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Achievable delay margin using LTI control for plants with unstable complex poles

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Abstract We consider the achievable delay margin of a real rational and strictly proper plant, with unstable complex poles, by a linear time-invariant (LTI) controller. The delay margin is defined as the largest time delay such that, for any delay less than this value, the closed-loop stability is maintained. Drawing upon a frequency domain method, particularly a bilinear transform technique, we provide an upper bound of the delay margin, which requires computing the maximum of a one-variable function. Finally, the effectiveness of the theoretical results is demonstrated through a numerical example.

Keywords delay margin, systems with time-delay, time-invariant systems, unstable complex poles, frequency domain method

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1 Introduction

The presence of time delay may cause poor performance even instability of the control system. Hence, there are many literature on analysis and control of time delay systems, e.g., see books [1–3], and survey papers [4–6]. One of the fundamental limitations of time delay systems is the achievable delay margin problem, which was one of the open problems in control posed in [7]. The delay margin is defined as the largest time delay such that, for any delay less than this value, there exists an LTI controller such that the closed-loop stability can be maintained [6, 8–11].

However, there are only a few results available in the literature on the achievable delay margin problem. For a stable plant, the zero controller can provide arbitrarily large delay margin. By using static state feedback, there would be a finite delay margin for plant with unstable poles [12]. As for LTI controllers, the time delay margin problem is first partly solved in [8], and some upper bounds on the achievable delay margin are provided. Recently, the technique has been extended to multi-input multi-output delay systems [10]. But these bounds are only proven to be tight in no more than one unstable pole and non-minimum phase zero. Under more general circumstances, these bounds may be crude. Based on the previous work, some tighter bounds have been given based on multiple unstable plant *real* poles [9].

In this paper, we study the achievable delay margin of plant with unstable *complex* poles. More specifically, for plants with both a pair of unstable complex conjugate poles and an unstable real pole (or a real non-minimum phase zero), we provide some novel upper bounds of the delay margin. To solve

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this problem, we transform the original problem into a one-dimensional optimization problem, and then this problem can be easily calculated.

Our results on delay margin focused on the use of LTI controller. For some nonlinear feedback controllers, such as adaptive controllers [13, 14], and nonlinear controllers [15, 16], the delay margin can be made infinite. Because these controllers are not easy to implement, it is still desired to study LTI controllers.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries. Section 3 presents our main results. A numerical example is given in Section 4 to verify the theoretical results. Finally, Section 5 contains conclusion. Two proofs are collected in the Appendix.

2 Preliminaries

In this section, we provide some notations, definitions and two technique lemmas.

Throughout the paper we adopt the following notations. Let $\Re$ be the set of real numbers, $\mathcal{C}$ the set of all complex numbers. For any complex number z, we denote its conjugate by $\bar{z}$. Let $\mathcal{C}^- := \{z : \mathrm{Re}(z) < 0\}$, $\mathcal{C}^+ := \{z : \mathrm{Re}(z) > 0\}$ and $\bar{\mathcal{C}}^+$ for its closure. H_∞ denotes the set of complex-valued functions, which are analytic and bounded in $\mathcal{C}^+$, and RH_∞ denotes the subset of real rational elements.

We first consider a single-input, single-output, real-rational and strictly proper plant $P_0(s)$. Combining this plant with a delay $\hat{\tau} > 0$, we denote a modified plant model set as

$$\mathcal{P}_{\hat{\tau}} := \{\mathrm{e}^{-\tau s} P_0(s) : \tau \in [0, \hat{\tau}]\}.$$

We then recall some definitions and methodology from [17]. The set of admissible controllers is the quotient field of RH_∞, which is denoted as $\mathcal{F}(RH_\infty)$. The controller $C \in \mathcal{F}(RH_\infty)$ stabilizes $P \in \mathcal{F}(H_\infty)$ if the transfer functions

$$(1 + PC)^{-1},\ P(1 + PC)^{-1},\ C(1 + PC)^{-1} \in H_\infty.$$

The controller C stabilizes $\mathcal{P}_{\hat{\tau}}$ if C stabilizes $\mathrm{e}^{-s\tau} P_0(s)$ for every $\tau \in [0, \hat{\tau}]$.

