \varphi_0 = -Sy_0 - s, \quad \xi_T = -L\varphi_T + Rx_T + r, \end{cases} \tag{12}$$

admits a unique solution $(\varphi, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta}) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^2) \times M^2_{\mathcal{F}}(0,T;\mathbb{R}^2) \times F^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$ such that

$$Q_t u_t - F_t\mathbb{E}[\varphi_t|\mathcal{F}_t^Y] + b_t\mathbb{E}[\xi_t|\mathcal{F}_t^Y] + q_t = 0 \tag{13}$$

with $\mathcal{F}_t^Y = \sigma\{Y_s^u; 0 \leqslant s \leqslant t\}$.

Proof. If u is the optimal control of Problem (LQC), it can be obtained by Lemma 3,

$$J[u] = \inf_{v\in\mathcal{U}_{ad}^0} J[v].$$

For any $v \in \mathcal{U}_{ad}$, let $(x^{u+\epsilon v}, y^{u+\epsilon v}, z^{u+\epsilon v}, \bar{z}^{u+\epsilon v}, k^{u+\epsilon v}, \bar{k}^{u+\epsilon v}) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^2)\times M^2_{\mathcal{F}}(0,T;\mathbb{R}^2)\times F^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$ be the solution of (9) corresponding to $u + \epsilon v, 0 < \epsilon < 1$. Then we introduce a variational equation:

$$\begin{cases} \dot{x}_t^1 = a_t x_t^1 + b_t v_t, \\ \mathrm{d}y_t^1 = -\Big\{A_t x_t^1 + B_t y_t^1 + C_t z_t^1 + \bar{C}_t\bar{z}_t^1 + \int_E D_t k^1_{(t,e)}\lambda_1(\mathrm{d}e) + \int_E \bar{D}_t\bar{k}^1_{(t,e)}\lambda_2(\mathrm{d}e) \\ \qquad\qquad + F_t v_t\Big\}\mathrm{d}t + z_t^1\mathrm{d}W_t + \bar{z}_t^1\mathrm{d}\bar{W}_t + \int_E k^1_{(t,e)}\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \bar{k}^1_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ x_0^1 = 0, \quad y_T^1 = Lx_T^1, \end{cases}$$

which admits a unique solution $(x^1, y^1, z^1, \bar{z}^1, k^1, \bar{k}^1) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^2) \times M^2_{\mathcal{F}}(0,T;\mathbb{R}^2) \times F^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$. By a similar argument like Lemma 1 in [23], we have

$$\lim_{\epsilon\to 0}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\bigg|\frac{x_t^{u+\epsilon v} - x_t^u}{\epsilon} - x_t^1\bigg|^2\bigg] = 0 \qquad \text{and} \qquad \lim_{\epsilon\to 0}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\bigg|\frac{y_t^{u+\epsilon v} - y_t^u}{\epsilon} - y_t^1\bigg|^2\bigg] = 0.$$

Next we give the variational equation of the cost functional,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}J[u+\epsilon v]|_{\epsilon=0} \\ &= \mathbb{E}\bigg[\int_0^T \Big\{(O_t x_t + o_t)x_t^1 + (P_t y_t + p_t)y_t^1 + (Q_t u_t + q_t)v_t\Big\}\mathrm{d}t + (Rx_T + r)x_T^1 + (Sy_0 + s)y_0^1\bigg]. \end{aligned} \tag{14}$$

In addition, as long as $(x, y, z, \bar{z}, k, \bar{k})$ is determined by (9), we can know that $(\varphi, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta}) \in S^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$ $\times M^2_{\mathcal{F}}(0,T;\mathbb{R}^2) \times F^2_{\mathcal{F}}(0,T;\mathbb{R}^2)$. Under Assumptions 1 and 2, we have the following equation by applying Itô's formula to $\xi_t x_t^1 + \varphi_t y_t^1$:

$$0 = \mathbb{E}\bigg[\int_0^T \big(Q_t u_t + q_t + b_t\mathbb{E}[\xi_t|\mathcal{F}_t^{Y^0}] - F_t\mathbb{E}[\varphi_t|\mathcal{F}_t^{Y^0}]\big)v_t\mathrm{d}t\bigg].$$

Hence,

$$Q_t u_t + q_t + b_t \mathbb{E}[\xi_t|\mathcal{F}_t^{Y^0}] - F_t \mathbb{E}[\varphi_t|\mathcal{F}_t^{Y^0}] = 0.$$

Because of $u \in \mathcal{U}_{ad}$, we can know that $\mathcal{F}_t^{Y^0} = \mathcal{F}_t^Y$ by Lemma 1. Thus we have desired results.

Then we give the sufficient condition as follows.

Theorem 2. Let Assumptions 1 and 2 hold. Assume that $u \in \mathcal{U}_{ad}$ satisfies

$$Q_t u_t - F_t \mathbb{E}[\varphi_t|\mathcal{F}_t^Y] + b_t \mathbb{E}[\xi_t|\mathcal{F}_t^Y] + q_t = 0,$$

where $(\varphi, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta})$ is a solution to (12). Then u is an optimal control of Problem (LQC).

Proof. For any admissible control v, we have

$$J[v] - J[u] := J_1 + J_2, \tag{15}$$

where

$$J_1 = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\{O_t(x_t^v - x_t)^2 + P_t(y_t^v - y_t)^2 + Q_t(v_t - u_t)^2\Big\}\mathrm{d}t + R(x_T^v - x_T)^2 + S(y_0^v - y_0)^2\bigg],$$

and

$$\begin{aligned} J_2 = \mathbb{E}\bigg[&\int_0^T \Big\{(O_t x_t + o_t)(x_t^v - x_t) + (P_t y_t + p_t)(y_t^v - y_t) + (Q_t u_t + q_t)(v_t - u_t)\Big\}\mathrm{d}t \\ &+ (R x_T + r)(x_T^v - x_T) + (S y_0 + s)(y_0^v - y_0)\bigg]. \end{aligned}$$

Obviously, $J_1 \geqslant 0$ for any admissible control v. Then it is enough to prove that $J_2 = 0$.

