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Further results on dynamic-algebraic Boolean control networks

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Abstract Restricted coordinate transformation, controllability, observability and topological structures of dynamic-algebraic Boolean control networks are investigated under an assumption. Specifically, given the input-state at some point, assume that the subsequent state is certain or does not exist. First, the system can be expressed in a new form after numbering the elements in admissible set. Then, restricted coordinate transformation is raised, which allows the dimension of new coordinate frame to be different from that of the original one. The system after restricted coordinate transformation is derived in the proposed form. Afterwards, three types of incidence matrices are constructed and the results of controllability, observability and topological structures are obtained. Finally, two practical examples are shown to demonstrate the theory in this paper.

Keywords controllability, dynamic-algebraic Boolean control network, fixed point and cycle, observability, restricted coordinate transformation

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1 Introduction

Since the appearance of systems biology, the research of relationships of genes, proteins or cells in biological systems has aroused much attention. In 1969, Kauffman proposed a useful model, Boolean networks (BNs), to describe connected feedback net of genes [1]. In BNs, the state of a node, which stands for the activity of a gene, is a Boolean function of the states of some nodes. The study of BNs becomes flourishing and many great progresses are made [2–5].

Semi-tensor product (STP), presented by Cheng, provides a systematic tool to analyze BNs [6, 7]. Besides, it is applied in other fields [8]. Boolean control networks (BCNs) arise when we introduce controls (inputs) to BNs. Controllability and observability are hot topics for control systems [9]. By means of STP, some fundamental problems in BNs and BCNs are addressed, including controllability and observability [10], realization [11], identification [12], stability and stabilization [13], decoupling problem [14, 15], l_1-gain analysis and model reduction [16]. In addition, observability of BCNs is further researched in [17–19]. Besides, the algorithm to calculate number of fixed points and cycles in BNs is given in [7]. Ref. [20] shows the structure of cycles in BCNs as compounded cycles. Fixed points and cycles in input space, state space and input-state space are all considered. Ref. [21] uses a different tool, incidence matrix, to discuss some problems in BCNs, such as controllability, observability and topological structures. The results are also generalized to mix-valued BCNs. State space coordinate transformation

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of BNs and BCNs is revealed in [11]. Using it, a BCN can be expressed in another coordinate frame and some special forms are possible to be realized such as Kalman decomposition.

In view of constraints, singular systems are proposed for different systems, such as singular time delay systems [22], singular fractional differential systems [23, 24], and singular Hamiltonian operators [25]. With some algebraic constraints, dynamic-algebraic Boolean networks (DABNs) are raised [26]. In fact, they are a type of singular Boolean networks (SBNs). Ref. [27] studies normalization, solvability, topological structures of SBNs. Ref. [28] discusses optimal control of singular Boolean control networks (SBCNs), while [29] considers function perturbations of singular Boolean networks. Three types of solutions to algebraic constraints are posed in [30], under which DABNs satisfying certain conditions can be transformed into standard BCNs.

For every control system, controllability, observability and topological structures are fundamental problems [31–34]. In this paper, controllability, observability and topological structures of dynamic-algebraic Boolean control networks are investigated under a new restricted coordinate transformation. An essential characteristic of dynamic-algebraic Boolean control networks (DABCNs) is that, given state and control at time t, the state at time $t+1$ may be not unique. In this paper, we discuss the special case that for any state and control at time t, the state in time $t+1$ is certain, if existing. Based on this assumption, we investigate restricted coordinate transformations, controllability, observability and topological structures of DABCNs. Restricted coordinate transformation is presented, which is peculiar to DABCNs. Different from coordinate transformation of conventional BNs and BCNs, it can change the dimension of coordinate frame and in the meantime make the transformed DABCNs equivalent to original one. Incidence matrix is allowed to act as a significant tool to reach conclusions about controllability, observability and topological structures of DABCNs.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries and specifies the system studied in this paper. Section 3 does the preparation work of numbering the elements of some sets and expressing the system in a new form. Section 4 defines restricted coordinate transformation, gives equivalent conditions for a mapping to be a restricted coordinate transformation, obtains the system after a restricted coordinate transformation, and provides an example to show these results. Section 5 derives the necessary and sufficient condition of controllability and the sufficient condition of observability by exploiting incidence matrix. Section 6 uses incidence matrix to find the number of input-state fixed points and cycles. Section 7 raises two practical examples to demonstrate the main results obtained in this paper. Section 8 makes concluding remarks.

2 Dynamic-algebraic boolean control networks

Firstly, some notations used in this paper are listed in the following.

- The i-th column of identify matrix I_n is symbolized by δ_n^i. Set

$$\Delta_n := \{\delta_n^1, \delta_n^2, \ldots, \delta_n^n\},$$

and $\Delta := \Delta_2$.

- $\mathcal{D} := \{0, 1\}$.
- Given a matrix M, $\mathrm{row}_i(M)$ is the i-th row of M, and $\mathrm{col}_i(M)$ is the i-th column of M. $\mathrm{Row}(M)$ is the set of rows of M, and $\mathrm{Col}(M)$ is the set of columns of M.
- Given a matrix $M \in \mathbb{R}_{n\times s}$, if $\mathrm{Col}(M) \subset \Delta_n$, then M is called a logical matrix. $\mathcal{L}_{n\times s}$ is the set of all $n \times s$ logical matrices. If a matrix $M = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_s}] \in \mathcal{L}_{n\times s}$, we simply define it as $M = \delta_n[i_1, i_2, \ldots, i_s]$.
- The cardinality (the number of the elements) of a set A is symbolized by $|A|$.
- The (i, j)-th element of a matrix M is denoted by $(M)_{ij}$.
- For a matrix M, $\mathrm{sgn}^+(M)$ is a matrix defined as follows:

$$(\mathrm{sgn}^+(M))_{ij} = \begin{cases} 1, & (M)_{ij} > 0, \\ 0, & \text{otherwise}. \end{cases}$$

We express that $M > 0$ if every entry of M is positive.

• For two matrices $M \in \mathbb{R}_{m\times s}$ and $N \in \mathbb{R}_{n\times s}$, the Khatri-Rao product of M, N, denoted by $M * N$, is a matrix of dimension $mn \times s$ subject to

$$\mathrm{col}_i(M * N) = \mathrm{col}_i(M) \otimes \mathrm{col}_i(N),$$

where $\otimes$ denotes Kronecker product [35].