If $C \in \mathcal{F}(RH_\infty)$ stabilizes P_0, the delay margin is

$$\mathrm{DM}(P_0, C) := \sup\{\hat{\tau} > 0 : C \text{ stabilizes } \mathcal{P}_{\hat{\tau}}\}.$$

The maximum delay margin achievable by a stabilizing controller is

$$\mathrm{DM}(P_0) := \sup\{\mathrm{DM}(P_0, C) : C \text{ stabilizes } P_0\}.$$

We first introduce a technique lemma, which is an extension of Proposition 5 in [8] and Lemma 1 in [9].

Lemma 1. Let $P_0(s)$ be a real rational and strictly proper plant with unstable complex poles $a \pm b\mathrm{j}$, which is stabilized by $C \in \mathcal{F}(RH_\infty)$. Suppose that the all-pass function is given as

$$B_\alpha(s) = \frac{Q_\alpha(s)}{\bar{Q}_\alpha(-s)},$$

and $Q_\alpha(s)$ has one of the following forms.

i) $Q_\alpha(s) = [(s - a)^2 + b^2] \cdot [(s + a)^2 + b^2 - \alpha] \cdot [1 - c_\alpha s]$.

ii) $Q_\alpha(s) = [(s - a)^2 + b^2] \cdot [(s + a)^2 + b^2 - \alpha] \cdot [1 - c_\alpha s] \cdot [z + s]$.

iii) $Q_\alpha(s) = [a + b\mathrm{j} - \alpha s] \cdot [a - b\mathrm{j} - \alpha s] \cdot [z + s]$.

In the above, c_α is a complex-valued continuous function of α, satisfying $c_0 = 0$, and $z > 0$ is a zero of $P_0(s)$. If there exists an $\hat{\alpha} > 0$ for which $B_{\hat{\alpha}}(s)$ and $P_0(s)$ have unstable pole-zero cancellation in $\mathcal{C}^+$ (excluding the obvious ones at $a \pm b\mathrm{j}$ and z), then there exists a critical value $\alpha^* \in (0, \hat{\alpha})$ and $\omega^* \in \Re$ so that

$$B_{\alpha^*}(\mathrm{j}\omega^*) P_0(\mathrm{j}\omega^*) C(\mathrm{j}\omega^*) = -1.$$

Proof. See Appendix A.

Remark 1. We shall apply the above lemma to establish some upper bounds of the achievable delay margin. Specifically, we use the above all-pass function $B_\alpha(s)$ to substitute for the exponential function $\mathrm{e}^{-\tau s}$ at $s=\mathrm{j}\omega$, then the analysis would become relatively easy. This idea can go back to the pseudo-delay method advanced by Rekasius [18], which has been successfully used for the stability analysis of time delay systems [19, 20].

The following lemma is cited from [8].

Lemma 2. Suppose that $\alpha,\beta>0$, Then $\arctan\alpha\leqslant\alpha$, and $|\arctan\alpha-\arctan\beta|\leqslant|\alpha-\beta|$.

3 Achievable delay margin

In this section, we first derive an upper bound of the delay margin of plant with an unstable real pole and a pair of unstable complex conjugate poles.

Theorem 1. Suppose that $P_0(s)$ has poles at $p>0$ and $a\pm b\mathrm{j}$ with $a\geqslant b>0$, then

$$\mathrm{DM}(P_0)\leqslant\sup_{\omega>0}f(\omega),$$

where

$$f(\omega)=\frac{2}{\omega}\arctan\left(\frac{\Lambda(\hat{\alpha},\omega)\omega^2}{p(\Lambda(\hat{\alpha},\omega)\omega+2a\hat{\alpha})}\right),\tag{1}$$

$\Lambda(\alpha,\omega)=\omega^2+4a^2+\alpha-2A$, $A=a^2+b^2$, $\hat{\alpha}=A^2/(2ap+A)$, and

$$f(\omega)<\frac{2}{p}.$$

Proof. Suppose that $C\in\mathcal{F}(RH_\infty)$ stabilizes P_0. Let $B_\alpha(s)=Q_\alpha(s)/\bar{Q}_\alpha(-s)$, with

$$Q_\alpha(s)=[(s-a)^2+b^2]\cdot[(s+a)^2+b^2-\alpha]\cdot[1-c_\alpha s],$$

and $c_\alpha=2a\alpha/(A^2-\alpha A)$, then c_α has the form required in Lemma 1.