Applying Itô's formula to $\varphi_t(y_t^v - y_t) + \xi_t(x_t^v - x_t)$ and taking expectations on both sides, we have

$$J_2 = \mathbb{E}\bigg[\int_0^T \Big(Q_t u_t - F_t \mathbb{E}[\varphi_t|\mathcal{F}_t^{Y^0}] + b_t \mathbb{E}[\xi_t|\mathcal{F}_t^{Y^0}] + q_t\Big)(v_t - u_t)\mathrm{d}t\bigg] = 0.$$

Assumption 3. $Q_t > 0$ and $1/Q_t$ are uniformly bounded, deterministic functions.

Similar to Wang et al. [16], we give the following corollary without proof.

Corollary 1. Let Assumptions 1–3 hold. If u is an optimal control of Problem (LQC), then u is unique.

3.2 Filtering

Let Assumptions 1–3 hold. Optimal condition (13) can be written as

$$u_t = \frac{1}{Q_t}\big(F_t \mathbb{E}[\varphi_t|\mathcal{F}_t^Y] - b_t \mathbb{E}[\xi_t|\mathcal{F}_t^Y] - q_t\big).$$

This demonstrates the importance of calculating the optimal filtering of (φ_t, ξ_t) depending on $\mathcal{F}_t^Y$ in order to compute u. Since (φ_t, ξ_t) is related to (x, y), we first need to calculate the optimal filtering for FBSDE (9) and (12). For any $v \in \mathcal{U}_{ad}$, let $\hat{\varsigma}_t = \mathbb{E}[\varsigma_t|\mathcal{F}_t^{Y^v}]$ with $\varsigma = x^0, x^v, y^v, z^v, \bar{z}^v, k^v, \bar{k}^v, \varphi, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta}$, $M, x^v y^v$ and $\gamma_t = \mathbb{E}[(x_t^v - \hat{x}_t^v)^2|\mathcal{F}_t^{Y^v}]$, $\widehat{(x_t^v)^i} = \mathbb{E}[(x_t^v)^i|\mathcal{F}_t^{Y^v}]$, $i = 1, 2, 3, \ldots$. Now we state the optimal filtering of (9), which is critical in representing the optimal control.

Lemma 4. Let Assumption 1 hold. For any $v \in \mathcal{U}_{ad}$, the optimal filtering $(\hat{x}^v, \hat{y}^v, \hat{z}^v, \hat{\bar{z}}^v, \hat{k}^v, \hat{\bar{k}}^v)$ of $(x^v, y^v, z^v, \bar{z}^v, k^v, \bar{k}^v)$ satisfying (9) with respect to $\mathcal{F}_t^{Y^v}$ satisfies

$$\begin{cases} \mathrm{d}\hat{x}_t^v = (a_t \hat{x}_t^v + b_t v_t + \bar{b}_t)\mathrm{d}t + \Big(\bar{c}_t + \dfrac{h_t \gamma_t}{\sigma_t}\Big)\mathrm{d}\tilde{W}_t + \displaystyle\int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}\hat{y}_t^v = -\Big\{A_t \hat{x}_t^v + B_t \hat{y}_t^v + C_t \hat{z}_t^v + \bar{C}_t \hat{\bar{z}}_t^v + F_t v_t + G_t + \displaystyle\int_E D_t \hat{k}_{(t,e)}^v \lambda_1(\mathrm{d}e) \\ \qquad\quad + \displaystyle\int_E \bar{D}_t \hat{\bar{k}}_{(t,e)}^v \lambda_2(\mathrm{d}e)\Big\}\mathrm{d}t + \hat{Z}_t^v \mathrm{d}\tilde{W}_t + \displaystyle\int_E \hat{\bar{k}}_{(t,e)}^v \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \hat{x}_0^v = x_0, \quad \hat{y}_T^v = L\hat{x}_T^v + \hat{M}, \end{cases} \tag{16}$$

where the conditional mean square error γ_t satisfies the following equation:

$$\begin{cases} \gamma_t = \left\{ 2a_t\gamma_t + c_t^2 - \dfrac{2\bar{c}_t h_t \gamma_t}{\sigma_t} - \dfrac{h_t^2\gamma_t^2}{\sigma_t^2} + \displaystyle\int_E f^2(t,e)\lambda_1(\mathrm{d}e) \right\}\mathrm{d}t + \dfrac{h_t}{\sigma_t}\Big(\widehat{x_t^3}^v - \widehat{x_t^2}^v \hat{x}_t^v - 2\gamma_t \hat{x}_t^v\Big)\mathrm{d}\tilde{W}_t, \\ \gamma_0 = 0, \end{cases} \tag{17}$$

the innovation processes for $\bar{W}_t$ can be calculated as

$$\tilde{W}_t = \int_0^t \frac{h_t}{\sigma_t}(x_s^v - \hat{x}_s^v)\mathrm{d}s + \bar{W}_t, \tag{18}$$

and

$$\hat{Z}_t^v = \hat{\bar{z}}_t^v + \frac{h_t}{\sigma_t}\Big(\widehat{x_t^v y_t^v} - \hat{x}_t^v \hat{y}_t^v\Big). \tag{19}$$

Proof. Since x_t^1 is $\mathcal{F}_t^{Y^0}$-adapted, by Lemma 1 we have

$$\hat{x}_t^v = \mathbb{E}[x_t^v|\mathcal{F}_t^{Y^v}] = \mathbb{E}[x_t^0|\mathcal{F}_t^{Y^0}] + x_t^1 = \hat{x}_t^0 + x_t^1.$$

Applying Theorem 214 in [19] to (5) and SDE in (4), then Eqs. (17), (18) and SDE in (16) are derived. Inspired by Wang et al. [24] and Wang et al. [16], for any $v \in \mathcal{U}_{ad}$, the BSDEP in (9) admits a unique solution $(y^v, z^v, \bar{z}^v, k^v, \bar{k}^v)$. Then we have

$$\begin{aligned} y_t^v = y_0^v - \int_0^t \Big\{ & A_s x_s^v + B_s y_s^v + C_s z_s^v + \bar{C}_s \bar{z}_s^v + \int_E D_s k_{(s,e)}^v \lambda_1(\mathrm{d}e) + \int_E \bar{D}_s \bar{k}_{(s,e)}^v \lambda_2(\mathrm{d}e) + F_s v_s \\ & + G_s \Big\}\mathrm{d}s + \int_0^t z_s^v \mathrm{d}W_s + \int_0^t \bar{z}_s^v \mathrm{d}\bar{W}_s + \int_0^t \int_E k_{(s,e)}^v \tilde{N}_1(\mathrm{d}s,\mathrm{d}e) + \int_0^t \int_e \bar{k}_{(s,e)}^v \tilde{N}_2(\mathrm{d}s,\mathrm{d}e), \end{aligned} \tag{20}$$