• Suppose there are two matrices A and B of dimensions $m \times n$ and $p \times q$, respectively, and the least common multiple of n and p is t. Then the semi-tensor product of A and B is defined as

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}) \in R_{mt/n\times qt/p}.$$

In fact, STP is a generalization of conventional matrix product. Therefore the symbol "$\ltimes$" can be omitted when no confusion is generated [6].

• Let A, B be two matrices with same dimensions and σ a binary operator. Then define $A\sigma B$ as a matrix satisfying

$$(A\sigma B)_{ij} = (A)_{ij}\sigma(B)_{ij}.$$

For example, assume

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$A\bar{\vee}B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The system studied in this paper is a DABCN with n network nodes, m input nodes and p outputs, which is depicted as follows:

$$\begin{cases} x_i(t+1) = f_i(x(t), u(t)), & i = 1, 2, \ldots, r, & (1) \\ \varphi_j(x(t), u(t)) = 1, & j = 1, 2, \ldots, q, & (2) \\ y_s(t) = g_s(x(t), u(t)), & s = 1, 2, \ldots, p, & (3) \end{cases}$$

where

$$\begin{aligned} & x(t) := (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathcal{D}^n, \\ & u(t) := (u_1(t), u_2(t), \ldots, u_m(t)) \in \mathcal{D}^m, \\ & y(t) := (y_1(t), y_2(t), \ldots, y_p(t)) \in \mathcal{D}^p, \\ & f := (f_1, f_2, \ldots, f_r) : \mathcal{D}^n \times \mathcal{D}^m \to \mathcal{D}^r \quad (1 \leqslant r \leqslant n) \end{aligned}$$

are logical state, logical input, logical output and a vector of logical functions, respectively. And in this system, (1) are called dynamic equations. Besides,

$$\begin{aligned} & \varphi_i : \mathcal{D}^n \to \mathcal{D}, \quad i = 1, 2, \ldots, q, \\ & \varphi_i(x(t)) = 1, \quad i = 1, 2, \ldots, q, \\ & g := (g_1, g_2, \ldots, g_p) : \mathcal{D}^n \times \mathcal{D}^m \to \mathcal{D}^p \end{aligned}$$

are logical functions, algebraic constraints and a vector of logical functions, respectively.

For each logical variable $x_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$, one can equivalently express 0 and 1 as $(0, 1)^{\mathrm{T}}$ and $(1, 0)^{\mathrm{T}}$ respectively. The later form is called vector form, while the former one is scalar form. Define the vector form of x_i, u_ω and y_s as X_i, U_ω and Y_s, respectively, $i = 1, 2, \ldots, n$, $\omega = 1, 2, \ldots, m$, $s = 1, 2, \ldots, p$.

Via STP, a logical function can be equivalently converted to an algebraic form, where logical variables appear in vector form [7]. A procedure to transfer algebraic form back to logical form is provided in [10]. Suppose that the algebraic from of system (1)–(3) is

$$\begin{cases} X^1(t+1) = LU(t)X(t), & (4)\\ HX(t) = \delta_{2^q}^1, & (5)\\ Y(t) = GU(t)X(t), & (6)\end{cases}$$

where $X = \ltimes_{i=1}^n X_i$, $X^1 = \ltimes_{i=1}^r X_i$, $U = \ltimes_{i=1}^m U_i$, $Y = \ltimes_{i=1}^p Y_i$, $L \in \mathcal{L}_{2^r \times 2^{n+m}}$, $H \in \mathcal{L}_{2^q \times 2^n}$, $G \in \mathcal{L}_{2^p \times 2^{n+m}}$.

We can derive that the solution set of algebraic constraint (5) is $\overline{\mathcal{X}}_a = \{\delta_{2^n}^i | \mathrm{col}_i(H) = \delta_{2^q}^1\}$, which is called admissible set of system (4)–(6). In scalar form, $\mathcal{X}_a = \{(x_1, x_2, \ldots, x_n) | \ltimes_{i=1}^n X_i \in \overline{\mathcal{X}}_a\}$ is the solution set of (2) and it is called the admissible set of system (1)–(3).

Given $x \in \mathcal{X}_a$, write that

$$\mathcal{U}_x = \left\{u | u \in \mathcal{D}^m,\ \exists x'_{r+1}, x'_{r+2}, \ldots, x'_n \ \text{ s.t. } \left(f(x,u), x'_{r+1}, x'_{r+2}, \ldots, x'_n\right) \in \mathcal{X}_a\right\}. \tag{7}$$

Let

$$V_a = \{(x,u) | x \in \mathcal{X}_a, u \in \mathcal{U}_x\}, \tag{8}$$

$$O_a = \{g(x,u) | (x,u) \in V_a\}. \tag{9}$$

The algebraic constraints determine that $u(t) \in \mathcal{U}_{x(t)}$, $t = 1, 2, \ldots$. For $x \in \mathcal{X}_a$ and $u \in \mathcal{U}_x$, define

$$N_{x,u} = \{(x'_1, x'_2, \ldots, x'_n) | (x'_1, x'_2, \ldots, x'_r) = f(x,u), (x'_1, x'_2, \ldots, x'_n) \in \mathcal{X}_a\}.$$

Note that $|N_{x,u}| > 0$ according to the definition of $\mathcal{U}_x$. For some $x \in \mathcal{X}_a$ and $u \in \mathcal{U}_x$, there may be $|N_{x,u}| \neq 1$. In this paper, we do not discuss this situation. In other words, the results are based on Assumption 1.

Assumption 1. $|N_{x,u}| = 1$, for any $x \in \mathcal{X}_a$ and $u \in \mathcal{U}_x$.

Remark 1. If Assumption 1 does not hold, then for $x(t) \in \mathcal{X}_a$ and $u(t) \in \mathcal{U}_x$, the state $x(t+1)$ may be not uniquely determined. Thus to define controllability, fixed point and cycles in a meaningful manner is a challenging problem. In fact, controllability involves for any two states, the system can be controlled from one to another. Besides, fixed point and cycles are one state or a set of states which the system would be attracted in after finite period of time. The details about controllability, fixed point and cycles are raised in Sections 5 and 6.