There exists an unstable pole-zero cancellation between $P_0(s)$ and $B_\alpha(s)$ at $s=p$ when $c_\alpha=1/p$, i.e.,

$$\frac{2a\alpha}{A^2-\alpha A}=\frac{1}{p}\leftrightarrow\alpha=\frac{A^2}{2ap+A}.$$

Then, according to Lemma 1, $\hat{\alpha}=A^2/(2ap+A)$, and there exists $\alpha^*\in(0,\hat{\alpha})$ and $\omega^*\in\Re$ so that

$$B_{\alpha^*}(\mathrm{j}\omega^*)P_0(\mathrm{j}\omega^*)C(\mathrm{j}\omega^*)=-1,$$

which means that $C(s)$ does not stabilize $B_{\alpha^*}(\mathrm{j}\omega^*)P_0(s)$. Note that $\omega\neq0$, for otherwise, $C(s)$ would not stabilize $P_0(s)$.

Below we want to show that $\mathrm{e}^{-\mathrm{j}\omega^*T}=B_{\alpha^*}(\mathrm{j}\omega^*)$ for some $T>0$, thereby

$$\mathrm{e}^{-\mathrm{j}\omega^*T}P_0(\mathrm{j}\omega^*)C(\mathrm{j}\omega^*)=-1,$$

which implies that $C(s)$ does not stabilize $\mathrm{e}^{-sT}P_0(s)$.

Note that $|B_{\alpha^*}(\mathrm{j}\omega^*)|=1=|\mathrm{e}^{-\mathrm{j}\omega^*T}|$ for all $T>0$, we only need to match their phase. Then, we make the following claim, which is proved in Appendix B.

Claim 1. Suppose the phase of $B_{\alpha^*}(\mathrm{j}\omega^*)$ and $\mathrm{e}^{-\mathrm{j}\omega^*T}$ are matched, then

$$-\omega^*T+2\mu\pi=2\arctan\left(\frac{\mathrm{Im}Q_{\alpha^*}(\mathrm{j}\omega^*)}{\mathrm{Re}Q_{\alpha^*}(\mathrm{j}\omega^*)}\right)\tag{2}$$

with $\mu = 0, \pm1, \pm2, ...$, $\mathrm{Im}Q_\alpha(\mathrm{j}\omega) = -c_\alpha\Lambda(\alpha,\omega)\omega^3$, and $\mathrm{Re}Q_\alpha(\mathrm{j}\omega) = \Lambda(\alpha,\omega)\omega^2 + 2a\alpha c_\alpha\omega^2 + A^2 - \alpha A$, $\alpha^* \in (0, \hat{\alpha})$.

In view of $a \geqslant b > 0$, and $\alpha^* < \hat{\alpha} = A^2/(2ap + A) < A$, it follows that $A^2 - \alpha^* A > 0$, $4a^2 - 2A \geqslant 0$, and $\Lambda(\alpha,\omega) = \omega^2 + 4a^2 + \alpha - 2A > 0$, then we obtain $\mathrm{Re}Q_{\alpha^*}(\mathrm{j}\omega^*) > 0$ and $\mathrm{Im}Q_{\alpha^*}(\mathrm{j}\omega^*)/\omega^* < 0$ for all $\alpha^* \in (0, \hat{\alpha})$ and $\omega^* \neq 0$. Therefore, by using (2), we obtain a smallest positive solution for T, denoted by T^*, i.e.,

$$T^* = \frac{2}{\omega^*}\arctan\left(\frac{-\mathrm{Im}Q_{\alpha^*}(\mathrm{j}\omega^*)}{\mathrm{Re}Q_{\alpha^*}(\mathrm{j}\omega^*)}\right).$$