and hence, the integral form of the BSDEP in (9) can be written as $y_t^v - Y_T^v$. Let (20) and (10) be the state and observation, respectively. From Theorem 214 in [19], we get

$$\begin{aligned} \hat{y}_t^v = y_0^v - \int_0^t \Bigg\{ & A_s \hat{x}_s^v + B_s \hat{y}_s^v + C_s \hat{z}_s^v + \bar{C}_s \hat{\bar{z}}_s^v + \int_E D_s \hat{k}_{(s,e)}^v \lambda_1(\mathrm{d}e) + \int_E \bar{D}_s \hat{\bar{k}}_{(s,e)}^v \lambda_2(\mathrm{d}e) \\ & + F_s v_s + G_s \Bigg\}\mathrm{d}s + \int_0^t \hat{Z}_s^v \mathrm{d}\tilde{W}_s + \int_0^t \int_E \hat{\bar{k}}_{(s,e)}^v \tilde{N}_2(\mathrm{d}s,\mathrm{d}e), \end{aligned}$$

with $y_0^v = \hat{y}_0^v$. Then it yields

$$\begin{aligned} \hat{y}_t^v = \hat{y}_T^v + \int_t^T \Bigg\{ & A_s \hat{x}_s^v + B_s \hat{y}_s^v + C_s \hat{z}_s^v + \bar{C}_s \hat{\bar{z}}_s^v + \int_E D_s \hat{k}_{(s,e)}^v \lambda_1(\mathrm{d}e) + \int_E \bar{D}_s \hat{\bar{k}}_{(s,e)}^v \lambda_2(\mathrm{d}e) \\ & + F_s v_s + G_s \Bigg\}\mathrm{d}s - \int_t^T \hat{Z}_s^v \mathrm{d}\tilde{W}_s - \int_t^T \int_E \hat{\bar{k}}_{(s,e)}^v \tilde{N}_2(\mathrm{d}s,\mathrm{d}e). \end{aligned}$$

Moreover

$$\hat{y}_T^v = \mathbb{E}[L x_T^v + M|\mathcal{F}_T^{Y^v}] = L\hat{x}_T^v + \hat{M},$$

where $\hat{x}_T^v$ is determined by (16).

As we all know, $(\hat{Z}^v, \hat{\bar{k}}^v)$ is a part of solution $(\hat{y}^v, \hat{Z}^v, \hat{\bar{k}}^v)$ to BSDEP (16), which can be computed by the Malliavin calculus. See Geiss and Steinicke [25] for details. In order to give the feedback representation of optimal control u, we need to calculate the filtering result of the adjoint equation.

Lemma 5. Let Assumptions 1 and 2 and $P_t = 0$ hold. The optimal filtering of $(\varphi, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta})$ depending on $\mathcal{F}_t^Y$ satisfies

$$\begin{cases} \mathrm{d}\hat{\varphi}_t = (B_t\hat{\varphi}_t - p_t)\mathrm{d}t + \Big\{ \bar{C}_t\hat{\varphi}_t + \dfrac{h_t}{\sigma_t}\big(\widehat{x_t\varphi_t} - \hat{x}_t\hat{\varphi}_t\big) \Big\}\mathrm{d}\tilde{W}_t + \displaystyle\int_E \bar{D}_t\hat{\varphi}_t \tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ -\mathrm{d}\hat{\xi}_t = \big(a_t\hat{\xi}_t + O_t\hat{x}_t + o_t - A_t\hat{\varphi}_t\big)\mathrm{d}t - \hat{\zeta}_t\mathrm{d}\tilde{W}_t - \displaystyle\int_E \hat{\bar{\vartheta}}_{(s,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \hat{\varphi}_0 = -Sy_0 - n, \quad \hat{\xi}_T = -L\hat{\varphi}_T + R\hat{x}_T + r, \end{cases} \tag{21}$$

with

$$\hat{\zeta}_t = \hat{\bar{\eta}}_t + \frac{h_t}{\sigma_t}\Big(\widehat{x_t\xi_t} - \hat{x}_t\hat{\xi}_t\Big),$$

where $(\hat{x}, \hat{y})$ satisfies (16) with $v = u$, and $\tilde{W}$ is the corresponding innovation process.

Theorem 3. Let Assumptions 1–3 and $P_t = 0$ hold. If $u_t = \frac{1}{Q_t}\big(F_t\hat{\varphi}_t - b_t\hat{\xi}_t - q_t\big)$ is the optimal control, then it can be expressed as

$$u_t = \frac{1}{Q_t}\Big[(F_t - b_t\Sigma_t)\hat{\varphi}_t - b_t\Pi_t\hat{x}_t - b_t\varrho_t - q_t\Big],$$

where $(\hat{x}, \hat{y}, \hat{Z}, \hat{\bar{k}})$, $(\hat{\varphi}, \hat{\xi}, \hat{\zeta}, \hat{\bar{\vartheta}})$, Π, Σ and ϱ are the solutions of (16) with $v = u$, (21), (25)–(27), respectively.

Proof. According to the form of terminal condition (12), we set

$$\xi_t = \Pi_t x_t + \Sigma_t\varphi_t + \varrho_t \tag{22}$$

with $\Pi_T = R$, $\Sigma_T = -L$ and $\varrho_T = r$, where Π, Σ and ϱ are deterministic differentiable functions. Applying Itô's formula to (22), we have

$$\begin{aligned} \mathrm{d}\xi_t = &\Big\{\dot{\Pi}_t x_t + \Pi_t\Big(a_t x_t + \frac{1}{Q_t}b_t\big((F_t - b_t\Sigma_t)\hat{\varphi}_t - b_t\Pi_t\hat{x}_t - b_t\varrho_t - q_t\big) + \bar{b}_t\Big) + \dot{\Sigma}_t\varphi_t \\ &+ \dot{\varrho}_t + \Sigma_t(B_t\varphi_t - p_t)\Big\}\mathrm{d}t + \big(\Pi_t c_t + \Sigma_t C_t\varphi_t\big)\mathrm{d}W_t + \big(\Pi_t\bar{c}_t + \Sigma_t\bar{C}_t\varphi_t\big)\mathrm{d}\bar{W}_t \\ &+ \int_E \big(\Pi_t f_{(s,e)} + \Sigma_t D_t\varphi_t\big)\tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \big(\Pi_t\bar{f}_{(s,e)} + \Sigma_t\bar{D}_t\varphi_t\big)\tilde{N}_2(\mathrm{d}t, \mathrm{d}e). \end{aligned} \tag{23}$$

