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A framework for stability analysis of high-order nonlinear systems based on the CMAC method

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Abstract A framework for analyzing the stability of a class of high-order minimum-phase nonlinear systems of relative degree two based on the characteristic model-based adaptive control (CMAC) method is presented. In particular, concerning the tracking problem for such high-order nonlinear systems, by introducing a consistency condition for quantitatively describing modeling errors corresponding to a group of characteristic models together with a certain kind of CMAC laws, we prove closed-loop stability and show that such controllers can make output tracking error arbitrarily small. Furthermore, following this framework, with a specific characteristic model and a golden-section adaptive controller, detailed sufficient conditions to stabilize such groups of high-order nonlinear systems are presented and, at the same time, tracking performance is analyzed. Our results provide a new perspective for exploring the stability of some high-order nonlinear plants under CMAC, and lay certain theoretical foundations for practical applications of the CMAC method.

Keywords characteristic model, characteristic model-based adaptive control (CMAC), consistency condition, stability, high-order nonlinear system

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1 Introduction

A model that is derived for facilitating controller design by taking into account plant dynamic properties, environmental characteristics, and control performance requirements can be called a characteristic model of a plant provided that the model and the plant are equivalent in output [1]. Generally, the resulting characteristic model can be represented by some low-order, time-varying difference equation, making low-order controller design easier. The characteristic model-based golden-section adaptive control (GSAC) method originally proposed in the 1990s by Wu, is a widely used low-order control law with simplicity of design, convenience of adjustment, and strong robustness [2, 3]. Until now, the idea of characteristic model-based adaptive control (CMAC) has been applied successfully to more than 400 systems belonging to 11 kinds of engineering plants in the fields of astronautics and industry (see, e.g., [4–8]). In particular, during reentry of the Shenzhou spacecraft and automatic rendezvous and docking of the Shenzhou spacecraft with Tiangong-1, control precision reached the top level in the world [1, 3, 4, 7].

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Apart from these practical applications, much progress has also been made in the theoretical research on the CMAC method. Among this work, related stability issues have been discussed actively. As cornerstone research, Xie and Wu proved the asymptotic stability of second-order, linear, time-invariant characteristic models under the golden-section controller with time-invariant parameters [9]. For characteristic models with time-varying parameters, various stability conditions in both the SISO and MIMO cases, corresponding to different adaptive control methods, were proposed by Wu and his co-workers by virtue of Lyapunov stability theory [1, 10, 11]. In addition, two practical engineering systems were investigated in [12, 13] and, in each case, the explicit characteristic model-based control law, together with the stability of the corresponding characteristic model under this controller, was presented. Moreover, on the basis of the above work, the stability of the closed-loop system composed of the plant and the CMAC was further investigated. Technically, such closed-loop stability issues are more complicated, as the characteristic modeling process has to be involved in the stability analysis. In recent work [14], the characteristic model-based pole placement method was developed to stabilize a class of SISO systems while the corresponding stability condition imposed on the modeling error was stringent. Following the existing framework for stabilization of sampled-data systems via their approximate discrete-time models (see, e.g., [15, 16]), closed-loop stability under the GSAC law was proved in [17] for a class of nonlinear systems with a relative degree of two and exponentially stable zero dynamics. Nevertheless, the persistent excitation (PE) condition was required. By introducing new techniques to remove the PE condition, our recent work [18] developed a group of sufficient conditions for stabilization of a class of second-order nonlinear systems via the GSAC which depend only on plant properties, controller parameters, and sampling period. As a consequence, all of the above work represents a case-by-case study in the sense that each stability result is achieved based on a certain specific characteristic model, along with some characteristic model-based controllers, that lacks generality and portability.

In this paper, from a more general perspective, we further investigate the above closed-loop stability issue, aiming to establish a framework used for analyzing the stability of some high-order nonlinear systems under the CMAC scheme. Specifically, for a class of high-order minimum-phase nonlinear systems of relative degree two, with a group of characteristic models in general form and certain kinds of sampled-data output feedback controllers based on such characteristic models, a framework will be established corresponding to the following basic question: "if this group of characteristic models can be stabilized by a characteristic model-based sampled-data output feedback controller, then under what condition can such a controller stabilize this class of high-order nonlinear systems?"

To that end, we have to overcome two major difficulties. Firstly, compared to the second-order case without internal dynamics, the coupling between internal and external states in the high-order case makes stability analysis more complicated since the stability of both states has to be established simultaneously. Even for minimum-phase systems, it is not acceptable to ignore the coupling between internal and external states groundlessly when designing the controller and demonstrating the stability of the closed-loop system. Secondly, as characteristic model-based controller design essentially belongs to the discrete-time design (DTD) approach for sampled-data systems [15], the corresponding closed-loop system has a hybrid structure, making the analyses more difficult. Moreover, regarding the DTD method, we want to emphasize that, although Nešić et al. have established a framework for the stability of sampled-data systems via their approximate discrete-time models, this framework cannot be inherited to solve our stability issue. One reason for this is that the Nešić's framework is founded on the basis of sampled-data state feedback controllers while, in CMAC, we consider sampled-data output feedback. Besides, in Nešić's framework, uniform boundedness with respect to the sampling period is a basic requirement for the applied sampled-data controller. Nevertheless, CMAC laws such as typical GSAC cannot meet this requirement. Consequently, new techniques must be introduced to overcome these difficulties. Also, noticing that many practical plants may be described by minimum-phase models with a relative degree of two (see, e.g., [19, 20]), it is of great significance, both theoretically and practically, to investigate this issue.

In the subsequent parts, we will formulate the problem in Section 2. Our main results including both a general framework for stability analysis and specific closed-loop stability results, together with their detailed proofs, are presented in Section 3. A numerical example is given in Section 4. Finally, Section 5 concludes this paper with some final remarks.

