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State feedback stabilization of stochastic nonlinear time-delay systems: a dynamic gain method

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Dear editor,

During the past few decades, a significant amount of attention has been focused on deterministic time-delay systems. These studies are based on all types of growth conditions on nonlinear functions. In a recent study [1], Zhang et al. obtained particularly good results using a dynamic gain method with wide practical use [2], and proved that restrictive growth conditions are unnecessary and can be removed in the case of state feedback control.

The problem regarding the controller design of stochastic nonlinear systems has been an active area of research; see [3–5] and the references therein. However, similar to deterministic systems, each of these studies also requires some growth conditions on nonlinear functions, such as [6, 7].

Considering stochastic nonlinear systems:

$$\begin{aligned} \mathrm{d}x_i &= (x_{i+1} + f_i(\bar{x}_i, \bar{x}_i(t-d)))\mathrm{d}t \\ &\quad + g_i^{\mathrm{T}}(\bar{x}_i, \bar{x}_i(t-d))\mathrm{d}\omega, \quad i = 1, \ldots, n-1, \\ \mathrm{d}x_n &= (u + f_n(x, x(t-d)))\mathrm{d}t \\ &\quad + g_n^{\mathrm{T}}(x, x(t-d))\mathrm{d}\omega, \\ x(s) &= \zeta(s), \quad s \in [-d, 0], \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $u \in \mathbb{R}$ are the system state and control inputs; $\bar{x}_i = (x_1, \ldots, x_i)^{\mathrm{T}}$; the drift terms $f_i : \mathbb{R}^{2i} \to \mathbb{R}$ and the diffusion terms $g_i = (g_{i1}, \ldots, g_{im})^{\mathrm{T}} : \mathbb{R}^{2i} \to \mathbb{R}^m$, $i = 1, \ldots, n$, are continuously differentiable with $f_i(0, 0) = 0$ and $g_i(0, 0) = 0$; and ω is an m-dimensional standard Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, and P being a probability measure.

The aim of this study is to construct a delay-independent, dynamic state feedback controller such that the equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability without imposing any growth conditions on the nonlinearities.

It should be noted that, although the dynamic gain design idea of the controller in this study stems from [1], it only considers deterministic systems. Compared with [1], the contributions and difficulties of this study are characterized through the following novel features:

(i) The main contribution of this study is to remove restrictive conditions in existing studies for a stochastic nonlinear time-delay system (1). Without imposing any growth conditions on the nonlinearities f_i and g_i, a delay-independent, dynamic state feedback controller is constructed to guarantee that the equilibrium at the origin of the closed-loop system is globally asymptotically stable in probability.

(ii) For a stochastic system (1), the appearance of the diffusion and Hessian terms will inevitably produce many more nonlinear terms, and dealing with them is not easy.

(iii) Because Eq. (1) describes a stochastic nonlinear time-delay system, its stability analysis is much more difficult than its deterministic counterpart.

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Preliminary results.

Lemma 1 ([8]). For positive real numbers m, n and a real-valued function $\pi(x,y) > 0$, $|x^m y^n| \leqslant \frac{m}{m+n}\pi(x,y)|x|^{m+n} + \frac{n}{m+n}\pi^{-\frac{m}{n}}(x, y)|y|^{m+n}$.

Lemma 2 ([9]). For a continuous function $f(x, y)$, there exist smooth functions $a(x) \geqslant 0$, $b(y) \geqslant 0$, $c(x) \geqslant 1$ and $d(y) \geqslant 1$ such that $|f(x,y)| \leqslant a(x)+b(y), |f(x,y)| \leqslant c(x)d(y)$.