In this case, for $x \in \mathcal{X}_a$ and $u \in \mathcal{U}_x$, let $F(x,u)$ be the unique element in $N_{x,u}$. The process to derive $F(x,u)$ is that, calculate $f(x,u)$ and find the unique $(x'_1, x'_2, \ldots, x'_n) \in \mathcal{X}_a$ determined by $(x'_1, x'_2, \ldots, x'_r) = f(x,u)$.

According to the discussion above, under Assumption 1, system (1)–(3) is expressed as this form:

$$\begin{cases} x(t+1) = F(x(t), u(t)), & (10)\\ y(t) = g(x(t), u(t)), & (11)\end{cases}$$

where $x(t) \in \mathcal{X}_a$, $u(t) \in \mathcal{U}_{x(t)}$.

3 Numbering the states and input-states

In this section, we number the elements of some sets preparing for the subsequent study. Thereby, an equivalent form of system (10) and (11) is raised.

For a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \in \mathcal{D}^s$, define a function

$$d(\alpha) = \alpha[2^{s-1} \ \ 2^{s-2} \ \ \cdots \ \ 2^0]^{\mathrm{T}} + 1. \tag{12}$$

We can see that the range of d is $\{1, 2, \ldots, 2^s\}$. In fact, $d(\cdot)$ is one-to-one. For any $\beta \in \{1, 2, \ldots, 2^s\}$, we can find $\alpha = d^{-1}(\beta)$ in the following process:

$$\beta_0 = \beta - 1, \quad \alpha_i = \left[\frac{\beta_{i-1}}{2^{s-i}}\right], \quad \beta_i = \beta_{i-1} - \alpha_i \times 2^{s-i}, \quad i = 1, 2, \ldots, s,$$

where in the second equation $[a]$ is the largest integer which is less than or equal to a [7].

Let $l = |\mathcal{X}_a|$. Number the elements of $\mathcal{X}_a$ so that

$$\mathcal{X}_a = \left\{x^{(1)}, x^{(2)}, \ldots, x^{(l)}\right\}$$

and the rule of numbering is $i < j$ when $d(x^{(i)}) < d(x^{(j)})$. Let $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)})$. For $i = 1, 2, \ldots, l$, let $\lambda_i = |\mathcal{U}_{x^{(i)}}|$. Write $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ and it is easy to see that $\lambda = |V_a|$. Number the elements of V_a so that

$$V_a = \left\{(x, u)^{(1)}, (x, u)^{(2)}, \ldots, (x, u)^{(\lambda)}\right\}$$

and the rule of numbering is $i < j$ when $d((x, u)^{(i)}) < d((x, u)^{(j)})$. Write $\mu = |O_a|$, and similarly number that

$$O_a = \{y^{(1)}, y^{(2)}, \ldots, y^{(\mu)}\}.$$

For any set A whose elements are numbered as $A = \{a^{(1)}, a^{(2)}, \ldots, a^{(|A|)}\}$, let the serial number of an element $a \in A$ be $n(a, A)$. That is, $n(a^{(i)}, A) = i$.

Construct an $l \times \lambda$ matrix Φ as follows:

$$(\Phi)_{ij} = \begin{cases} 1, & x^{(i)} = F\big((x, u)^{(j)}\big), \\ 0, & \text{otherwise}. \end{cases}$$

And a $\mu \times \lambda$ matrix Ψ is established satisfying

$$(\Psi)_{ij} = \begin{cases} 1, & y^{(i)} = g\big((x, u)^{(j)}\big), \\ 0, & \text{otherwise}. \end{cases}$$

According to the construction of matrices above, system (10) and (11) can be written as

$$\begin{cases} \delta_l^{n(x(t+1), \mathcal{X}_a)} = \Phi \delta_\lambda^{n((x(t), u(t)), V_a)}, & (13) \\ \delta_\mu^{n(y(t), O_a)} = \Psi \delta_\lambda^{n((x(t), u(t)), V_a)}, & (14) \end{cases}$$

where $x(t) \in \mathcal{X}_a$, $u(t) \in \mathcal{U}_{x(t)}$.

Afterwards, this form is employed to investigate restricted coordinate transformation, controllability, observability, fixed points and cycles of DABCNs.

4 Restricted coordinate transformation

In conventional BCNs, coordinate transformation allows the system to be expressed under new coordinate frame. In this section, we propose restricted coordinate transformation of DABCNs. It is a generalized coordinate transformation and sufficient to equivalently transform a system to be depicted under new coordinate frame.

Suppose that there is a mapping

$$z = h(x), \tag{15}$$

where $x \in \mathcal{X}_a$, $h : \mathcal{D}^n \to \mathcal{D}^k$. Set

$$\mathcal{Z}_a = \{h(x) | x \in \mathcal{X}_a\}, \quad \overline{V}_a = \{(h(x), u) | (x, u) \in V_a\}.$$

Mapping (15) is called restricted coordinate transformation if $h|_{\mathcal{X}_a}$, which is the restriction of h to $\mathcal{X}_a$, is one-to-one. It is easy to be verified that it is a restricted coordinate transformation if and only if $|\mathcal{Z}_a| = |\mathcal{X}_a|$. When $n = k$ and $\mathcal{X}_a = \mathcal{D}^n$, a restricted coordinate transformation is a coordinate transformation [11].

Let $\eta = |\mathcal{Z}_a|$, and number the elements of $\mathcal{Z}_a$ such that

$$\mathcal{Z}_a = \left\{z^{(1)}, z^{(2)}, \ldots, z^{(\eta)}\right\}.$$

The rule of numbering is the same as that of $\mathcal{X}_a$ and V_a. Construct an $\eta \times l$ matrix H as

$$(H)_{ij} = \begin{cases} 1, & z^{(i)} = h\big(x^{(j)}\big), \\ 0, & \text{otherwise}. \end{cases}$$

Then there is

$$\delta_\eta^{n(z,\mathcal{Z}_a)} = H\delta_l^{n(x,\mathcal{X}_a)}. \tag{16}$$

According to the definition of restricted coordinate transformation, it is natural that mapping (15) is a restricted coordinate transformation if and only if H is nonsingular. Furthermore, it is obvious that H is permutation matrix. Thus $H^{-1} = H^{\mathrm{T}}$.