Notations. In the following discussion, we use the common definitions of class $\mathcal{K}$ and $\mathcal{K}_\infty$ given in [21]. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{N}$ the set of natural numbers. $\|\cdot\|$ is the Euclidean vector norm or the spectral matrix norm, and $\|\cdot\|_1$ the sum vector norm (or l_1 norm) [22]. $\mathbf{0}_{m\times n}$ denotes the m-by-n zero matrix, and I_n the n-dimensional identity matrix. For any matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_i(M)\,(i=1,\ldots,n)$ denotes the ith eigenvalue of M. Let $T>0$ be the sampling period, $\bullet(k) \triangleq \bullet(t)|_{t=kT}$, $k\in\mathbb{N}$. For any given function $\kappa(t)$, $t\in D_\kappa$, the expression $\kappa(t)=O(1)$, $t\in D_\kappa$ means that there exists a positive constant $M_\kappa>0$ such that $|\kappa(t)|\leqslant M_\kappa$, $\forall t\in D_\kappa$.

2 Problem formulation

We consider the following high-order SISO nonlinear system of relative degree two:

$$\begin{cases} \dot{x} = Ax + B\left[a(x,z) + b(x,z)u + d(t)\right], \\ \dot{z} = f_0(x,z), & t\geqslant 0, \\ y = Cx, \end{cases} \tag{1}$$

where $x=[x_1\ x_2]^{\mathrm{T}}\in\mathbb{R}^2$ and $z\in\mathbb{R}^m$ are the state variables, $u\in\mathbb{R}$ is the control input, $y\in\mathbb{R}$ is the measured output, $a(x,z)$, $b(x,z)$, and $f_0(x,z)$ are nonlinear functions that may contain unknown dynamics, and $d(t)$ is an external disturbance. In addition, the triple (A,B,C) is as follows:

$$A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad C=[\,1\ \ 0\,].$$

Throughout the paper, we need the following assumptions:

Assumption 1. The unknown function $a(x,z)$ is differentiable and globally Lipschitz. Moreover, there exists $L>0$ such that for any $(x^{\mathrm{T}},z^{\mathrm{T}})^{\mathrm{T}},(x_*^{\mathrm{T}},z_*^{\mathrm{T}})^{\mathrm{T}}\in\mathbb{R}^{m+2}$,

$$|a(x,z)-a(x_*,z_*)|\leqslant L\left(\|x-x_*\|_1+\|z-z_*\|_1\right). \tag{2}$$

Assumption 2. The unknown control gain $b(x,z)$ is continuous, belongs to a bounded interval which does not contain 0, and has a known sign. Also, the disturbance $d(t)$ is bounded. Without loss of generality, let $\underline{b}$, $\bar{b}$, and d be positive constants satisfying

$$0<\underline{b}\leqslant b(x,z)\leqslant\bar{b},\quad \forall(x,z),\quad \sup_{t\geqslant 0}|d(t)|\leqslant d. \tag{3}$$

Assumption 3. There exists a continuously differentiable function $V_z(z):\mathbb{R}^m\to\mathbb{R}_+$ such that

$$c_1\|z\|^2\leqslant V_z(z)\leqslant c_2\|z\|^2,\quad \dot{V}_z(z)\leqslant -c_3\|z\|^2+c_4\|x\|^2,\quad \forall x\in\mathbb{R}^2,\ z\in\mathbb{R}^m, \tag{4}$$

where $c_i>0\,(i=1,\ldots,4)$ are positive constants.

Remark 1. Notice that the external dynamics of system (1) has integrators in a series structure, which appears to limit the applicable scope of what we can obtain in this paper. Nevertheless, for affine nonlinear systems in general form $\dot{\zeta}=f(\zeta)+g(\zeta)u$, $y=h(\zeta)$, we know from input-output linearization theory (see [21,23]) that it can be transformed into the system (1) by a change of variables $(x,z)=T_\zeta(\zeta)$, provided that such systems have relative degree two and satisfy certain conditions. Hence, many systems in the above general form can also be dealt with by the method we propose in this paper.

Remark 2. Assumption 3 ensures that the system $\dot{z}=f_0(x,z)$ with input x is input-to-state stable (ISS), and that its solution satisfies $\|z(t)\|\leqslant\sqrt{c_2/c_1}\exp\left[-(c_3t)/(2c_2)\right]\|z(0)\|+c_z\sup_{0\leqslant\tau\leqslant t}\|x(\tau)\|$, $\forall t\geqslant 0$, where $c_z=\sqrt{c_2c_4/(c_1c_3)}$.

Here we consider the output tracking problem for a minimum-phase uncertain nonlinear system (1). More specifically, our control objective is to develop a sampled-data output feedback controller to make sure that, for all initial states in any given compact set, the state signals $(x(t), z(t))$ are bounded, and the output $y(t)$ tracks the reference trajectory generated from the target system

$$\dot{x}^*(t) = A_m x^*(t) + Br(t), \quad y^*(t) = Cx^*(t), \quad t \geqslant 0, \tag{5}$$

where $x^*(t) = [x_1^*(t)\ x_2^*(t)]^{\mathrm{T}} \in \mathbb{R}^2$ is the state, $r(t) \in \mathbb{R}$ is the input signal satisfying $|r(t)| \leqslant \bar{r}$ with $\bar{r} > 0$ as a known constant, and A_m is a Hurwitz matrix defined by

$$A_m = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{6}$$

Hence, it is evident that the trajectories of the target system (5) are bounded for any bounded initial states that satisfy

$$\sup_{t\geqslant 0} \|x^*(t)\| \leqslant M^*, \tag{7}$$

where $M^* > 0$ is a constant that depends on the parameters $\{A_m, \bar{r}\}$ and the initial values $x^*(0)$.