Lemma 3 ([9]). For the $\mathcal{C}^1$ function $f_i(\bar{x}_i, \bar{x}_i(t-d))$ with $f_i(0,0)=0$, there exist smooth functions $\bar{\gamma}_{ij}(x_j) \geqslant 0$ and $\bar{\gamma}^*_{ij}(x_j(t-d)) \geqslant 0$, $j=1,\dots,i$, such that $|f_i(\cdot)| \leqslant \sum_{j=1}^{i}(\bar{\gamma}_{ij}(x_j)|x_j| + \bar{\gamma}^*_{ij}(x_j(t-d))|x_j(t-d)|)$.

Remark 1. By Lemma 3, it is easy to see that continuously differentiable nonlinear functions f_i and g_{ik} can be transformed into

$$\begin{aligned}|f_i| \leqslant& \sum_{j=1}^{i}(\bar{\gamma}_{ij}(x_j)|x_j| \\ &+\bar{\gamma}^*_{ij}(x_j(t-d))|x_j(t-d)|),\ i=1,\dots,n,\\ |g_{ik}| \leqslant& \sum_{j=1}^{i}(\lambda_{i,k,j}(x_j)|x_j| + \lambda^*_{i,k,j}(x_j(t-d)) \\ &\cdot|x_j(t-d)|),\ k=1,\dots,m,\end{aligned} \tag{2}$$

where $\bar{\gamma}_{ij}(\cdot) \geqslant 0$, $\bar{\gamma}^*_{ij}(\cdot) \geqslant 0$, $\lambda_{i,k,j}(\cdot) \geqslant 0$, $\lambda^*_{i,k,j}(\cdot) \geqslant 0$ are smooth functions.

Design and analysis of dynamic state feedback controller. For simplicity, for any vector $a = (a_1,\dots,a_n)^{\mathrm{T}}$, we define $\bar{a}_i = (a_1,\dots,a_i)^{\mathrm{T}}$.

Initial step. Take $\xi_1 = x_1$ and $V_1(x_1, l_1) = \frac{1}{4}(1+\frac{1}{l_1})\xi_1^4$, where $l_1(t) \geqslant 1$ is a dynamic gain to be designed in Step 2. Then

$$\begin{aligned}\mathcal{L}V_1 \leqslant& \left(1+\frac{1}{l_1}\right)\xi_1^3 x_2^* + 2|\xi_1^3\xi_2| \\ &+2|\xi_1^3 f_1| + 3\xi_1^2|g_1^{\mathrm{T}} g_1| - \frac{\dot{l}_1}{4l_1^2}\xi_1^4,\end{aligned} \tag{3}$$

where $\xi_2 = x_2 - x_2^*$. By (2) and Lemmas 1 and 2,

$$\begin{aligned}&2|\xi_1^3\xi_2| \leqslant \frac{3}{2}\xi_1^4 + \frac{1}{2}\xi_2^4,\\ &2|\xi_1^3 f_1| \leqslant 2\xi_1^4\bar{\gamma}_{11}(x_1) + \xi_1^4 \\ &\qquad +4\xi_1^4(t-d)\bar{\gamma}^{*4}_{11}(x_1(t-d)),\\ &3\xi_1^2|g_1^{\mathrm{T}} g_1| \leqslant \xi_1^4\Psi_{11}(x_1) \\ &\qquad +\xi_1^4(t-d)\Psi^*_{11}(x_1(t-d)),\end{aligned} \tag{4}$$

where $\Psi_{11}(\cdot) \geqslant 0$ and $\Psi^*_{11}(\cdot) \geqslant 0$ are smooth functions. Construct the Lyapunov-Krasovskii (L-K) functional $V_{1\mathrm{LK}} = V_1(x_1, l_1) + \int_{t-d}^{t}\xi_1^4(s)(\bar{\gamma}^{*4}_{11}(s) + \Psi^*_{11}(s))\mathrm{d}s$. By (4), the following is obtained:

$$\begin{aligned}\mathcal{L}V_{1\mathrm{LK}} \leqslant& \left(1+\frac{1}{l_1}\right)\xi_1^3 x_2^* + \xi_2^4 + \xi_1^4(3+2\bar{\gamma}_{11}(x_1) \\ &+4\bar{\gamma}^{*4}_{11}(x_1) + \Psi_{11}(x_1) + \Psi^*_{11}(x_1)) - \frac{\dot{l}_1}{4l_1^2}\xi_1^4.\end{aligned} \tag{5}$$