Now we build a matrix to describe the corresponding relationship between (x, u) and (z, u). Divide the identity matrix

$$I_\lambda = \begin{bmatrix} I_{\lambda_1} & O_{\lambda_1\times\lambda_2} & \cdots & O_{\lambda_1\times\lambda_l} \\ O_{\lambda_2\times\lambda_1} & I_{\lambda_2} & \cdots & O_{\lambda_2\times\lambda_l} \\ \vdots & \vdots & & \vdots \\ O_{\lambda_l\times\lambda_1} & O_{\lambda_l\times\lambda_2} & \cdots & I_{\lambda_l} \end{bmatrix} \triangleq \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_l \end{bmatrix}. \tag{17}$$

For $i = 1, 2, \ldots, l$, write $h_i = n(h(x^{(i)}), \mathcal{Z}_a)$.

Suppose that mapping (15) is restricted coordinate transformation. For $j = 1, 2, \ldots, \eta$, let $h'_j = n(h^{-1}(z^{(j)}), \mathcal{X}_a)$. Then it holds that

$$H = \delta_l[h_1, h_2, \ldots, h_l], \quad H^{-1} = \delta_\eta[h'_1\ h'_2\ \cdots\ h'_\eta].$$

Set $K = [J_{h'_1}^{\mathrm{T}}\ J_{h'_2}^{\mathrm{T}}\ \cdots\ J_{h'_l}^{\mathrm{T}}]^{\mathrm{T}}$. Theorem 1 is obtained.

Theorem 1. Equality

$$\delta_\lambda^{n((h(x),u),\overline{V}_a)} = K\delta_\lambda^{n((x,u),V_a)}$$

holds for restricted coordinate transformation (15) and the matrix K above.

Proof. Suppose that $z^{(q)} = h(x^{(p)})$, $(x^{(p)}, u^{(s)}) = (x, u)^{(t)}$, $(z^{(q)}, u^{(s)}) = (z, u)^{(v)}$. Then it is straightforward that $q = h_p$, $p = h'_q$, $t = \sum_{i=1}^{p-1} \lambda_i + s$, $v = \sum_{i=1}^{q-1} \lambda_{h'_i} + s$. Because $\mathrm{col}_t(K)$ is a unit vector, it is sufficient to show that $(K)_{tv} = 1$. For division (17), in the t-th column of identity matrix I_λ, the only one element 1 is located in the block J_p. After elementary row transformation, I_λ is converted to K. In the t-th column of K, the only one element 1 is located in block $J_p = J_{h'_q}$. Specifically, it is in the $(\sum_{i=1}^{q-1} \lambda_{h'_i} + s)$-th row. Let $v = \sum_{i=1}^{q-1} \lambda_{h'_i} + s$, so there is $(K)_{tv} = 1$.

Notably K is a permutation matrix, so $K^{-1} = K^{\mathrm{T}}$. After restricted coordinate transformation (15), system (13) is transformed to

$$\delta_\eta^{n(z(t+1),\mathcal{Z}_a)} = H\delta_l^{n(x(t+1),\mathcal{X}_a)} = H\Phi\delta_\lambda^{n((x(t),u(t)),V_a)} = H\Phi K^{\mathrm{T}}\delta_\lambda^{n((z(t),u(t)),\overline{V}_a)} \triangleq \overline{\Phi}\delta_\lambda^{n((z(t),u(t)),\overline{V}_a)}, \tag{18}$$

where $(z(t), u(t)) \in \overline{V}_a$. Similarly, system (14) is transformed to

$$\delta_\eta^{n(y(t),O_a)} = \Psi\delta_\lambda^{n((x(t),u(t)),V_a)} = \Psi K^{\mathrm{T}}\delta_\lambda^{n((z(t),u(t)),\overline{V}_a)} \triangleq \overline{\Psi}\delta_\lambda^{n((z(t),u(t)),\overline{V}_a)}, \tag{19}$$

where $(z(t), u(t)) \in \overline{V}_a$.

Remark 2. After restricted coordinate transformation, the dimension of new coordinate frame, k, is not necessarily equal to that of the former one, n. It is different from that of standard BNs [11].

Example 1 shows the result in this section.

Example 1. Consider the following DABCN:

$$\begin{cases} x_1(t+1) = u(t), \\ x_2(t+1) = x_1(t) \vee x_3(t), \\ x_1(t) \leftrightarrow x_2(t) = 0, \\ x_3(t) = 0, \\ y = x_1(t). \end{cases} \tag{20}$$

From the algebraic constraints we can derive that $\mathcal{X}_a = \{(0,1,0),(1,0,0)\}$. Numbering the elements we get $x^{(1)} = (0,1,0)$, $x^{(2)} = (1,0,0)$. Then $\mathcal{U}_{x^{(1)}} = \{1\}$, $\mathcal{U}_{x^{(2)}} = \{0\}$. Next, we have

$$V_a = \{(0,1,0,1),(1,0,0,0)\}, \quad (x,u)^{(1)} = (0,1,0,1), \quad (x,u)^{(2)} = (1,0,0,0),$$
$$O_a = \{0,1\}, \quad y^{(1)} = 0, \quad y^{(2)} = 1.$$

Because

$$f\big((x,u)^{(1)}\big) = (1,0), \quad f\big((x,u)^{(2)}\big) = (0,1), \quad g\big((x,u)^{(1)}\big) = 0, \quad g\big((x,u)^{(2)}\big) = 1,$$

it is trivial that

$$N_{(x,u)^{(1)}} = \{(1,0,0)\}, \quad N_{(x,u)^{(2)}} = \{(0,1,0)\}.$$

Thus Assumption 1 holds for the system. With

$$\Phi = \delta_2[2\ 1], \quad \Psi = \delta_2[1\ 2],$$

we can express system (20) as form (13) and (14). Consider a restricted coordinate transformation $z = h(x)$, where $x \in \mathcal{X}_a$, as

$$h(0,1,0) = (1,0), \quad h(1,0,0) = (0,1).$$

Then $\mathcal{Z}_a = \{(0,1),(1,0)\}$ and after numbering the elements we have $z^{(1)} = (0,1)$, $z^{(2)} = (1,0)$. The transformation is restricted coordinate transformation. Calculate that $\overline{V}_a = \{(0,1,0),(1,0,1)\}$, $(z,u)^{(1)} = (0,1,0)$, $(z,u)^{(2)} = (1,0,1)$. Establish that

$$H = \delta_2[2\ 1], \quad K = \delta_2[2\ 1].$$

The system after restricted coordinate transformation is

$$\delta_\eta^{n(z(t+1),\mathcal{Z}_a)} = H\Phi K^{\mathrm{T}} \delta_\lambda^{n((z(t),u(t)),\overline{V}_a)},$$
$$\delta_\eta^{n(y(t),O_a)} = \Psi K^{\mathrm{T}} \delta_\lambda^{n((z(t),u(t)),\overline{V}_a)},$$

where $(z(t),u(t)) \in \overline{V}_a$.