Note that this output tracking problem can be converted into a stabilizing one for the corresponding error system, making it convenient to design the controller and analyze control performance. By setting $e_i = x_i - x_i^*$ $(i = 1, 2)$, $y_e = y - y^*$, and combining (1) and (5), we have error dynamics is as follows:

$$\dot{e}_1(t) = e_2(t), \quad \dot{e}_2(t) = a_e(e_1, e_2, z, t) + b_e(e_1, e_2, z, t)u(t), \quad \dot{z}(t) = f_0^e(e_1, e_2, z, t), \quad y_e(t) = e_1(t), \tag{8}$$

where $b_e(e_1, e_2, z, t) \triangleq b(e_1 + x_1^*(t), e_2 + x_2^*(t), z)$, $f_0^e(e_1, e_2, z, t) \triangleq f_0(e_1 + x_1^*(t), e_2 + x_2^*(t), z)$, and

$$a_e(e_1, e_2, z, t) \triangleq a(e_1 + x_1^*(t), e_2 + x_2^*(t), z) + k_1 x_1^*(t) + k_2 x_2^*(t) - r(t) + d(t), \tag{9}$$

in which $a(\cdot), b(\cdot), f_0(\cdot)$, and $d(t)$ are given in (1), and $r(t)$, k_1, k_2 are given in (5) and (6), respectively. Moreover, according to Assumptions 1 and 2, and from (7) and (9), it is not difficult to verify that the nonlinear function $a_e(\cdot)$ defined by (9) has the following properties:

(A1) $a_e(e_1, e_2, z, t)$ is globally Lipschitz in (e_1, e_2, z) uniformly in t. Also, for any (e_1', e_2', z') and (e_1'', e_2'', z''),

$$\begin{aligned} |a_e(e_1', e_2', z', t) - a_e(e_1'', e_2'', z'', t)| &= |a(e_1' + x_1^*(t), e_2' + x_2^*(t), z', t) - a(e_1'' + x_1^*(t), e_2'' + x_2^*(t), z'', t)| \\ &\leqslant L(|e_1' - e_1''| + |e_2' - e_2''| + \|z' - z''\|_1), \quad \forall t \geqslant 0. \end{aligned} \tag{10}$$

(A2) $a_e(0, 0, 0, t)$ is uniformly bounded, and there exists a positive number $M_{a0} > 0$ that depends on the parameters $\{L,\ d,\ k_1,\ k_2,\ \bar{r},\ M^*\}$ such that

$$\sup_{t\geqslant 0} |a_e(0, 0, 0, t)| \leqslant M_{a0}. \tag{11}$$

(A3) $a_e(e_1, e_2, z, t)$ is differentiable with respect to its arguments (e_1, e_2, z), and

$$|a_e(e_1, e_2, z, t_1) - a_e(e_1, e_2, z, t_2)| \leqslant M_{a1}, \quad \forall (e_1, e_2, z^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m+2}, \quad \forall t_1, t_2 \geqslant 0, \tag{12}$$

where $M_{a1} > 0$ is a constant depending on $\{L,\ d,\ k_1,\ k_2,\ \bar{r},\ M^*\}$.

Consequently, our control objective is converted into designing a sampled-data output feedback controller such that $y_e(t) \to 0$ as $t \to \infty$. Centering around this objective, our main results are presented in the subsequent section.

3 Main results

Now we concentrate on the stabilization of high-order nonlinear error systems (8) of relative degree two via the CMAC method. Specifically, a general framework for stability analysis with respect to the CMAC method as well as specific closed-loop stability results for this error system (8) are established below, successively. In the following, if there are no special instructions, we always assume that the sampling period $T \in (0, T_{\max}]$ is bounded with an upper bound $T_{\max} > 0$, and that signals satisfy $y_e(-1) = e_1(-1) = e_1(0) - Te_2(0)$, $e_1(i) = 0\,(\forall i < -1)$, as well as $e_2(i) = u(i) = 0\,(\forall i < 0)$.

3.1 A general framework for stability analysis

Basically, the framework for stability analysis to be established shortly arises from exploration of the following question: if a characteristic model of a plant is stabilized by some characteristic model-based adaptive controller, then under what conditions can such a CMAC law stabilize the plant? To answer this, analyzing quantitatively the modeling error of the characteristic model becomes a key point. Along the way, a consistency condition is introduced first to quantitatively describe the permissible modeling error of the characteristic model. Then, using this consistency condition, our framework for stability analysis will be established.

3.1.1 *Consistency condition*

In order to present the consistency condition, two elements need to be specified beforehand — the form of characteristic models and the scope of the sampled-data output feedback controls. Here we focus on the commonly used class of second-order characteristic models, which is described as follows:

$$\begin{cases} e_1(k+1) = f_1(k)e_1(k) + f_2(k)e_1(k-1) + g_0(k)u(k) + g_1(k)u(k-1), & (13) \\ Te_2(k+1) = \nu(e_1(k+1), e_1(k), z, u(k), k+1), & (14) \end{cases}$$

where the coefficients $\{f_1(k), f_2(k), g_0(k), g_1(k)\}$ satisfy

$$\begin{aligned} &|f_1(k) - 2| \leqslant \epsilon_1(T), \quad |f_2(k) + 1| \leqslant \epsilon_2(T), \\ &0 < \epsilon_{01}(T^2) \leqslant g_0(k) \leqslant \epsilon_{02}(T^2), \quad 0 < \epsilon_{11}(T^2) \leqslant g_1(k) \leqslant \epsilon_{12}(T^2), \end{aligned} \tag{15}$$