In view of (5) and $l_1 \geqslant 1$, the virtual controller

$$\begin{aligned}x_2^* =& -\xi_1\Big(\frac{n-1}{4} + 4 + 2\bar{\gamma}_{11}(x_1) + 4\bar{\gamma}^{*4}_{11}(x_1) \\ &+\Psi_{11}(x_1) + \Psi^*_{11}(x_1)\Big) \triangleq -\xi_1\beta_1(x_1)\end{aligned} \tag{6}$$

leads to $\mathcal{L}V_{1\mathrm{LK}} \leqslant -\xi_1^4 - \frac{n-1}{4}\xi_1^4 + \xi_2^4 - \frac{\dot{l}_1}{4l_1^2}\xi_1^4$.

Step 2. Construct L-K functional

$$\begin{aligned}&V_2 = V_{1\mathrm{LK}} + \frac{1}{l_1}W_2(x_1,x_2) + \frac{1}{l_1 l_2}\left(\frac{\xi_1^4}{4} + W_2(x_1,x_2)\right),\\ &W_2 = \frac{1}{4}\xi_2^4 = \frac{1}{4}(x_2 - x_2^*)^4,\end{aligned} \tag{7}$$

with $l_2(t) \geqslant 1$ being a dynamic gain to be designed in the next step. Construct L-K functional $V_{2\mathrm{LK}} = V_2 + \int_{t-d}^{t}\xi_2^4(s)A(s)\mathrm{d}s + \int_{t-d}^{t}\frac{1}{l_1(s)}\xi_1^4(s)B(s)\mathrm{d}s$, where $A(\cdot)$, $B(\cdot)$ are appropriate nonnegative smooth functions. Then,

$$\begin{aligned}&\mathcal{L}V_{2\mathrm{LK}} \leqslant -\xi_1^4 - \frac{n-1}{4}\xi_1^4 - \frac{\dot{l}_1}{4l_1^2}\xi_1^4 + \frac{1}{l_1}\left(1+\frac{1}{l_2}\right)\xi_2^3 \\ &\cdot(x_3^* + x_3 - x_3^*) + \frac{1}{l_1}\xi_1^4[C(x_1)+B(x_1)] + \Big(\frac{1}{l_1(t)} \\ &-\frac{1}{l_1(t-d)}\Big)\xi_1^4(t-d)B(x_1(t-d)) + \xi_2^4[1+D(x_1,x_2) \\ &+A(x_1,x_2)] - \frac{\dot{l}_1}{l_1^2}W_2(\cdot) - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{l_1^2 l_2^2}\left(\frac{\xi_1^4}{4} + W_2(\cdot)\right),\end{aligned} \tag{8}$$

where $C(\cdot) \geqslant 0$, $D(\cdot) \geqslant 0$ are smooth functions. The gain update law is then designed as follows:

$$\begin{aligned}&\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(x_1), 0\}, \quad l_1(0) = 1,\\ &\rho_1(x_1) = 4[C(x_1) + B(x_1)].\end{aligned} \tag{9}$$

From (9), it is easy to verify that the gain l_1 has the following properties:

$$\begin{aligned}&0 \leqslant \dot{l}_1 \leqslant l_1\rho_1(x_1), \quad \dot{l}_1 \geqslant -l_1^2 + l_1\rho_1(x_1),\\ &l_1(t) \geqslant l_1(t-d) \geqslant 1.\end{aligned} \tag{10}$$