Because $c_\alpha = 2a\alpha/(A^2 - \alpha A)$ and $\Lambda(\alpha,\omega)$ are increasing functions of α, $c_\alpha < c_{\hat{\alpha}}$ and $\Lambda(\alpha,\omega) < \Lambda(\hat{\alpha},\omega)$ for all $\alpha \in (0, \hat{\alpha})$. Given any $\omega > 0$, we have

$$2a\alpha c_\alpha\omega^2 + A^2 - \alpha A = 2a\alpha\omega^2\frac{2a\alpha}{A^2 - \alpha A} + A^2 - \alpha A \geqslant 2a\hat{\alpha}\omega,$$

then

$$\begin{aligned} T^* &= \frac{2}{\omega^*}\arctan\left(\frac{c_{\alpha^*}\Lambda(\alpha,\omega^*){\omega^*}^3}{\Lambda(\alpha,\omega^*){\omega^*}^2 + 2a\alpha^* c_{\alpha^*}{\omega^*}^2 + A^2 - \alpha^* A}\right) \\ &< \frac{2}{\omega^*}\arctan\left(\frac{c_{\hat{\alpha}}\Lambda(\hat{\alpha},\omega^*){\omega^*}^3}{\Lambda(\hat{\alpha},\omega^*){\omega^*}^2 + 2a\hat{\alpha}\omega^*}\right) \\ &= \frac{2}{\omega^*}\arctan\left(\frac{\Lambda(\hat{\alpha},\omega^*){\omega^*}^2}{p(\Lambda(\hat{\alpha},\omega^*)\omega^* + 2a\hat{\alpha})}\right) \\ &= f(\omega^*), \end{aligned}$$

where we used the fact that $c_{\hat{\alpha}} = 1/p$. Hence, $C(s)$ cannot stabilize $\mathrm{e}^{-sT^*}P_0(s)$, so

$$\mathrm{DM}(P_0, C) < T^* < f(\omega^*).$$

Since this holds for every $C \in \mathcal{F}(RH_\infty)$, and $f(\omega^*)$ is an even function of ω^*, we obtain

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} f(\omega).$$

It follows from Lemma 2 that

$$f(\omega) = \frac{2}{\omega}\arctan\left(\frac{\Lambda(\hat{\alpha},\omega)\omega^2}{p(\Lambda(\hat{\alpha},\omega)\omega + 2a\hat{\alpha})}\right) < \frac{2\Lambda(\hat{\alpha},\omega)\omega}{p(\Lambda(\hat{\alpha},\omega)\omega + 2a\hat{\alpha})} < \frac{2}{p},$$

which completes the proof of this theorem.

Remark 2. From Theorem 1, we see that $\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} f(\omega) \leqslant 2/p$. Note that $2/p$ is the bound obtained in Theorem 7 of [8], which studied plant with a real unstable pole at p. Then, we give an improved (smaller) bound when the plant has also a pair of unstable complex poles.

Now, we extend the above result to plant with a real non-minimum phase zero, an unstable real pole, and a pair of unstable complex conjugate poles.

Theorem 2. Suppose that $P_0(s)$ has a real zero at $z > 0$, and poles at $p > 0$ and $a \pm b\mathrm{j}$ with $a \geqslant b > 0$, $z > p$, then

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} g(\omega),$$

where

$$g(\omega) = f(\omega) - \frac{2}{\omega}\arctan\frac{\omega}{z} < \frac{2}{p} - \frac{2}{z}, \tag{3}$$

and $f(\omega)$ is defined in (1).

Proof. Let $B_\alpha(s) = Q_\alpha(s)/\bar{Q}_\alpha(-s)$, with

$$Q_\alpha(s) = [(s-a)^2 + b^2] \cdot [(s+a)^2 + b^2 - \alpha] \cdot [1 - c_\alpha s] \cdot [z + s],$$

and $c_\alpha = 2a\alpha/(A^2 - \alpha A)$. From the proof of Theorem 1, we have

$$T^* < \frac{2}{\omega^*} \arctan\left(\frac{\Lambda(\hat{\alpha}, \omega^*)\omega^{*2}}{p(\Lambda(\hat{\alpha}, \omega^*)\omega^* + 2a\hat{\alpha})}\right) - \frac{2}{\omega^*} \arctan \frac{\omega^*}{z} = f(\omega^*) - \frac{2}{\omega^*} \arctan \frac{\omega^*}{z}.$$

The proof of (3) is obtained from Lemma 2. The remainder of the argument is quite similar to that given in Theorem 1, so we omit the details of which.