in which $\epsilon_1(\cdot), \epsilon_2(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ are class $\mathcal{K}$ functions, and the functions $\epsilon_{ij}(T^2)\,(i = 0, 1, j = 1, 2)$ are of the same order as T^2 for any $T \in (0, T_{\max}]$, whose values might also depend on the system states. Moreover, the function $\nu(\cdot)$ is required to have the following property: for any $T \in (0, T_{\max}]$ and constant $C_\nu \geqslant 0$, if $\|(e_1(k+1), e_1(k), \bar{\epsilon}(T^2)u(k))\|_1 \leqslant C_\nu$ in which $\bar{\epsilon}(T^2) \triangleq \max\{\epsilon_{02}(T^2), \epsilon_{12}(T^2)\}$, then

$$|\nu(e_1(k+1), e_1(k), z, u(k), k+1)| \leqslant \tau_\nu(C_\nu), \quad \forall z \in \mathbb{R}^m, \ \forall k \geqslant 0, \tag{16}$$

where $\tau_\nu(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a class $\mathcal{K}$ function.

We remark that the second-order time-varying difference equation (13) is established for designing sampled-data output controllers. Besides, compared to the normal characteristic model in the existing characteristic modeling theory (see [1]), the additional equation (14), as part of our characteristic model, is also taken into account. As a matter of fact, this equation (14) plays an important role in analyzing the stability of the state e_2. We also remark that, since the system (1) we consider in this paper is minimum-phase and satisfies Assumption 3, the equation describing the internal dynamics z is not involved in the above characteristic model (13) and (14). The details to support this statement can be found in the proofs of our main results. As for other more general minimum-phase systems, when deriving characteristic models, the questions of whether it is necessary to take internal dynamics z into account and how to compress the information z into the coefficients of model (13) are still open.

Now having the characteristic model (13) and (14), we further need the exact discrete-time model of the system (8), which can be denoted by

$$e_1(k+1) = F_1^{ex}(e_1, e_2, z, u(k), k), \quad e_2(k+1) = F_2^{ex}(e_1, e_2, z, u(k), k). \tag{17}$$

Corresponding to the above characteristic model (13) and (14), we also introduce an equivalence form of the exact discrete-time model (17) represented by

$$\begin{cases} e_1(k+1) = f_1(k)e_1(k) + f_2(k)e_1(k-1) + g_0(k)u(k) + g_1(k)u(k-1) + \Delta_1(k+1), & (18)\\ Te_2(k+1) = \nu(e_1(k+1), e_1(k), z, u(k), k+1) + \Delta_2(k+1), & (19)\end{cases}$$

where the time-varying parameters $\{f_1(k),\ f_2(k),\ g_0(k),\ g_1(k)\}$ have the properties given in (15), $\nu(\cdot)$ is the function defined by (14) which satisfies (16), and $\Delta_1(k+1)$, $\Delta_2(k+1)$ are the modeling errors that depend on both the system states and the control input.

Another element that needs to be specified is the scope of sampled-data output feedback controllers. In view of the fact that the model (13) used for the controller design is linear, and from the practicality of control laws, we take the following set as our admissible control set:

$$\mathcal{U} \triangleq \left\{ u = \{u_T(k), k \geqslant 0\} :\ u_T(k) = \sum_{i=k-p}^{k} \alpha_{T,k-i}(k) y_e(i),\ \forall k \geqslant 0 \right\}, \tag{20}$$

where, at each step $k \geqslant 0$, the control signal $u_T(k)$ is a linear combination of the past and present output signals $\{y_e(k), \ldots, y_e(k-p)\}$ $(p \geqslant 0)$, and the time-varying control parameters $\{\alpha_{T,i}(k),\ i = 0, \ldots, p\}$ usually depend on the coefficients of the characteristic model, which may be calculated by some parameter estimator. Moreover, by the properties (15), we also set that, for all $T \in (0, T_{\max}]$,

$$\sup_{k \geqslant 0} \max_{0 \leqslant j \leqslant p} \left\{ \bar{\epsilon}(T^2) |\alpha_{T,j}(k)|,\ T^2 |\alpha_{T,j}(k)| \right\} \leqslant \bar{\alpha}_u, \tag{21}$$

where $\bar{\alpha}_u > 0$ is a constant, and $\bar{\epsilon}(T^2) = \max\{\epsilon_{01}(T^2), \epsilon_{02}(T^2)\}$ in which $\epsilon_{01}(T^2), \epsilon_{02}(T^2)$ are given by (15). Obviously, the widely used discrete-time PD control, golden-section adaptive control, and minimum-variance adaptive control all belong to the above admissible control set. Thus, the sampled-data controllers we are concerned with not only have a simple linear structure, bearing practicality in engineering, but also have generality to a certain extent. In the subsequent discussion, to simplify notation we drop the subscript T in the variables $u_T(k)$ and $\alpha_{T,i}(k)$ if no ambiguity is caused.

We are now in a position to provide the consistency condition. To describe it more clearly and concisely, we utilize the ordered pair $\{F^{cm},\ u\}$ to represent the family containing the characteristic model (13) and (14) and some admissible sampled-data feedback control $u \in \mathcal{U}$. Similarly, the family $\{F^{ex},\ u\}$ corresponds to the exact discrete-time model (17), or the equivalent one (18) and (19), and the same admissible control u. In addition, the following two simple definitions are also needed.

Definition 1. A vector $(a_0, \ldots, a_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$ is stable if the roots of the corresponding polynomial $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ are all located in the unit circle.

Definition 2. For any n-dimensional real column vectors $A' = (a_1', \ldots, a_n')^{\mathrm{T}}, A'' = (a_1'', \ldots, a_n'')^{\mathrm{T}} \in \mathbb{R}^n$,

- $A' > A''$ if their elements satisfy $a_i' > a_i''$, $\forall 1 \leqslant i \leqslant n$;
- $A' \geqslant A''$ if their elements satisfy $a_i' \geqslant a_i''$, $\forall 1 \leqslant i \leqslant n$. In particular, if $a_i' \geqslant 0\,(\forall 1 \leqslant i \leqslant n)$, we also say that the vector A' is non-negative.