Comparing the above equality with (12), it yields

$$\begin{aligned} &\eta_t = \Pi_t c_t + \Sigma_t C_t\varphi_t, \quad \bar{\eta}_t = \Pi_t\bar{c}_t + \Sigma_t\bar{C}_t\varphi_t, \\ &\vartheta_{(s,e)} = \Pi_t f_{(s,e)} + \Sigma_t D_t\varphi_t, \quad \bar{\vartheta}_{(s,e)} = \Pi_t\bar{f}_{(s,e)} + \Sigma_t\bar{D}_t\varphi_t. \end{aligned}$$

Taking $\mathbb{E}[\cdot|\mathcal{F}_t^Y]$ on the drift term of (23) and comparing with the drift term of (21), we obtain

$$\begin{aligned} &\dot{\Pi}_t\hat{x}_t + \Pi_t\Big(a_t\hat{x}_t + \frac{1}{Q_t}b_t\big((F_t - b_t\Sigma_t)\hat{\varphi}_t - b_t\Pi_t\hat{x}_t - b_t\varrho_t - q_t\big) + \bar{b}_t\Big) + \dot{\Sigma}_t\hat{\varphi}_t + \dot{\varrho}_t + \Sigma_t(B_t\hat{\varphi}_t - p_t) \\ &\quad = -(a_t\Pi_t + O_t)\hat{x}_t - (a_t\Sigma_t - A_t)\hat{\varphi}_t - a_t\varrho_t - o_t. \end{aligned} \tag{24}$$

Comparing the coefficients of $\hat{x}_t$ and $\hat{\varphi}_t$ in (24), we have

$$\begin{cases} \dot{\Pi}_t + 2a_t\Pi_t - \dfrac{1}{Q_t}b_t^2\Pi_t^2 + O_t = 0, \\ \Pi_T = R, \end{cases} \tag{25}$$

$$\begin{cases} \dot{\Sigma}_t + \Big(a_t + B_t - \dfrac{1}{Q_t}b_t^2\Pi_t\Big)\Sigma_t + \dfrac{1}{Q_t}b_tF_t\Pi_t - A_t = 0, \\ \Sigma_T = -L, \end{cases} \tag{26}$$

and

$$\begin{cases} \dot{\varrho}_t + \Big(a_t - \dfrac{1}{Q_t}b_t^2\Pi_t\Big)\varrho_t - \Sigma_t p_t - \dfrac{1}{Q_t}b_t q_t\Pi_t + \Pi_t\bar{b}_t + o_t = 0, \\ \varrho_T = r. \end{cases} \tag{27}$$

It is easy to see that there are unique solutions to the above three ordinary differential equations (ODEs). Thus we have

$$u_t = \frac{1}{Q_t}\Big[(F_t - b_t\Sigma_t)\hat{\varphi}_t - b_t\Pi_t\hat{x}_t - b_t\varrho_t - q_t\Big].$$

4 Two special cases of problem LQC

4.1 LQ optimal control for SDEP

We consider a partially observed stochastic system with jumps and verify the correctness of Theorem 3 by another method.

$$\begin{cases} \mathrm{d}x_t^v = (a_t x_t^v + b_t v_t + \bar{b}_t)\mathrm{d}t + c_t \mathrm{d}W_t + \bar{c}_t \mathrm{d}\bar{W}_t + \int_E f_{(s,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(s,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}Y_t^v = (h_t x_t^v + g_t v_t + \bar{g}_t)\mathrm{d}t + \sigma_t \mathrm{d}\bar{W}_t + \int_E \kappa_{(s,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ x_0^v = x_0, \quad Y_0^v = 0, \end{cases} \tag{28}$$

where Y^v is the observed process. The cost functional is as follows:

$$J[v] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big(O_t (x_t^v)^2 + Q_t v_t^2\Big)\mathrm{d}t + R(x_T^v)^2\bigg]. \tag{29}$$

Our goal is to find a $u \in \mathcal{U}_{ad}$ such that $J[u] = \inf_{v \in \mathcal{U}_{ad}} J[v]$ subject to (28) and (29). Then we use the backward separation techniques to solve this problem and we give the feedback representation of optimal control. We shall solve this problem in three steps.

Step 1. According to Theorem 1, we know that the optimal control needs to satisfy $u_t = -\frac{b_t}{Q_t}\hat{\xi}_t$, with ξ satisfing the following FBSDEP:

$$\begin{cases} \mathrm{d}x_t = \Big(a_t x_t - \frac{b_t^2}{Q_t}\hat{\xi}_t + \bar{b}_t\Big)\mathrm{d}t + c_t \mathrm{d}W_t + \bar{c}_t \mathrm{d}\bar{W}_t + \int_E f_{(t,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}\xi_t = -(a_t \xi_t + O_t x_t)\mathrm{d}t + \eta_t \mathrm{d}W_t + \bar{\eta}_t \mathrm{d}\bar{W}_t + \int_E \vartheta_{(t,e)} \tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{\vartheta}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \bar{x}_0 = x_0, \quad \bar{\xi}_T = R\bar{x}_T. \end{cases} \tag{30}$$

Step 2. Since $\hat{\xi}_t$ is $\mathcal{F}_t^Y$-adapted, similar to Lemma 4, we derive that

$$\begin{cases} \mathrm{d}\hat{x}_t = \Big(a_t \hat{x}_t - \frac{b_t^2}{Q_t}\hat{\xi}_t + \bar{b}_t\Big)\mathrm{d}t + \Big(\bar{c}_t + \frac{h_t \gamma_t}{\sigma_t}\Big)\mathrm{d}\tilde{W}_t + \int_E \bar{f}_{(t,e)} \tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \hat{x}_0 = x_0. \end{cases} \tag{31}$$