Remark 3. From Theorem 2, we see that $\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} f(\omega) \leqslant 2/p - 2/z$. Note that $2/p - 2/z$ is the upper bound of the achievable delay margin obtained in Theorem 15 of [8], which studied plant with a pole at $p > 0$ and a zero at $z > 0$. Then, we give an improved (smaller) bound when the plant has also a pair of unstable complex poles.

In the following, we derive an upper bound of the delay margin of plant with a real non-minimum phase zero and a pair of unstable complex conjugate poles.

Theorem 3. Suppose that $P_0(s)$ has a zero at $z > 0$, and complex poles at $a \pm b\mathrm{j}$ with $0 < a < 2z$, then

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} f(\omega), \tag{4}$$

where

$$f(\omega) = \frac{2}{\omega}\left[\arctan\left(\frac{\omega+b}{a}\right) + \arctan\left(\frac{\omega-b}{a}\right) - \arctan\frac{\omega}{z}\right],$$

and

$$\frac{4a}{a^2+b^2} - \frac{2}{z} \leqslant f(\omega) \leqslant \frac{4}{a} - \frac{2}{z}. \tag{5}$$

Proof. Let $B_\alpha(s) = Q_\alpha(s)/\bar{Q}_\alpha(-s)$, with

$$Q_\alpha(s) = [a + b\mathrm{j} - \alpha s] \cdot [a - b\mathrm{j} - \alpha s] \cdot [z + s]. \tag{6}$$

Along the proof of Theorem 1, by using Lemma 1, we can obtain (4). The proof of (5) is trivial. The readers can refer to Theorem 2 in [10], where without $\arctan(\omega/z)$ and $2/z$.

Remark 4. The all-pass function $B_\alpha(s)$ in (6) is inspired by the work of [10], which gave some delay radii and bounds of multi-input multi-output systems. It should be point out that there is a small mistake in the Corollary 1 of [10], the condition is $a < z$ (a is referred to as $\mathrm{Re}(p)$ in the paper).

Remark 5. When $b = 0$, i.e., there are one real unstable pole and one real non-minimum phase zero. From Theorem 3, we obtain the delay margin $\mathrm{DM}(P_0) \leqslant 4/a - 2/z$, while from Theorem 15 of [8], the delay margin $\mathrm{DM}(P_0) \leqslant 2/a - 2/z$. So Theorem 3 may give some conservative results.

Remark 6. The merit of the above methods is to construct a suitable all-passtransfer function according to the unstable poles in the plant. The present work is helpful in dealing with more general cases, such as plants involving multiple unstable real and complex poles, and even non-minimum phase zeros.

For example, when a plant with two pairs of unstable complex poles at $a \pm b\mathrm{j}$ and $c \pm d\mathrm{j}$, we can design an all-pass transfer function $B_\alpha(s) = Q_\alpha(s)/\bar{Q}_\alpha(-s)$, with

$$Q_\alpha(s) = [(s-a)^2 + b^2] \cdot [(s+a)^2 + b^2 + \alpha] \cdot [1 - c_\alpha s],$$

where

$$c_\alpha = \frac{2a\alpha}{A(A-\alpha)\cos\phi} \mathrm{e}^{\mathrm{j}\phi},$$

$A = a^2 + b^2$, and $\phi = \tan^{-1}(d/c)$. Following the above method, we can obtain some estimates of the upper bound of the achievable delay margin.

4 Numerical example

In this section, we provide a plant with unstable complex poles, obtain some upper bounds of their achievable delay margin by using the existing methods and the proposed methods, and give some comparisons between them.

Consider a plant $P_0(s)$ with poles at 2, $2 \pm \mathrm{j}$ and zero at $z = 5$.

In the following, we first consider the unstable poles, then consider both the pole and the zero.

1) As the plant has poles at 2 and $2 \pm \mathrm{j}$, by using Theorem 7 and Theorem 9 in [8], we obtain $\mathrm{DM}(P_0) \leqslant 1$, and $\mathrm{DM}(P_0) \leqslant 1.4283$, respectively.