Then the consistency condition that quantitatively describes the admissible modeling error of the characteristic model is given as follows.

Consistency. The family $\{F^{cm}, u\}$ is said to be consistent with $\{F^{ex}, u\}$ if there exists a non-negative integer $\bar{p}_1 \in \mathbb{N}$ such that the corresponding modeling errors $\Delta_1(k+1)$, $\Delta_2(k+1)$ satisfy: for each compact set $\Omega \subset \mathbb{R}^{(m+2)(\bar{p}_1+1)}$, there exist class $\mathcal{K}$ functions $\varrho_{i0}(\cdot),\ \varrho_j(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+\ (i = 1, 2; j = 0, \ldots, \bar{p}_1)$, and a non-negative stable vector $(\lambda_0^*, \ldots, \lambda_{\bar{p}_1}^*)^{\mathrm{T}} \in \mathbb{R}^{\bar{p}_1+1}$, such that for all $T \in (0, T_{\max}]$, and any $k \geqslant 0$, when $[e_1(k), \ldots, e_1(k-\bar{p}_1), Te_2(k), \ldots, Te_2(k-\bar{p}_1), Tz^{\mathrm{T}}(k), \ldots, Tz^{\mathrm{T}}(k-\bar{p}_1)]^{\mathrm{T}} \in \Omega$, we have

$$|\Delta_1(k+1)| \leqslant \varrho_{10}(T), \tag{22}$$

$$|\Delta_2(k+1)| \leqslant \textstyle\sum_{j=0}^{\bar{p}_1} \left[\varrho_{\bar{p}_1-j}(T) + \lambda_{\bar{p}_1-j}^*\right] T|e_2(k-j)| + \varrho_{20}(T). \tag{23}$$

Furthermore, this group of conditions (22) and (23) with respect to the modeling errors $\Delta_1(k)$ and $\Delta_2(k)$ are called the consistency condition.

As can be seen, the above consistency condition explicitly illuminates the upper bounds of the acceptable modeling errors between $\{F^{cm}, u\}$ and $\{F^{ex}, u\}$. Also, such bounds decrease with the reduction of the sampling period T, which will help to prove the closed-loop stability with regard to the plant (1).

3.1.2 *Main theorems*

Now, based on the above consistency condition, we present the stability of the closed-loop system composed of the admissible sampled-data feedback controller (20) and the system (1).

Theorem 1. Consider a high-order nonlinear plant (1) satisfying Assumptions 1–3, the characteristic model (13) and (14), as well as the admissible sampled-data output feedback controller (20). If

(C1) the family $\{F^{cm}, u\}$ is consistent with $\{F^{ex}, u\}$, and

(C2) there exists $T_* > 0$ such that for all $T \in (0, T_*)$ the discrete-time closed-loop system comprised of the characteristic model (13) and the controller (20) is exponentially stable,

then for any $\rho_0 > 0$, and any initial state $\|(x_1(0), x_2(0), z^{\mathrm{T}}(0))\|_1 \leqslant \rho_0$, there exist $T^* < T_*$ and a class $\mathcal{K}$ function $\varrho_e^*(\cdot)$ such that for each $T \in (0, T^*)$ the trajectories of the closed-loop system composed of this controller (20) and the plant (1) satisfy the following properties:

(1) boundedness:

$$\sup_{t\geqslant 0}\,[|x_1(t)| + T|x_2(t)| + T\|z(t)\|_1] = O(1); \tag{24}$$

(2) Tracking performance:

$$\limsup_{t\to\infty}\,|y(t) - y^*(t)| = O(\varrho_e^*(T)). \tag{25}$$

Furthermore, if $x(0) = x^*(0)$, then there exists $\varrho^{e*}(\cdot) \in \mathcal{K}$ such that $\sup_{t\geqslant 0}|y(t) - y^*(t)| = O(\varrho^{e*}(T))$.

Remark 3. This result illustrates that, for such minimum-phase nonlinear systems (1) with a relative degree of two, an answer to that basic question we mentioned earlier in Subsection 3.1 is the consistency condition, i.e., the condition (C1). Clarifying this, the framework we pursue is established.

Remark 4. The transient performance of the closed-loop system relates to its initial state, and it can be improved as the absolute value of the initial state decreases. Also, for any bounded initial state, the tracking error enters ultimately into a neighborhood of the origin with a radius of $O(\varrho_e^*(T))$.

Remark 5. Theoretically, the steady tracking error can be made arbitrarily small by making the sampling period T small enough. However, in practice, there are some limitations on the value of the sampling period due to various physical constraints. The corresponding stability issue when the sampling period is prescribed beforehand remains for further investigation.

Next, we prove Theorem 1. To this end, we first analyze the properties of the solution of the error system (8). For any given sampled-data control signals $\{u(k),\, k \geqslant 0\}$, the solution of system (8) satisfies

$$\begin{cases} e_1(t) = e_1(k) + (t-kT)e_2(k) + \displaystyle\int_{kT}^{t}(t-\tau)[a_e(e_1(\tau), e_2(\tau), z(\tau), \tau) \\ \qquad\qquad + b_e(e_1(\tau), e_2(\tau), z(\tau), \tau)u(k)]\mathrm{d}\tau, \\ e_2(t) = e_2(k) + \displaystyle\int_{kT}^{t}[a_e(e_1(\tau), e_2(\tau), z(\tau), \tau) + b_e(e_1(\tau), e_2(\tau), z(\tau), \tau)u(k)]\,\mathrm{d}\tau, \\ z(t) = z(k) + \displaystyle\int_{kT}^{t} f_0^e(e_1(\tau), e_2(\tau), z(\tau), \tau)\mathrm{d}\tau, \quad \forall t \in [kT, (k+1)T]. \end{cases} \tag{26}$$

One property of this solution is given as follows.