Solving the BSDEP in (30) and taking $\mathbb{E}[\cdot|\mathcal{F}_t^Y]$ on both side, we have

$$\hat{\xi}_t = R\mathrm{e}^{\int_t^T a_s \mathrm{d}s}\mathbb{E}[x_T|\mathcal{F}_t^Y] + \int_t^T \mathrm{e}^{\int_t^s a_r \mathrm{d}r} O_s \mathbb{E}[x_s|\mathcal{F}_t^Y]\mathrm{d}s. \tag{32}$$

We now claim that $\hat{\xi}_t = \Pi_t \hat{x}_t + \varrho_t$, and set $\alpha_t = \frac{b_t^2}{Q_t}\Pi_t$, $\beta_t = \bar{b}_t - \frac{b_t^2}{Q_t}\varrho_t$. Let $\Phi(s,t)$ be the fundamental solution of

$$\frac{\mathrm{d}\Phi(s,t)}{\mathrm{d}s} = \begin{pmatrix} a_t - \alpha_t & 0 \\ -\alpha_t & a_t \end{pmatrix} \Phi(s,t). \tag{33}$$

From (30) and (31), we get

$$\begin{aligned} \begin{pmatrix} \hat{x}_s \\ x_s \end{pmatrix} = {} & \Phi(s,t)\begin{pmatrix} \hat{x}_t \\ x_t \end{pmatrix} + \int_t^s \Phi(s,r)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\beta_r \mathrm{d}r + \int_t^s \Phi(s,r)\begin{pmatrix} \bar{c}_r + \frac{h_r \gamma_r}{\sigma_r} & 0 & 0 \\ 0 & c_r & \bar{c}_r \end{pmatrix}\mathrm{d}\begin{pmatrix} \tilde{W}_r \\ W_r \\ \bar{W}_r \end{pmatrix} \\ & + \int_t^s \int_E \Phi(s,r)\begin{pmatrix} 0 & \bar{f}_{(r,e)} \\ f_{(r,e)} & \bar{f}_{(r,e)} \end{pmatrix}\begin{pmatrix} \tilde{N}_1(\mathrm{d}r, \mathrm{d}e) \\ \tilde{N}_2(\mathrm{d}r, \mathrm{d}e) \end{pmatrix}. \end{aligned} \tag{34}$$

Then we have

$$\begin{aligned}\mathbb{E}[x_s|\mathcal{F}_t^Y] &= \begin{pmatrix}0 & 1\end{pmatrix}\Phi(s,t)\begin{pmatrix}1\\1\end{pmatrix}\hat{x}_t + \int_t^s \begin{pmatrix}0 & 1\end{pmatrix}\Phi(s,r)\begin{pmatrix}1\\1\end{pmatrix}\beta_r \mathrm{d}r \\ &= \mathrm{e}^{\int_t^s (a_r-\alpha_r)\mathrm{d}r}\hat{x}_t + \int_t^s \mathrm{e}^{\int_r^s (a_\mu-\alpha_\mu)\mathrm{d}\mu}\beta_r \mathrm{d}r.\end{aligned} \tag{35}$$

Substituting (35) into (32), we have $\hat{\xi}_t = \bar{\Pi}_t\hat{x}_t + \bar{\varrho}_t$, with

$$\bar{\Pi}_t = R\mathrm{e}^{\int_t^T (2a_s-\alpha_s)\mathrm{d}s}\hat{x}_t + \int_t^T \mathrm{e}^{\int_r^T (a_s-\alpha_s)\mathrm{d}s\beta_r\mathrm{d}r}, \tag{36}$$

$$\bar{\varrho}_t = R\mathrm{e}^{\int_t^T a_s\mathrm{d}s}\int_t^T \mathrm{e}^{\int_s^T (a_r-\alpha_r)\mathrm{d}r}\beta_s\mathrm{d}s + \int_t^T \mathrm{e}^{\int_t^s a_r\mathrm{d}r}O_s\int_t^s \mathrm{e}^{\int_r^s (a_\mu-\alpha_\mu)\mathrm{d}\mu}\beta_r\mathrm{d}r\mathrm{d}s. \tag{37}$$

We know that Eqs. (25) and (27) are reduced to

$$\begin{cases}\dot{\Pi}_t + 2a_t\Pi_t - \dfrac{b_t^2}{Q_t}\Pi_t^2 + O_t = 0,\\ \Pi_T = R,\end{cases} \tag{38}$$

$$\begin{cases}\dot{\varrho}_t + a_t\varrho_t - \dfrac{b_t^2}{Q_t}\Pi_t\varrho_t + \Pi_t\bar{b}_t = 0,\\ \varrho_T = 0.\end{cases} \tag{39}$$

We know that Eqs. (36) and (37) are the solutions to (38) and (39), respectively. From the existence and uniqueness of solutions to (38) and (39), we know that $\Pi(\cdot) \equiv \bar{\Pi}(\cdot)$, $\varrho(\cdot) \equiv \bar{\varrho}(\cdot)$. Then the candidate optimal control u can be rewritten as

$$u_t = -\frac{b_t}{Q_t}\Pi_t\hat{x}_t - \frac{b_t}{Q_t}\varrho_t. \tag{40}$$

Step 3. We should prove that u is the unique optimal control under the additional assumption $h \equiv 0$. Since $\hat{x}_t^v \perp (x_t^v - \hat{x}_t^v)$, the cost functional (29) can be rewritten as

$$J[v] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big(O_t(\hat{x}_t^v)^2 + Q_tv_t^2\Big)\mathrm{d}t + R(\hat{x}_T^v)^2\bigg] + \frac{1}{2}\int_0^T O_t\gamma_t\mathrm{d}t + \frac{1}{2}R\gamma_T, \tag{41}$$

where γ satisfies (17). Noticing $h \equiv 0$, then γ is the solution of ODE, so γ is a deterministic function. Then for any $v \in \mathcal{U}_{ad}$,

$$J[v] - J[u] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big(O_t(\hat{x}_t^v - \hat{x}_t)^2 + Q_t(v_t - u_t)^2\Big)\mathrm{d}t + R(\hat{x}_T^v - \hat{x}_T)^2\bigg] + \Xi,$$

where

$$\Xi = \mathbb{E}\bigg[\int_0^T \Big(O_t\hat{x}_t(\hat{x}_t^v - \hat{x}_t) + Q_tu_t(v_t - u_t)\Big)\mathrm{d}t + R\hat{x}_T(\hat{x}_T^v - \hat{x}_T)\bigg]. \tag{42}$$