5 Controllability and observability

This section studies the controllability and observability of DABCNs. Controllability of DABCNs is defined as follows.

Definition 1. Consider system (10) and (11).

(1) For $x^{(i)}, x^{(j)} \in \mathcal{X}_a$, $x^{(j)}$ is said to be reachable from $x^{(i)}$ at the s-th step ($s = 1,2,\ldots$), if for $x(0) = x^{(i)}$ we can find a sequence of controls $u(t) \in U_{x(t)}$, $t = 0,1,2,\ldots,s-1$, such that $x(s) = x^{(j)}$.

(2) For $x^{(i)}, x^{(j)} \in \mathcal{X}_a$, $x^{(j)}$ is said to be reachable from $x^{(i)}$, if there exists a positive integer s such that $x^{(j)}$ is reachable from $x^{(i)}$ at the s-th step.

(3) The system is said to be controllable at $x^{(i)}$ if any $x^{(j)} \in \mathcal{X}_a$ is reachable from $x^{(i)}$.

(4) The system is said to be controllable, if for any $x^{(i)} \in \mathcal{X}_a$, the system is controllable at $x^{(i)}$.

Then we define incidence matrix to derive related results. The definition and application of incidence matrices here are similar to those in [21]. Divide matrix Φ as

$$\Phi = [\Phi_1 \ \Phi_2 \ \cdots \ \Phi_l],$$

where Φ_i is of dimension $l \times \lambda_i$, $i = 1, 2, \ldots, l$. Suppose that $x^{(s')} = F(x^{(s)}, u^{(t)})$, $(x^{(s)}, u^{(t)}) = (x, u)^{(v)}$. Then $v = \sum_{i=1}^{s-1} \lambda_i + t$. Thus $(\Phi)_{s'v} = 1$ and $(\Phi_s)_{s't} = 1$. We can draw the conclusion that for any $i, j \in \{1, 2, \ldots, l\}$, relation $\delta_l^j \in \Phi_i$ holds if and only if there exists $u \in \mathcal{U}_{x^{(i)}}$ such that $x^{(j)} = F(x^{(i)}, u)$.

Construct the $l \times l$ state incidence matrix Γ_1 and $\lambda \times \mu$ output incidence matrix Γ_2 as

$$(\Gamma_1)_{ij} = \begin{cases} 1, & \delta_l^j \in \Phi_i, \\ 0, & \text{otherwise}, \end{cases} \qquad (\Gamma_2)_{ij} = \begin{cases} 1, & \delta_\mu^j \in \Psi_i, \\ 0, & \text{otherwise}. \end{cases}$$

Remark 3. It may happen that $\mathcal{U}_{x^{(i)}} = \emptyset$ for some $x^{(i)} \in \mathcal{X}_a$. If so, it holds that $\mathrm{row}_i(\Gamma_1) = \mathrm{row}_i(\Gamma_2) = \mathbf{0}$.

The property in Proposition 1 is used to obtain the final results.

Proposition 1. Given system (10) and (11), $x^{(j)}$ is reachable from $x^{(i)}$ at the second step if and only if $(\Gamma_1^2)_{ij} > 0$, and there are $(\Gamma_1^2)_{ij}$ routes for $x^{(i)}$ to reach $x^{(j)}$.

Proof. $(\Gamma_1^2)_{ij} = \sum_{\kappa=1}^{l} (\Gamma_1)_{i\kappa}(\Gamma_1)_{\kappa j}$. There exist $u(0) \in U_{x^{(i)}}$ and $u(1) \in U_{x^{(\kappa)}}$ such that $x(1) = x^{(\kappa)}$ and $x(2) = x^{(j)}$ if and only if $(\Gamma_1)_{i\kappa} = 1$ and $(\Gamma_1)_{\kappa j} = 1$. Thus this proposition is clear.

Theorem 2 reveals the necessary and sufficient conditions of controllability.

Theorem 2. Consider system (10) and (11).

(1) $x^{(j)}$ is reachable from $x^{(i)}$ at the s-th step if and only if

$$(\Gamma_1^s)_{ij} > 0,$$

and there are $(\Gamma_1^s)_{ij}$ routes for $x^{(i)}$ to reach $x^{(j)}$.

(2) $x^{(j)}$ is reachable from $x^{(i)}$ if and only if

$$\sum_{s=1}^{l-1} (\Gamma_1^s)_{ij} > 0.$$

(3) The system is controllable at $x^{(i)}$ if and only if

$$\sum_{s=1}^{l-1} \mathrm{row}_i(\Gamma_1^s) > 0.$$

(4) The system is controllable if and only if

$$\sum_{s=1}^{l-1} \Gamma_1^s > 0.$$

Proof. Consider Proposition 1 and the first result is easy to be verified by induction. Via Cayley-Hamiltan theorem, if $(\Gamma_1^s)_{ij} = 0$, $s = 1, 2, \ldots, l-1$, then $(\Gamma_1^s)_{ij} = 0$, $s = l, l+1, \ldots$. And the last three results are obvious.

Definition 2 and sufficient conditions of observability are given as follows.

Definition 2. (1) For $x^{(i)}, x^{(j)} \in \mathcal{X}_a$, $i \neq j$, $x^{(i)}$ and $x^{(j)}$ are said to be distinct, if when initial states $x(0)$ are given differently as $x^{(i)}$ and $x^{(j)}$, respectively, there exists an $s \in \{0, 1, \ldots\}$ and we can choose $u(0), u(1), \ldots, u(s)$ such that $y(s)$ are different.