Using the relationship (10), we arrive at

$$\begin{aligned}&-\frac{\dot{l}_1}{4l_1^2}\xi_1^4 \leqslant \frac{\xi_1^4}{4} - \frac{1}{4l_1}\xi_1^4\rho_1(x_1),\ \frac{1}{l_1(t)} - \frac{1}{l_1(t-d)} \leqslant 0,\\ &-\frac{\dot{l}_1}{l_1^2}W_2(\cdot) - \frac{\dot{l}_1 l_2 + l_1\dot{l}_2}{l_1^2 l_2^2}\left(\frac{\xi_1^4}{4} + W_2(\cdot)\right) \\ &\quad \leqslant -\frac{\dot{l}_2}{l_1 l_2^2}\left(\frac{\xi_1^4}{4} + W_2(\cdot)\right).\end{aligned} \tag{11}$$

Using Lemma 1, the following is obtained:

$$2|\xi_2^3(x_3 - x_3^*)| \leqslant \xi_2^4 + c_3\xi_3^4, \tag{12}$$

where $\xi_3 = x_3 - x_3^*$, and c_3 is a positive constant.

Substituting (9), (11), and (12) into (8) and designing a virtual controller as

$$\begin{aligned}x_3^* =& -l_1\xi_2[n+1+A(x_1,x_2) + D(x_1,x_2)] \\ \triangleq& -l_1\xi_2\beta_2(x_1,x_2),\end{aligned} \tag{13}$$

the following is obtained:

$$\mathcal{L}V_{2\mathrm{LK}} \leqslant -\sum_{j=1}^{2}\xi_j^4 - (n-2)\left(\frac{\xi_1^4}{4}+\xi_2^4\right) + c_3\xi_3^4 - \frac{\dot{l}_2}{l_1 l_2^2}\left(\frac{\xi_1^4}{4}+W_2(\cdot)\right). \quad (14)$$

Using the recursive method, we can obtain the dynamic gains $l_1,\ldots,l_{n-1}$ and a series of virtual controllers $x_1^*,\ldots,x_{n+1}^*$ given by

$$\begin{aligned}
&\dot{l}_1 = \max\{-l_1^2 + l_1\rho_1(x_1), 0\},\\
&\dot{l}_k = \max\{-l_k^2 + l_k\rho_k(\bar{l}_{k-1}, \bar{x}_k), 0\},\ k = 2,\ldots,n-1,\\
&x_1^* = 0, \quad x_2^* = -\xi_1\beta_1(x_1),\\
&x_j^* = -l_1\cdots l_{j-2}\xi_{j-1}\beta_{j-1}(\bar{l}_{j-3}, \bar{x}_{j-2}),\\
&\xi_{j-1} = x_{j-1} - x_{j-1}^*, \quad j = 1,\ldots,n, \qquad (15)
\end{aligned}$$

with $\rho_k(\cdot) > 0$, $\beta_j(\cdot) > 0$ being smooth functions. The controller

$$u = -l_1\cdots l_{n-1}\beta_n(\bar{l}_{n-2}, x) \quad (16)$$

renders $\mathcal{L}V_{n\mathrm{LK}} \leqslant -\sum_{i=1}^n \xi_i^4 \leqslant 0$.

Theorem 1. For a stochastic nonlinear time-delay system (1), there is a delay-independent, dynamic state feedback controller (16) such that the closed-loop system has a unique solution in $[-d,\infty)$, and the equilibrium at the origin of the closed-loop systems is globally asymptotically stable (GAS) in probability.

Proof. The proof is divided into two steps.