In order to show that u in (40) is indeed an optimal control, we need to prove $\Xi \geqslant 0$. Applying Itô's formula to $\hat{\xi}_t(\hat{x}_t^v - \hat{x}_t)$, we get

$$\Xi = \mathbb{E}\bigg[\int_0^T (Q_tu_t + b_t\hat{\xi}_t)(v_t - u_t)\mathrm{d}t\bigg] = 0.$$

Next, let us calculate the value of $J[u]$. Substituting (40) to (41), we get

$$J[u] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \bigg(O_t + \frac{b_t^2}{Q_t}\Pi_t^2\bigg)\hat{x}_t^2 + \frac{b_t^2}{Q_t}\varrho_t^2 + 2\frac{b_t^2}{Q_t}\Pi_t\varrho_t\hat{x}_t\mathrm{d}t + R\hat{x}_T^2\bigg] + \frac{1}{2}\int_0^T O_t\gamma_t\mathrm{d}t + \frac{1}{2}R\gamma_T. \tag{43}$$

Applying Itô's formula to $\frac{1}{2}\Pi_t\hat{x}_t^2 + \varrho_t\hat{x}_t$, we have

$$\begin{aligned}\mathbb{E}\bigg[\frac{1}{2}\Pi_T\hat{x}_T^2 + \varrho_T\hat{x}_T\bigg] = \mathbb{E}\bigg[\frac{1}{2}\Pi_0\hat{x}_0^2 + \varrho_0\hat{x}_0 + \int_0^T \bigg\{ -\frac{1}{2}\bigg(O_t + \frac{b_t^2}{Q_t}\Pi_t^2\bigg)\hat{x}_t^2 - \frac{b_t^2}{Q_t}\Pi_t\hat{x}_t\varrho_t \\ + \frac{1}{2}\Pi_t c_t^2 + \bigg(\bar{b}_t - \frac{b_t^2}{Q_t}\varrho_t\bigg)\varrho_t + \frac{1}{2}\int_E \Pi_t\bar{f}_{(t,e)}^2\lambda_2(\mathrm{d}e)\bigg\}\mathrm{d}t\bigg].\end{aligned} \tag{44}$$

Combining (43) and (44), we get

$$\begin{aligned}J[u] = \mathbb{E}\bigg[\int_0^T \bigg(\bar{b}_t - \frac{b_t^2}{2Q_t}\varrho_t\bigg)\varrho_t + \frac{1}{2}c_t^2\Pi_t + \frac{1}{2}\int_E \Pi_t\bar{f}_{(t,e)}^2\lambda_2(\mathrm{d}e)\mathrm{d}t\bigg] \\ + \frac{1}{2}\int_0^T O_t\gamma_t\mathrm{d}t + \frac{1}{2}R\gamma_T + \frac{1}{2}\Pi_0 x_0^2 + \varrho_0 x_0.\end{aligned} \tag{45}$$

Therefore, we have the following theorem.

Theorem 4. If Assumptions 1–3 and $h_t = 0$ hold, the optimal control u and the optimal cost functional $J[u]$ are given by (40) and (45), respectively.

4.2 LQ optimal control of FBSDE driven by Poisson jump

Consider the following stochastic control system:

$$\begin{cases}\mathrm{d}x_t^v = \big\{a_t x_t^v + b_t v_t + \bar{b}_t\big\}\mathrm{d}t + \displaystyle\int_E f_{(t,e)}\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \bar{f}_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \mathrm{d}y_t^v = -\bigg\{A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t\bar{z}_t^v + F_t v_t + G_t + \displaystyle\int_E D_t k_{(t,e)}^v\lambda_1(\mathrm{d}e) \\ \qquad + \displaystyle\int_E \bar{D}_t\bar{k}_{(t,e)}^v\lambda_2(\mathrm{d}e)\bigg\}\mathrm{d}t + \int_E k_{(t,e)}^v\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_e \bar{k}_{(t,e)}^v\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ x_0^v = x_0, \quad y_T^v = Lx_T^v + M,\end{cases} \tag{46}$$

along with the cost functional,

$$J[v] = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\{O_t(x_t^v)^2 + Q_t v_t^2 + 2o_t x_t^v + 2p_t y_t^v + 2q_t v_t\Big\}\mathrm{d}t + R(x_T^v)^2 + 2rx_T^v + S(y_0^v)^2 + 2sy_0^v\bigg]. \tag{47}$$

Suppose that $\tilde{N}(\cdot,\cdot)$ is the observed process. This can be seen as the case of (10) with $h_t = g_t = \bar{g}_t = \sigma_t = 0$ and $\kappa_{(t,\cdot)} = 1$. Clearly, $\mathcal{F}_t^{Y^v} = \mathcal{F}_t^{Y^0} = \sigma\{\tilde{N}_2([0,s],A), 0 \leqslant s \leqslant t, A \in \mathcal{E}\}$. The observation seems simple, but the Wonham separation principle is still invalid; thus the resulting conclusions are not trivial. Now we give the following necessary condition.

Theorem 5. Let Assumptions 1 and 2 hold. Suppose that u is an optimal control and $(x, y, k, \bar{k})$ is the corresponding optimal state. Then the FBSDEP

$$\begin{cases}\mathrm{d}\varphi_t = \big(B_t\varphi_t - p_t\big)\mathrm{d}t + \displaystyle\int_E \varphi_t D_t\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \varphi_t\bar{D}_t\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \mathrm{d}\xi_t = -\Big\{a_t\xi_t + O_t x_t + o_t - A_t\varphi_t\Big\}\mathrm{d}t + \displaystyle\int_E \vartheta_{(t,e)}\tilde{N}_1(\mathrm{d}t,\mathrm{d}e) + \int_E \bar{\vartheta}_{(t,e)}\tilde{N}_2(\mathrm{d}t,\mathrm{d}e), \\ \varphi_0 = -Sy_0 - s, \quad \xi_T = -L\varphi_T + Rx_T + r,\end{cases} \tag{48}$$

admits a unique solution $(\varphi, \xi, \vartheta, \bar{\vartheta}) \in S_{\mathcal{F}}^2(0,T;\mathbb{R}^2) \times F_{\mathcal{F}}^2(0,T;\mathbb{R}^2)$ such that

$$Q_t u_t - F_t\mathbb{E}[\varphi_t|\mathcal{F}_t^Y] + b_t\mathbb{E}[\xi_t|\mathcal{F}_t^Y] + q_t = 0. \tag{49}$$

Then we give the sufficient condition for optimal control without proof.