By using Theorem 1 in this paper, we have

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} \frac{2}{\omega} \arctan \Delta_0(\omega) \approx 0.6481,$$

with

$$\Delta_0(\omega) = \frac{(\omega^2 + 103/13)\omega^2}{2(\omega^2 + 103/13)\omega + 200/13}. \tag{7}$$

The upper bound of the delay margin is obtained at $\omega \approx 2.1$.

Since $0.6481 < \min\{1, 1.4283\}$, as stated in Remark 2, we can obtain a tighter upper bound by using Theorem 1.

2) As the plant has zero at $z = 5$ and pole at 2, by using Theorem 15 in [8],

$$\mathrm{DM}(P_0) \leqslant \frac{2}{2} - \frac{2}{5} = 0.6.$$

As the plant has zero at $z = 5$ and poles at $2 \pm \mathrm{j}$, by using Theorem 3,

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} f(\omega) \approx 1.2,$$

where

$$f(\omega) = \frac{2}{\omega} \left[\arctan\left(\frac{\omega+1}{2}\right) + \arctan\left(\frac{\omega-1}{2}\right) - \arctan\frac{\omega}{5} \right],$$

and the upper bound is obtained as $\omega \to 0$.

When considering all the plant poles and zero, by using Theorem 2,

$$\mathrm{DM}(P_0) \leqslant \sup_{\omega>0} \frac{2}{\omega} \left(\arctan \Delta_0(\omega) - \arctan\frac{\omega}{3} \right) \approx 0.2709,$$

where $\Delta_0(\omega)$ is defined in (7), and the upper bound is obtained at $\omega \approx 2.257$.

Since $0.2709 < 0.6 < 1.2$, as stated in Remarks 3 and 5, Theorem 2 gives a more tight result than Theorem 15 in [8], while Theorem 3 may give a conservative result.

From above, we notice that

$$0.2709 < \min\{1, 1.4283, 0.6481, 0.6, 1.2\},$$

then Theorem 2 gives better results than any others. Moreover, it is reasonable that the more unstable poles and non-minimum phase zeros the plant have, the less input delay the plant can tolerate.

5 Conclusion

We extended the achievable delay margin for plant with unstable complex poles. In particular, we mainly studied the case of plants with a real pole and a pair of complex conjugate poles. This is significantly more complex than plant with only one unstable real pole or one pair of unstable complex conjugate poles. An improved upper bound of the delay margin has been given by using frequency domain method. The result shows that the delay margin can in general be computed by solving an optimization problem, which is maximizing a real function with only one variable.

In the future, we will focus on designing a rational LTI controller to satisfy delay margin larger than or equal to a given value τ ($\tau < \mathrm{DM}(P_0)$). We think this will be more practical in real applications although the work is more challenging.

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Appendix A Proof of Lemma 1

We first write $P_\alpha(s) = P_0(s)B_\alpha(s)$ as the ratio of two polynomials, which have no common zeros in $\bar{\mathcal{C}}^+$ for small $\alpha \geqslant 0$.

If i) holds, write

$$P_0(s) = g \cdot \frac{\bar{n}_0(s)}{[(s-a)^2+b^2]\cdot \bar{d}_0(s)}$$

with $\bar{n}_0(s)$ and $[(s-a)^2+b^2]\cdot\bar{d}_0(s)$ monic and coprime. Let

$$\begin{cases} n_\alpha(s) :=[(s+a)^2+b^2-\alpha]\cdot(1-c_\alpha s)\cdot\bar{n}_0(s), \\ d_\alpha(s) :=[(s+a)^2+b^2]\cdot[(s-a)^2+b^2-\alpha]\cdot(1+\bar{c}_\alpha s)\cdot\bar{d}_0(s). \end{cases}$$

If ii) holds, write

$$P_0(s) = g \cdot \frac{\bar{n}_0(s)(z-s)}{[(s-a)^2+b^2]\cdot \bar{d}_0(s)}$$

with $\bar{n}_0(s)(z-s)$ and $[(s-a)^2+b^2]\cdot\bar{d}_0(s)$ monic and coprime. Let

$$\begin{cases} n_\alpha(s) :=[(s+a)^2+b^2-\alpha]\cdot(1-c_\alpha s)\cdot[z+s]\cdot\bar{n}_0(s), \\ d_\alpha(s) :=[(s+a)^2+b^2]\cdot[(s-a)^2+b^2-\alpha]\cdot(1+\bar{c}_\alpha s)\cdot\bar{d}_0(s). \end{cases}$$