(2) System (10) and (11) is said to be observable if any two $x^{(i)}, x^{(j)} \in \mathcal{X}_a$, $i \neq j$, are distinct.

Divide Ψ as

$$\Psi = [\Psi_1 \ \Psi_2 \ \cdots \ \Psi_l],$$

where Ψ_i is of dimension $\mu \times \lambda_i$, $i = 1, 2, \ldots, l$. We can derive that for any $i \in \{1, 2, \ldots, l\}$ and $j \in \{1, 2, \ldots, \mu\}$, relation $\delta_\mu^j \in \Phi_i$ holds if and only if there exists $u \in \mathcal{U}_{x^{(i)}}$ such that $y^{(j)} = g(x^{(i)}, u)$.

Proposition 2. Given $x(0) = x^{(i)}$, $y^{(j)}$ and an integer s, there exists a sequence of controls $u(t) \in U_{x(t)}$, $t = 0, 1, 2, \ldots, s$, such that $y(s) = y^{(j)}$, if and only if $(\Gamma_1^s \Gamma_2)_{ij} > 0$.

Proof. $(\Gamma_1^s \Gamma_2)_{ij} = \sum_{\kappa=1}^{l} (\Gamma_1^s)_{i\kappa} (\Gamma_2)_{\kappa j}$. $(\Gamma_1^s)_{i\kappa} > 0$ means $x^{(\kappa)}$ is reachable from $x^{(i)}$ at the s-th step. $(\Gamma_2)_{\kappa j} > 0$ means if $x(s) = x^{(\kappa)}$, there exists a $u(s) \in \mathcal{U}_{x(s)}$ such that $y(s) = y^{(j)}$. Then the result is clear.

Theorem 3. Consider system (10) and (11).

(1) $x^{(i)}$ and $x^{(j)}$ are distinct if there exists an $s \in \{0, 1, \ldots, l-1\}$ such that

$$\mathrm{row}_i(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)) \neq \mathrm{row}_j(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)).$$

(2) System (10) and (11) is observable if for any $i, j \in \{1, 2, \ldots, l\}$, $i \neq j$, there exists an $s \in \{0, 1, \ldots, l-1\}$ such that

$$\mathrm{row}_i(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)) \neq \mathrm{row}_j(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)).$$

Proof. Suppose that there exists an $s \in \{0, 1, \ldots, l-1\}$ such that

$$\mathrm{row}_i(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)) \neq \mathrm{row}_j(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)).$$

Without loss of generality, assume that $(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2))_{ik} = 1$ and $(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2))_{jk} = 0$. Then according to Proposition 2, for initial state $x^{(i)}$, there exists a sequence of controls $u(t)$, $t = 0, 1, 2, \ldots, s$, such that $y(s) = y^{(k)}$, while $y(s) \neq y^{(k)}$ for initial state $x^{(j)}$ with the same control sequence. Thus the first result holds. Combine it with Definition 2, and then the second result is trivial.

Theorem 4 is an equivalent expression of Theorem 3.

Theorem 4. Consider system (10) and (11).

(1) $x^{(i)}$ and $x^{(j)}$ are distinct if

$$\vee_{s=0}^{l-1} [\mathrm{row}_i(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)) \bar{\vee} \mathrm{row}_j(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2))] \neq \mathbf{0}.$$

(2) System (10) and (11) is observable if

$$\wedge_{i=1}^{l} \wedge_{j=1, j\neq i}^{l} \vee_{s=0}^{l-1} [\mathrm{row}_i(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2)) \bar{\vee} \mathrm{row}_j(\mathrm{sgn}^+(\Gamma_1^s \Gamma_2))] \neq \mathbf{0}.$$

6 Fixed points and cycles

The fixed points and cycles considered in this section are defined in the following.

Definition 3. Let $s \in \{1, 2, \ldots, \lambda\}$. Suppose there is a set of input-states

$$C = \big\{ \big(x^{(i_1)}, u^{(j_1)}\big), \big(x^{(i_2)}, u^{(j_2)}\big), \ldots, \big(x^{(i_s)}, u^{(j_s)}\big) \big\} \subset V_a$$

satisfying

$$x^{(i_{\mu+1})} = F(x^{(i_\mu)}, u^{(j_\mu)}), \quad \mu = 1, 2, \ldots, s-1,$$

and

$$x^{(i_1)} = F\big(x^{(i_s)}, u^{(j_s)}\big).$$

Then we call C a cycle of length s. Furthermore, when $s = 1$, we call $(x^{(i_1)}, u^{(j_1)})$ a fixed point.

Ref. [20] researched fixed points and cycles in input-state space within the framework of STP, so the inputs and states appear in vector form. The definition here is equivalent to that in [20] and the inputs and states arise in scalar form. To show the number of fixed points and cycles, we construct $\lambda \times \lambda$ input-state incidence matrix Γ_3 as follows. For $i \in \{1, 2, \dots, \lambda\}$, $j \in \{1, 2, \dots, l\}$, $t \in \{1, 2, \dots, \lambda_j\}$ and $v = \sum_{\mu=1}^{j-1} \lambda_\mu + t$, there is

$$(\Gamma_3)_{iv} = \begin{cases} 1, & \mathrm{col}_i(\Phi) = \delta_l^j, \\ 0, & \text{otherwise.} \end{cases}$$

It comes without doubt that $(\Gamma_3)_{iv} = 1$ if and only if $x^{(j)} = F((x, u)^{(i)})$, where i, j, v are integers assigned above. Theorem 5 presents the number of fixed points and cycles of DABCNs.

Theorem 5. Consider system (10) and (11).

(1) The number of fixed points is

$$N_1 = \mathrm{tr}(\Gamma_3).$$

(2) The number of length s cycles, N_s, is inductively determined by

$$N_s = \frac{\mathrm{tr}(\Gamma_3^s) - \sum_{k \in \mathcal{P}(s)} k N_k}{s}, \quad s = 2, 3, \dots, \lambda,$$

where $\mathcal{P}(s)$ is the set of proper factors of s.