Theorem 6. Let Assumptions 1 and 2 hold. If $u \in \mathcal{U}_{ad}$ satisfies

$$Q_t u_t - F_t\mathbb{E}[\varphi_t|\mathcal{F}_t^Y] + b_t\mathbb{E}[\xi_t|\mathcal{F}_t^Y] + q_t = 0,$$

where $(\varphi, \xi, \vartheta, \bar{\vartheta})$ is a solution to (48), then u is an optimal control.

According to Theorem 19.6 in Liptser and Shiryaev [26], we get the following filtering estimation. The proof is similar to Lemmas 4 and 5, so we omit it.

Theorem 7. For any $v \in \mathcal{U}_{ad}$, the optimal filters $(\hat{x}^v, \hat{y}^v, \hat{k}^v, \hat{\bar{k}}^v)$ and $(\hat{\varphi}, \hat{\xi}, \hat{\vartheta}, \hat{\bar{\vartheta}})$ of the solutions $(x^v, y^v, k^v, \bar{k}^v)$ and $(\varphi, \xi, \vartheta, \bar{\vartheta})$ to (46) and (48) with respect to $\mathcal{F}_t^{Y^v}$ and $\mathcal{F}_t^Y$ satisfy

$$\begin{cases} \mathrm{d}\hat{x}_t^v = (a_t\hat{x}_t^v + b_tv_t + \bar{b}_t)\mathrm{d}t + \int_E \bar{f}_{(t,e)}\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}\hat{y}_t^v = -\left\{A_t\hat{x}_t^v + B_t\hat{y}_t^v + \int_E D_t\hat{k}_{(t,e)}^v\lambda_1(\mathrm{d}e) + \int_E \bar{D}_t\hat{\bar{k}}_{(t,e)}^v\lambda_2(\mathrm{d}e) + F_tv_t + G_t\right\}\mathrm{d}t + \int_E \hat{\bar{k}}_{(t,e)}^v\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \hat{x}_0^v = x_0, \quad \hat{y}_T^v = L\hat{x}_T^v + \hat{M}, \end{cases} \tag{50}$$

and

$$\begin{cases} \mathrm{d}\hat{\varphi}_t = (B_t\hat{\varphi}_t - p_t)\mathrm{d}t + \int_E \bar{D}_t\hat{\varphi}_t\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}\hat{\xi}_t = -\big(a_t\hat{\xi}_t + O_t\hat{x}_t + o_t - A_t\hat{\varphi}_t\big)\mathrm{d}t + \int_E \hat{\bar{\vartheta}}_{(t,e)}\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \hat{\varphi}_0 = -Sy_0 - n, \quad \hat{\xi}_T = -L\hat{\varphi}_T + R\hat{x}_T + r, \end{cases} \tag{51}$$

respectively.

Then the optimal control is

$$u_t = \frac{1}{Q_t}(F_t\hat{\varphi}_t - b_t\hat{\xi}_t - q_t). \tag{52}$$

Similar to Corollary 1, we also obtain the uniqueness of the optimal control u. The feedback optimal control can be obtained directly from Theorem 3. Thus we omit it.

Remark 3. If the state equation and observation equation have no jumps, that is $f = \bar{f} = D = \bar{D} = k = \bar{k} = \kappa = 0$, then the main results including optimal control and its feedback representation degenerate into the main result in Wang et al. [16].

5 Application in finance

In this section, we solve the financial mathematics problem raised at the beginning of this paper. Firstly, we give the following assumption.

Assumption 4. Let $\tilde{g}(t, x, y, z, \bar{z}, k, \bar{k}, v) = \tilde{B}_ty + \tilde{F}_tv$. Suppose that $\tilde{B}_t$, $\tilde{F}_t$ and $\bar{q}_t$ belong to $L^\infty(0, T; \mathbb{R})$. $\kappa_{(t,\cdot)} \geqslant -1$, $\bar{s} \geqslant 0$ and $\bar{r}$ are constants.

Solving BSDEP (3) with Assumption 4, we have

$$\mathbb{E}[y_t^v] = \mathbb{E}\left[x_T^v \mathrm{e}^{\int_t^T \tilde{B}_s\mathrm{d}s} + \int_t^T \tilde{F}_s\mathrm{e}^{\int_t^s \tilde{B}_u\mathrm{d}u}v_s\mathrm{d}s\right].$$

Then Problem (SRU) is equivalent to the following Problem (LQCU).

Problem 3 (LQCU)**.** Find an admissible control $u \in \mathcal{U}_{ad}$ to minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T (Q_tv_t^2 + 2\rho_tv_t)\mathrm{d}t + R(x_T^v)^2 + 2\phi x_T^v\right], \tag{53}$$

subject to (1) and (2) with

$$\rho_t = -Q_t\bar{q}_t - \bar{s}\tilde{F}_t\mathrm{e}^{\int_0^t \tilde{B}_s\mathrm{d}s} \quad \text{and} \quad \phi = -R\bar{r} - \bar{s}\mathrm{e}^{\int_0^T \tilde{B}_s\mathrm{d}s}.$$

It is easy to know that $\rho_t \in L^\infty(0, T; \mathbb{R})$ and ϕ is a constant.