If iii) holds, write

$$P_0(s)=g\cdot\frac{\bar{n}_0(s)(z-s)}{\bar{d}_0(s)}$$

with $\bar{n}_0(s)(z-s)$ and $\bar{d}_0(s)$ monic and coprime. Let

$$\begin{cases} n_\alpha(s) :=[a+b\mathrm{j}-\alpha s]\cdot[a-b\mathrm{j}-\alpha s]\cdot[z+s]\cdot\bar{n}_0(s), \\ d_\alpha(s) :=[a-b\mathrm{j}+\alpha s]\cdot[a+b\mathrm{j}+\alpha s]\cdot\bar{d}_0(s). \end{cases}$$

Then, combing the above three cases, we can write

$$P_\alpha(s)=P_0(s)B_\alpha(s)=g\cdot\frac{n_\alpha(s)}{d_\alpha(s)}.$$

It is straightforward to verify that $n_\alpha(s)$ and $d_\alpha(s)$ have no common zeros in $\bar{\mathcal{C}}^+$ for small $\alpha\geqslant 0$. But this property would be lost at $\hat{\alpha}>0$ for there is an unstable pole-zero cancellation in $P_\alpha(s)$. Based on the continuity of the zeros of the characteristic polynomial as a function of the parameter α, there exist an $\alpha^*\in[0,\hat{\alpha})$ so that the polynomial has a zero on the imaginary axis, i.e., $s=\mathrm{j}\omega^*$, which means that $B_{\alpha^*}(\mathrm{j}\omega^*)P_0(\omega^*)C(\omega^*)=-1$. Thus, the proof is completed.

Appendix B Proof of Claim 1

Following the notation of $A=a^2+b^2$, and

$$Q_\alpha(s)=[a+b\mathrm{j}-\alpha s]\cdot[a-b\mathrm{j}-\alpha s]\cdot[z+s],$$

at $s=\mathrm{j}\omega$, we have

$$\begin{aligned} Q_\alpha(\mathrm{j}\omega) =&[(\mathrm{j}\omega-a)^2+b^2][(\mathrm{j}\omega+a)^2+b^2-\alpha][1-\mathrm{j}\omega c_\alpha] \\ =&[A-\omega^2-2a\omega\mathrm{j}][A-\omega^2+2a\omega\mathrm{j}-\alpha][1-\mathrm{j}\omega c_\alpha] \\ =&[(A-\omega^2)^2-\alpha(A-\omega^2)+4a^2\omega^2+2a\alpha\omega\mathrm{j}](1-\mathrm{j}c_\alpha\omega) \\ =&[\omega^4+\omega^2(4a^2+\alpha-2A+2a\alpha)+A^2-\alpha A] \\ &-\mathrm{j}[c_\alpha\omega^5+c_\alpha\omega^3(4a^2+\alpha-2A)+c_\alpha(A^2-\alpha A)-2a\alpha]. \end{aligned}$$

Since

$$c_\alpha=\frac{2a\alpha}{A^2-\alpha A},$$

so $c_\alpha(A^2-\alpha A)-2a\alpha=0$, and note that $\Lambda(\alpha,\omega)=\omega^2+4a^2+\alpha-2A$, then from the above equation, we have

$$\begin{aligned} Q_\alpha(\mathrm{j}\omega) =&[\omega^4+\omega^2(4a^2+\alpha-2A+2a\alpha)+A^2-\alpha A]-\mathrm{j}[c_\alpha\omega^5+c_\alpha\omega^3(4a^2+\alpha-2A)] \\ =&[\Lambda(\alpha,\omega)\omega^2+2a\alpha c_\alpha\omega^2+A^2-\alpha A]-\mathrm{j}c_\alpha\Lambda(\alpha,\omega)\omega^3 \\ =&\mathrm{Re}Q_\alpha(\mathrm{j}\omega)-\mathrm{j}\mathrm{Im}Q_\alpha(\mathrm{j}\omega), \end{aligned}$$

as claimed.