Step I: We first prove the almost sure boundedness of $x_1(t),\ldots,x_n(t)$, $l_1(t),\ldots,l_{n-1}(t)$ in $[-d,\infty)$. It follows from Itô's formula that

$$\begin{aligned}&\mathrm{E}\{V_{n\mathrm{LK}}(x(\sigma_r\wedge t), l(\sigma_r\wedge t), \sigma_r\wedge t)\}\\ &\leqslant V_{n\mathrm{LK}}(x(0), l(0), 0), \qquad (17)\end{aligned}$$

where $\sigma_r \triangleq \inf\{t \geqslant 0 : |x(t)| \geqslant r\}$, $r \geqslant 0$, $g = (g_1,\ldots,g_n)^{\mathrm{T}}$. From the definition of $V_{n\mathrm{LK}}$,

$$\begin{aligned}&\mathrm{E}\{V_{n\mathrm{LK}}(x(\sigma_r\wedge t), l(\sigma_r\wedge t), \sigma_r\wedge t)\}\\ &\geqslant \mathrm{E}\left(\left(\frac{1}{4}x_1^4(s) + \frac{1}{4l_1(s)}(x_2(s)-x_2^*(s))^4 + \cdots\right.\right.\\ &\left.\left.\left. + \frac{1}{4l_1(s)\cdots l_{n-1}(s)}(x_n(s)-x_n^*(s))^4\right)\right|_{s=\sigma_r\wedge t}\right). \quad (18)\end{aligned}$$

Letting $r \to \infty$, it is easy to see from (17) and (18) that $x_1(t)$, $\frac{1}{4l_1(t)}(x_2(t)-x_2^*(t))^4$, $\ldots$, $\frac{1}{4l_1(t)\cdots l_{n-1}(t)}(x_n(t)-x_n^*(t))^4$ are bounded almost surely in $[-d,\sigma_\infty)$. It is clear from (8) that the gain $l_1(t)$ is monotonously non-deceasing. We claim that $l_1(t)$ is bounded almost surely in $[-d,\sigma_\infty)$. If not, suppose $\lim_{t\to\sigma_\infty} l_1(t) = \infty$. Through continuity, $\rho_1(x_1)$ is almost surely bounded from the almost sure boundedness of x_1. Thus, there is a time instant $0 < T_1 < \sigma_\infty$ such that $-l_1^2(t) + l_1(t)\rho_1(x_1(t)) \leqslant 0$ on $[T_1,\sigma_\infty)$. From (10), $\dot{l}_1(t) = \max\{-l_1^2(t) + l_1(t)\rho_1(x_1(t)), 0\} = 0$ in $[T_1,\sigma_\infty)$, which contradicts $\lim_{t\to\sigma_\infty} l_1(t) = \infty$. Hence, $l_1(t)$ is almost surely bounded in $[-d,\sigma_\infty)$, and thus so is $x_2(t)$ based on the definition of $x_2^*(t)$. It is not difficult to see that $\sigma_\infty = \infty$. Hence, it is recursively easy to obtain $l_2(t)$, $x_3(t),\ldots,l_{n-1}(t), x_n(t)$, which are bounded almost surely in $[-d,\infty)$.

Step II: It is easy to see that $V_{n\mathrm{LK}} \geqslant \frac{1}{4}\xi_1^4(t) + \frac{1}{4l_1(t)}\xi_2^4(t) + \cdots + \frac{1}{4l_1(t)\cdots l_{n-1}(t)}\xi_n^4(t)$. In view of the almost sure boundedness of l_i and $l_i \geqslant 1$, there is a class $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $V_{n\mathrm{LK}} \geqslant \alpha_1(|x(t)|)$. From the almost sure boundedness of l_i and $l_i \geqslant 1$, it is easy to see that there is a class $\mathcal{K}_\infty$ function $\alpha_2(\cdot)$ such that $V_{n\mathrm{LK}} \leqslant \alpha_2\left(\sup_{-d\leqslant\tau\leqslant 0}|x(\tau+t)|\right)$.

Hence, by Theorem 2 in [3], Theorem 1 holds.

Remark 2. From the controller design and stability analysis, we can see that the choice of Lyapunov-Krasovskii function is important.

Conclusion. This study solved the important problem of state feedback stabilization of stochastic nonlinear time-delay systems without imposing any growth conditions on the nonlinearities.

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