Lemma 1. Consider the solution (26) of the error system (8). Suppose that Assumptions 1–3 are satisfied; then, for any sampled-data control signals $\{u(k), k \geqslant 0\}$,

$$\begin{aligned} \omega_k(t) \leqslant\, & p_\omega(t-kT)\{|e_1(k)| + |e_2(k)| + (t-kT)|e_2(k)| + [(t-kT) + (t-kT)^2/2][\bar{b}|u(k)| \\ & + M_{a0} + L\sqrt{mc_2/c_1}\|z(k)\|_1 + \sqrt{mc_4/c_1}M^*L(t-kT)^{1/2}]\}, \end{aligned} \tag{27}$$

$$\|z(t)\|_1 \leqslant \sqrt{\frac{mc_2}{c_1}}\exp\left\{-\frac{c_3}{2c_2}(t-kT)\right\}\|z(k)\|_1 + \sqrt{\frac{mc_4}{c_1}}M^*(t-kT)^{1/2} + \sqrt{\frac{mc_4}{c_1}}(t-kT)^{1/2}\omega_k(t), \tag{28}$$

hold for all $t \in [kT, (k+1)T]$ and all $k \geqslant 0$, where

$$\omega_k(t) \triangleq \sup_{kT \leqslant s \leqslant t} [|e_1(s)| + |e_2(s)|], \quad p_\omega(s) = \exp\left\{L(s^2/2 + s)\left(1 + \sqrt{mc_4 s/c_1}\right)\right\} \ (s \geqslant 0), \tag{29}$$

in which L, $\bar{b}$, c_i $(i = 1, \ldots, 4)$ are constants given in (2)–(4), m is the dimension of the internal dynamics z, and M^*, M_{a0} are positive numbers defined by (7) and (11), respectively.

Proof. See Appendix A.

Now we show the detailed proof of Theorem 1.

Proof of Theorem 1. We split the proof into two parts. Firstly, we prove, under the sampled-data controller (20), that the sampling signals $(x_1(k), x_2(k), z(k))$ are bounded and give their explicit boundaries. Then the boundedness of trajectories, together with tracking performance, is presented.

Part I. Recall that our sampled-data output feedback controller is as follows:

$$u(k) = \sum_{i=k-p}^{k} \alpha_{k-i}(k) e_1(i). \tag{30}$$

Then the closed-loop system consisting of the controller (30) and the characteristic model (13) is

$$e_1(k+1) = f_1(k)e_1(k) + f_2(k)e_1(k-1) + g_0(k) \sum_{i=k-p}^{k} \alpha_{k-i}(k)e_1(i) + g_1(k) \sum_{i=k-1-p}^{k-1} \alpha_{k-1-i}(k-1)e_1(i). \tag{31}$$

By introducing $E_1(k) \triangleq [e_1(k), \ldots, e_1(k-1-p)]^{\mathrm{T}} \in \mathbb{R}^{p+2}$, the above discrete-time closed-loop system (31) can be equivalently described as

$$E_1(k+1) = \begin{bmatrix} \alpha_A(k) \\ I_{p+1} \quad \mathbf{0}_{(p+1)\times 1} \end{bmatrix} E_1(k) \triangleq A(k)E_1(k), \quad \forall k \geqslant 0, \tag{32}$$

where $\alpha_A(k) = [f_1(k), f_2(k), \mathbf{0}_{1\times p}] + g_0(k)[\alpha_0(k), \ldots, \alpha_p(k), 0] + g_1(k)[0, \alpha_0(k-1), \ldots, \alpha_p(k-1)] \in \mathbb{R}^{1\times(p+2)}$. Also, we introduce the state transition matrix sequence with respect to the system (32),

$$\Phi(k+1, i) = A(k)\Phi(k, i), \quad \Phi(i, i) = I_{p+2}, \quad \forall k \geqslant i \geqslant 0. \tag{33}$$

Then it follows from condition (C2) that, there exist $M_c > 1$, $\lambda_c \in (0, 1)$ such that for all $T \in (0, T_*)$,

$$\|\Phi(k+1, i)\| \leqslant M_c \lambda_c^{k+1-i}, \quad \forall k \geqslant i \geqslant 0, \tag{34}$$

where the constants M_c, λ_c depend on the system structure and the control parameters.

Next, with the consistency condition (C1) and the stability property given by (34), we further analyze the stability of the closed-loop system comprised of the system (8) and the sampled-data controller (30). We set the initial state of the system (8) to satisfy

$$|e_1(0)| + |e_2(0)| \leqslant \rho_e, \quad \|z(0)\|_1 \leqslant \rho_0, \tag{35}$$

where $\rho_e > 0$ is determined by the initial state of both the plant (1) and the target system (5). In the following, we will demonstrate that there exists $T^* \in (0, 1]$ such that, for all $T \in (0, T^*)$, the sampling signals $(e_1(k), e_2(k), z(k))$ satisfy the following properties:

$$\sup_{k\geqslant 0} |e_1(k)| \leqslant \rho_1, \quad \sup_{k\geqslant 0} T|e_2(k)| \leqslant \rho_2, \quad \sup_{k\geqslant 0} T\|z(k)\|_1 \leqslant \rho_3, \tag{36}$$

where $\rho_1 \geqslant \rho_e$, $\rho_2 \geqslant \rho_e$, and $\rho_3 \geqslant \rho_0$ are positive numbers whose explicit values will be determined later.