The proof is similar to the parallel results in standard BNs without control [7]. Ref. [21] initiates using input-state incidence matrix to find the number of fixed points and cycles in BCNs. Ref. [20] provides complete process to find fixed points and cycles in input-state space. For fixed points and cycles in DABCNs, the number of them is revealed in Theorem 5.

7 Examples

In this section, two practical examples are presented to demonstrate the above results.

Example 2. BNs are established to simulate the system that the biochemical oscillator regulates the mitosis and DNA replication of cell cycles [28,36]. Set cyclin as x_1, cyclin-dependent kinases (cdk) as x_2, and cdk-activated ubiquitin ligase as x_3. In view of environment and human behaviour, we add another cyclin symbolized by u. The state iteration is expressed as

$$\begin{cases} x_1(t+1) = \neg x_3(t) \vee u(t), \\ x_2(t+1) = x_1(t), \\ 1 = x_2(t) \leftrightarrow x_3(t). \end{cases} \tag{21}$$

And it is shown in Figure 1 in [28]. From the algebraic constraints we can derive that $\mathcal{X}_a = \{(0,0,0), (0,1,1), (1,0,0), (1,1,1)\}$. Number the elements that $x^{(1)} = (0,0,0)$, $x^{(2)} = (0,1,1)$, $x^{(3)} = (1,0,0)$, $x^{(4)} = (1,1,1)$. Afterwards, we have

$$V_a = \{(0,0,0,0), (0,0,0,1), (0,1,1,0), (0,1,1,1), (1,0,0,0), (1,0,0,1), (1,1,1,0), (1,1,1,1)\},$$
$$(x,u)^{(1)} = (0,0,0,0), \quad (x,u)^{(2)} = (0,0,0,1), \quad (x,u)^{(3)} = (0,1,1,0), \quad (x,u)^{(4)} = (0,1,1,1),$$
$$(x,u)^{(5)} = (1,0,0,0), \quad (x,u)^{(6)} = (1,0,0,1), \quad (x,u)^{(7)} = (1,1,1,0), \quad (x,u)^{(8)} = (1,1,1,1).$$

Calculate that

$$f\big((x,u)^{(1)}\big) = (1,0), \quad f\big((x,u)^{(2)}\big) = (1,0), \quad f\big((x,u)^{(3)}\big) = (0,0), \quad f\big((x,u)^{(4)}\big) = (1,0),$$
$$f\big((x,u)^{(5)}\big) = (1,1), \quad f\big((x,u)^{(6)}\big) = (1,1), \quad f\big((x,u)^{(7)}\big) = (0,1), \quad f\big((x,u)^{(8)}\big) = (1,1).$$

Next, we verify whether Assumption 1 holds.

$$N_{(x,u)^{(1)}} = \{(1,0,0)\}, \quad N_{(x,u)^{(2)}} = \{(1,0,0)\}, \quad N_{(x,u)^{(3)}} = \{(0,1,1)\}, \quad N_{(x,u)^{(4)}} = \{(1,0,0)\},$$

$$N_{(x,u)^{(5)}} = \{(1,1,1)\},\quad N_{(x,u)^{(6)}} = \{(1,1,1)\},\quad N_{(x,u)^{(7)}} = \{(0,1,1)\},\quad N_{(x,u)^{(8)}} = \{(1,1,1)\}.$$

Therefore Assumption 1 holds for the system. Thus

$$\Phi = \delta_4[3\ 3\ 1\ 3\ 4\ 4\ 2\ 4].$$

Hence we can express the system as

$$\delta_l^{n(x(t+1),\mathcal{X}_a)} = \Phi\delta_\lambda^{n((x(t),u(t)),V_a)},$$

where $x(t) \in \mathcal{X}_a$, $u(t) \in \mathcal{U}_{x(t)}$. The incidence matrix

$$\Gamma_1 = \begin{bmatrix} 0\ 0\ 1\ 0 \\ 1\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ 0\ 1\ 0\ 1 \end{bmatrix}.$$

And we can see that

$$\Gamma_1^2 = \begin{bmatrix} 0\ 0\ 0\ 1 \\ 0\ 0\ 1\ 1 \\ 0\ 1\ 0\ 1 \\ 1\ 1\ 1\ 1 \end{bmatrix},\quad \Gamma_1^3 = \begin{bmatrix} 0\ 1\ 0\ 1 \\ 0\ 1\ 0\ 2 \\ 1\ 1\ 1\ 1 \\ 1\ 1\ 2\ 2 \end{bmatrix},\quad \Gamma_1 + \Gamma_1^2 + \Gamma_1^3 = \begin{bmatrix} 0\ 1\ 1\ 2 \\ 1\ 1\ 2\ 3 \\ 1\ 2\ 1\ 3 \\ 2\ 3\ 3\ 4 \end{bmatrix}.$$

According to Theorem 2, $x^{(1)}$ is not reachable from $x^{(1)}$. The system is not controllable. Next, we obtain that

$$\Gamma_3 = \begin{bmatrix} 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \end{bmatrix}.$$

According to Theorem 5, the number of fixed points, N_1, and the numbers of length s ($s = 2, 3, \ldots, 8$) cycles, N_s, are gotten:

$$N_1 = 1,\quad N_2 = 0,\quad N_3 = 2,\quad N_4 = 6,\quad N_5 = 6,\quad N_6 = 7,\quad N_7 = 18,\quad N_8 = 33.$$

Example 3. The decomposition of logic circuit is widely studied [37–39]. The decomposition of a programmable logic array (PLA) into two cascaded PLAs is investigated in [40]. Ref. [41] demonstrates that any logical function can be implemented by an AND-OR two-level circuit, and it is usually realized as PLAs. Figure 1 represents a case of PLA. If we consider (β_0, β_1) as a vector of logical variable, the value (1, 1) never appears [40]. It can be verified that the logic circuit in Figure 2 constitutes a BCN if we let the clock "CLK" run according to the discrete time t. Here $u(t)$ is the logical input, $\beta(t) := (\beta_0(t), \beta_1(t))$ is the logical state, and $y(t)$ is the logical output. The BN is expressed as

$$\begin{cases} \beta_0(t+1) = (\beta_0(t) \wedge \beta_1(t)) \vee u(t), \\ \beta_1(t+1) = \neg(\beta_0(t) \vee \beta_1(t) \vee u(t)), \\ 1 = \neg(\beta_0(t) \wedge \beta_1(t)), \\ y(t) = \beta_0(t) \rightarrow \beta_1(t). \end{cases} \tag{22}$$

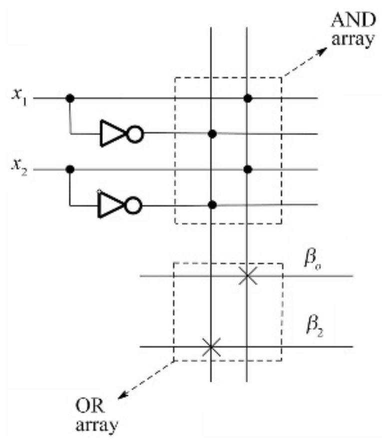

Figure 1 PLA leads to state constraint.