By Theorem 2, if u is an optimal control, we have

$$\begin{cases} \mathrm{d}x_t = (r_tx_t + b_tv_t - \bar{b}_t)\mathrm{d}t + c_t\mathrm{d}W_t + \bar{c}_t\mathrm{d}\bar{W}_t + \int_E f_{(t,e)}\tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{f}_{(t,e)}\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \mathrm{d}\xi_t = -r_t\xi_t\mathrm{d}t + \eta_t\mathrm{d}W_t + \bar{\eta}_t\mathrm{d}\bar{W}_t + \int_E \vartheta_{(t,e)}\tilde{N}_1(\mathrm{d}t, \mathrm{d}e) + \int_E \bar{\vartheta}_{(t,e)}\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ x_0 = x_0, \quad \xi_T = Rx_T + \phi, \end{cases} \tag{54}$$

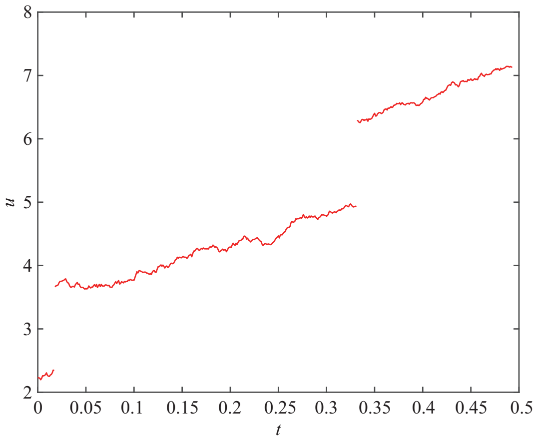

Figure 1 (Color online) Optimal control.

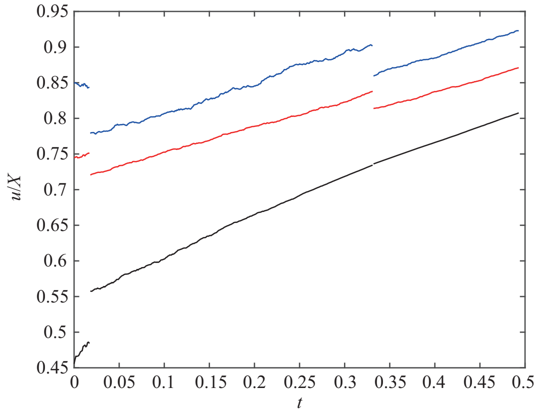

Figure 2 (Color online) Optimal investment proportion: $r = 0.2$ (blue), $r = 0.4$ (red) and $r = 0.8$ (black).

which admits a unique solution $(x, \xi, \eta, \bar{\eta}, \vartheta, \bar{\vartheta}) \in S^2_{\mathcal{F}}(0, T; \mathbb{R}^2) \times M^2_{\mathcal{F}}(0, T; \mathbb{R}^2) \times F^2_{\mathcal{F}}(0, T; \mathbb{R}^2)$ such that

$$u_t = -\frac{1}{Q_t}\Big(b_t \mathbb{E}[\xi_t|\mathcal{F}^Y_t] + \rho_t\Big).$$

Moreover, the optimal control u can be written as follows by Theorem 3:

$$u_t = -\frac{1}{Q_t}\Big[b_t(\Pi_t \hat{x}_t + \varrho_t) + \rho_t\Big], \tag{55}$$

where Π, ϱ and $\hat{x}$ satisfy the following equations:

$$\begin{cases} \dot{\Pi}_t + 2r_t\Pi_t - \dfrac{1}{Q_t}b_t^2\Pi_t^2 = 0, \\ \Pi_T = R, \end{cases}$$

$$\begin{cases} \dot{\varrho}_t + \Big(r_t - \dfrac{1}{Q_t}b_t^2\Pi_t\Big)\varrho_t - \dfrac{1}{Q_t}b_t\rho_t\Pi_t - \Pi_t\bar{b}_t = 0, \\ \varrho_T = \phi, \end{cases}$$

and

$$\begin{cases} \mathrm{d}\hat{x}^v_t = \Big\{\Big(r_t - \dfrac{1}{Q_t}b_t^2\Pi_t\Big)\hat{x}^v_t - \dfrac{1}{Q_t}b_t(b_t\varrho_t + \rho_t) - \bar{b}_t\Big\}\mathrm{d}t + \Big(\bar{c}_t + \dfrac{h_t\gamma_t}{\sigma_t}\Big)\mathrm{d}\tilde{W}_t + \displaystyle\int_E \bar{f}_{(t,e)}\tilde{N}_2(\mathrm{d}t, \mathrm{d}e), \\ \hat{x}^v_0 = x_0. \end{cases} \tag{56}$$

Applying Itô's formula to $(\hat{x}_t)^2$ with the BDG inequality, we obtain $\mathbb{E}[\sup_{0\leqslant t\leqslant T}(\hat{x}_t)^2] < +\infty$. And u is adapted to $\mathcal{F}^Y_t$ and $\mathcal{F}^{Y^0}_t$. Then we can verify that u is an admissible control. Corollary 1 implies that u satisfying (55) is the unique optimal strategy.

Example 1. Assume $b_t = \bar{c}_t = Q_t = \tilde{B}_t = \tilde{F}_t = R = \bar{s} = 1$, $\bar{b}_t = 0.4$, $h_t = \bar{r} = 0$, $\bar{f}_{(t,\cdot)} = \lambda = 2$, $\bar{q}_t = 0.6$, $T = 0.5$.

If we assume that $r_t = 0.2$, we get the optimal strategy u in Figure 1. It shows one orbit of the optimal strategy path. It is clearly shown that the optimal strategy u is discontinuous. According to (55) and (56), we can know that the optimal strategy is indeed discontinuous. And whenever a jump occurs, the optimal strategy will also change dramatically.

We also get the optimal investment proportion corresponding to the interest rates $r = 0.2, 0.4, 0.8$ in Figure 2. It shows that the larger the interest rate is, the smaller the optimal investment proportion is. This is because the company will invest more money in the money account after the interest rate increases.

6 Conclusion

This paper investigates the LQ optimal control problem for partially observed FBSDEP with correlated noise. Because the observation equation is driven by Brownian motions and Poisson random measures, the observation Y^v is no longer a Brownian motion. Additionally, the drift term of the observation Y^v is linear with respect to the state x^v and control v, so the observation does not satisfy the condition for using the Girsanov transformation. Thus, the Girsanov transformation is invalid. We apply a backward separation approach to decompose the state and the observation to overcome the circular dependence. We obtain the necessary and sufficient conditions as well as the feedback representation of optimal control by combining the backward separation approach with the variational method and stochastic filtering. Then, we present two special cases to demonstrate the significance of our results. Finally, we give a financial example and derive the explicitly optimal control. In the future, we will study the backward separation approach of stochastic systems with regime-switching or random coefficients.

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