We adopt the contradiction argument. Suppose that the above statement is not correct. Notice that the initial state satisfies $|e_1(0)| \leqslant \rho_1$, $T|e_2(0)| \leqslant \rho_2$, $T\|z(0)\|_1 \leqslant \rho_3$; then it follows that there must exist a sampling time $k_0 \geqslant 0$ such that the values $(e_1(k_0+1),\ e_2(k_0+1),\ z(k_0+1))$ do not have the properties given by Eq. (36). Let $k^*+1 \geqslant 1$ be the first time such that the state signals $(e_1(k), e_2(k), z(k))$ go beyond the bounds given in (36). Therefore, for all $0 \leqslant k \leqslant k^*$, we have

$$|e_1(k)| \leqslant \rho_1,\ T|e_2(k)| \leqslant \rho_2,\ T\|z(k)\|_1 \leqslant \rho_3, \tag{37}$$

while, at the sampling time k^*+1, there are only three cases given below:

$$|e_1(k^*+1)| > \rho_1, \quad \text{or} \quad T|e_2(k^*+1)| > \rho_2, \quad \text{or} \quad T\|z(k^*+1)\|_1 > \rho_3.$$

Now let us discuss the above three cases one by one.

Case 1. $|e_1(k^*+1)| > \rho_1$.

Firstly, by putting the sampled-data controller (30) into the exact discrete-time model (18), we obtain

$$E_1(k+1) = A(k)E_1(k) + B_0\Delta_1(k+1), \tag{38}$$

where $A(k)$ is defined in (32), $B_0 \triangleq [1, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^{p+2}$, and $\Delta_1(k+1)$ is given in (18). From this, we calculate an upper bound of $|e_1(k^*+1)|$. For the term $\Delta_1(k+1)$, noticing that for any $0 \leqslant k \leqslant k^*$ the states satisfy the boundedness given by (37), then according to the consistency condition (22), we know that there exists a class $\mathcal{K}$ function $\varrho_{10}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|\Delta_1(k+1)| \leqslant \varrho_{10}(T), \quad \forall 0 \leqslant k \leqslant k^*. \tag{39}$$

Again from the Eq. (38) we obtain that

$$E_1(k+1) = \Phi(k+1, 0)E_1(0) + \sum_{i=0}^{k} \Phi(k+1, i+1)B_0\Delta_1(i+1), \tag{40}$$

where $\Phi(k, i)$ is defined by (33). Substituting (34) and (39) into (40), it follows that for any $0 \leqslant k \leqslant k^*$,

$$\begin{aligned} \|E_1(k+1)\| &\leqslant \|\Phi(k+1, 0)\| \cdot \|E_1(0)\| + \sum_{i=0}^{k} \|\Phi(k+1, i+1)\| \cdot \|B_0\| \cdot |\Delta_1(i+1)| \\ &\leqslant M_c\lambda_c^{k+1}\|E_1(0)\| + \sum_{i=0}^{k} M_c\lambda_c^{k-i}\varrho_{10}(T) \leqslant M_c\|E_1(0)\| + M_c/(1-\lambda_c)\varrho_{10}(T). \end{aligned} \tag{41}$$

So according to the initial condition (35) and our appointment $e_1(-1) = e_1(0) - Te_2(0)$, and noticing that $\varrho_{10}(\cdot)$ belongs to class $\mathcal{K}$, we conclude that there exists $T_1 \leqslant \min\{T_*, 1\}$, such that for all $T \in (0, T_1)$,

$$\|E_1(k^*+1)\| < 2M_c\rho_e \triangleq \rho_1, \tag{42}$$

which contradicts $|e_1(k^*+1)| > \rho_1$.

Case 2. $T|e_2(k^*+1)| > \rho_2$.

Similarly to Case 1, we need only to calculate an upper bound of $T|e_2(k^*+1)|$. Firstly, by the exact discrete-time model (19), the consistency condition (23), as well as the boundedness (37), we obtain that

$$T|e_2(k+1)| \leqslant \sum_{j=0}^{\bar{p}_1} \left[\varrho_{\bar{p}_1-j}(T) + \lambda^*_{\bar{p}_1-j}\right] T|e_2(k-j)| + |\nu(e_1(k+1), e_1(k), z, u(k), k+1)| + \varrho_{20}(T), \tag{43}$$

holds for all $0 \leqslant k \leqslant k^*$, where $\bar{p}_1 \in \mathbb{N}$ is a positive integer, $(\lambda^*_0, \ldots, \lambda^*_{\bar{p}_1})^{\mathrm{T}}$ is an non-negative stable vector, $\varrho_i(\cdot)\,(i = 0, \ldots, \bar{p}_1)$ and $\varrho_{20}(\cdot)$ are class $\mathcal{K}$ functions, and the function $\nu(\cdot)$ is given in (14) satisfying (16). Thus, in order to analyze the above inequality, let us introduce the state vector $E_2(k) \triangleq [|e_2(k)|, \ldots, |e_2(k-\bar{p}_1)|]^{\mathrm{T}} \in \mathbb{R}^{\bar{p}_1+1}$, and the matrices

$$A_\lambda \triangleq \begin{bmatrix} \Lambda^* \\ I_{\bar{p}_1} \quad \mathbf{0}_{\bar{p}_1\times 1} \end{bmatrix}, \quad A_e(T) \triangleq A_\lambda + \begin{bmatrix} \bar{\varrho}(T) \\ \mathbf{0}_{\bar{p}_1\times(\bar{p}_1+1)} \end{bmatrix} \in \mathbb{R}^{(\bar{p}_1+1)\times(\bar{p}_1+1)},$$