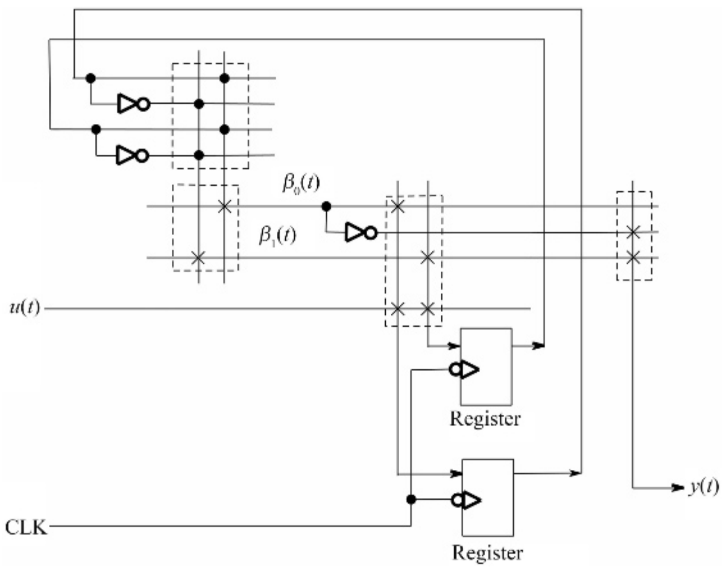

Figure 2 Logic circuit with feedback path.

From the algebraic constraints we can derive that $\mathcal{X}_a = \{(0,0), (0,1), (1,0)\}$. Number the elements that $\beta^{(1)} = (0,0)$, $\beta^{(2)} = (0,1)$, $\beta^{(3)} = (1,0)$. Next, we have

$$
\begin{aligned}
&V_a = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1)\},\\
&(\beta,u)^{(1)} = (0,0,0), \quad (\beta,u)^{(2)} = (0,0,1), \quad (\beta,u)^{(3)} = (0,1,0),\\
&(\beta,u)^{(4)} = (0,1,1), \quad (\beta,u)^{(5)} = (1,0,0), \quad (\beta,u)^{(6)} = (1,0,1),\\
&O_a = \{0,1\}, \quad y^{(1)} = 0, \quad y^{(2)} = 1.
\end{aligned}
$$

Then it is obtained that

$$
\begin{aligned}
&N_{(x,u)^{(1)}} = \{(0,1)\}, \quad N_{(x,u)^{(2)}} = \{(1,0)\}, \quad N_{(x,u)^{(3)}} = \{(0,0)\},\\
&N_{(x,u)^{(4)}} = \{(1,0)\}, \quad N_{(x,u)^{(5)}} = \{(0,0)\}, \quad N_{(x,u)^{(6)}} = \{(1,0)\},
\end{aligned}
$$

so Assumption 1 holds for the system. In the similar process to that in Example 2, we can get

$$\Phi = \delta_3[2\ 3\ 1\ 3\ 1\ 3], \quad \Psi = \delta_2[2\ 2\ 2\ 2\ 1\ 1].$$

Hence we can express the system as

$$\delta_l^{n(\beta(t+1),\mathcal{X}_a)} = \Phi\delta_\lambda^{n((\beta(t),u(t)),V_a)}, \quad \delta_\mu^{n(y(t),O_a)} = \Psi\delta_\lambda^{n((\beta(t),u(t)),V_a)},$$

where $\beta(t) \in \mathcal{X}_a$, $u(t) \in \mathcal{U}_{\beta(t)}$. The conclusion about controllability, fixed points and cycles can be drawn via the same method as in Example 2. Consider the problem on observability.

$$\Gamma_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Calculate that

$$\Gamma_1\Gamma_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Gamma_1^2\Gamma_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

There exists an integer 0 such that

$$\mathrm{row}_1(\mathrm{sgn}^+(\Gamma_1^0\Gamma_2)) \neq \mathrm{row}_3(\mathrm{sgn}^+(\Gamma_1^0\Gamma_2)).$$

According to Theorem 3, $\beta^{(1)}$ and $\beta^{(3)}$ are distinct. Similarly, $\beta^{(2)}$ and $\beta^{(3)}$ are distinct. But the sufficient condition where $\beta^{(1)}$ and $\beta^{(2)}$ are distinct does not hold. Thus we cannot judge whether the system is observable via Theorem 3.

8 Conclusion

DABCNs have been investigated under Assumption 1. In this case, given $(x(t), u(t)) \in V_a$, the state $x(t+1)$ is uniquely determined. Next, restricted coordinate transformation has been proposed. It is obviously a generalization of coordinate transformation of conventional BNs and BCNs. The necessary and sufficient condition for a mapping to be a restricted coordinate transformation has been discussed. The system after restricted coordinate transformation has been expressed in an equivalent new form. Different from the coordinate transformation in standard BNs and BCNs, the dimension of coordinate frame may be changed. Incidence matrices are introduced to derive the equivalent condition of controllability and sufficient condition of observability. Furthermore, they have been used to obtain the number of input-state fixed points and cycles. Considering merely admissible input-states makes incidence matrices to be properly established and applied.

When Assumption 1 does not hold, given a state and an input at time t, the state at time $t+1$ is uncertain. In this case, how to define controllability, observability, fixed points and cycles becomes a challenging problem. It is beyond the scope of the discussion in this paper and deserves further study.

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