where $\Lambda^* \triangleq [\lambda^*_{\bar{p}_1}, \dots, \lambda^*_0]$ and $\bar{\varrho}(T) \triangleq [\varrho_{\bar{p}_1}(T), \dots, \varrho_0(T)]$. So Eq. (43) can be equivalently described by

$$TE_2(k+1) \leqslant A_e(T)TE_2(k) + B_e\left[|\nu(e_1(k+1), e_1(k), z, u(k), k+1)| + \varrho_{20}(T)\right], \quad \forall 0 \leqslant k \leqslant k^*, \tag{44}$$

where $B_e \triangleq [1, 0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^{\bar{p}_1+1}$, and the notation "$\leqslant$" represents the relation between vectors defined in Definition 2. Now we proceed to calculate an upper bound of $T|e_2(k^*+1)|$. At first, let us focus on the matrix function $A_e(T)$. Since the vector $(\lambda^*_0, \dots, \lambda^*_{\bar{p}_1})^{\mathrm{T}}$ is non-negative and stable, it follows that the spectral radius of A_λ satisfies $\lambda^e_0 \triangleq \max_{1\leqslant i\leqslant \bar{p}_1+1}\{|\lambda_i(A_\lambda)|\} < 1$. From this, and according to the continuous dependence of matrix eigenvalues on its elements (see, e.g., the Ostrowski theorem [24]), we know that the eigenvalues $\{\lambda_i(A_e(T)),\ i = 1, \dots, \bar{p}_1+1\}$ of the matrix $A_e(T)$ have the following property: for any given $\lambda^e_1 \in (\lambda^e_0, 1)$, there exists a sampling period $T_2 \leqslant T_1$ such that, for all $T \in (0, T_2)$,

$$\max_{1\leqslant i\leqslant \bar{p}_1+1}\{|\lambda_i(A_e(T))|\} < \lambda^e_1 < 1. \tag{45}$$

By this, we can further obtain (see Lemma 2.4.1 in [25])

$$\|A^k_e(T)\| \leqslant M^e_A(\lambda^e_2)^k, \quad \forall T \in (0, T_2), \quad \forall k \geqslant 0, \tag{46}$$

where

$$M^e_A = \sqrt{\bar{p}_1+1}(1+2/\epsilon_e)^{\bar{p}_1}, \quad \lambda^e_2 = \lambda^e_1 + \epsilon_e \cdot \sup_{T\in(0,T_2]}\|A_e(T)\| < 1, \tag{47}$$

in which $\epsilon_e > 0$ is a proper small constant. Then, for the function $\nu(\cdot)$, by combining (16), (37), the controller (30), its property (21), and the boundedness (42) that was proved in Case 1, we have

$$|\nu(e_1(k+1), e_1(k), z, u(k), k+1)| \leqslant \tau_\nu(\rho_\nu), \quad \forall 0 \leqslant k \leqslant k^*, \tag{48}$$

where $\rho_\nu = [2 + \bar{\alpha}_u(p+1)]\rho_1$, in which $\bar{\alpha}_u$, ρ_1 are given by (21) and (42). Thus, for the inequality (44), noticing that all the elements of $A_e(T)$ are non-negative, and combining the property (48), we have

$$TE_2(k+1) \leqslant A_e(T)^{k+1}TE_2(0) + \sum_{i=0}^{k} A_e(T)^{k-i}B_e[\tau_\nu(\rho_\nu) + \varrho_{20}(T)], \quad \forall 0 \leqslant k \leqslant k^*,$$

which together with (46) allows the further derivation that $\forall 0 \leqslant k \leqslant k^*$,

$$\begin{aligned} T\|E_2(k+1)\| &\leqslant T\|A_e(T)^{k+1}\|\|E_2(0)\| + \sum_{i=0}^{k}\|A_e(T)^{k-i}\| \cdot \|B_e\| \cdot [\tau_\nu(\rho_\nu) + \varrho_{20}(T)] \\ &\leqslant TM^e_A(\lambda^e_2)^{k+1}\|E_2(0)\| + \sum_{i=0}^{k} M^e_A(\lambda^e_2)^{k-i}[\tau_\nu(\rho_\nu) + \varrho_{20}(T)] \\ &\leqslant TM^e_A(\lambda^e_2)^{k+1}\|E_2(0)\| + \frac{M^e_A}{1-\lambda^e_2}[\tau_\nu(\rho_\nu) + \varrho_{20}(T)]. \end{aligned} \tag{49}$$

Consequently, from the initial condition (35) and the fact that $\varrho_{20}(\cdot)$ is a class $\mathcal{K}$ function, we obtain that there exists $T_3 \leqslant T_2$ such that for all $T \in (0, T_3)$,

$$T\|E_2(k^*+1)\| < 2M^e_A/(1-\lambda^e_2)\tau_\nu(\rho_\nu) \triangleq \rho_2, \tag{50}$$

which is in contradiction with $T|e_2(k^*+1)| > \rho_2$.

Case 3. $T\|z(k^*+1)\|_1 > \rho_3$.

Firstly, according to Assumption 3, and the boundedness (7) of the target system's trajectory, we know that

$$\|z(t)\|_1 \leqslant \sqrt{mc_2/c_1}\exp\{-c_3t/(2c_2)\}\|z(0)\|_1 + \sqrt{mc_2c_4/(c_1c_3)}\left\{M^* + \sup_{0\leqslant\tau\leqslant t}[|e_1(\tau)| + |e_2(\tau)|]\right\},\ \forall t \geqslant 0, \tag{51}$$

where c_i $(i = 1, \ldots, 4)$ are given in (4), m is the dimensionality of the internal dynamics z, and M^* is given by (7). Now we calculate an upper bound of $T\|z(k^*+1)\|_1$. From (51), it is obvious that we only need to calculate an upper bound of $\omega_k(t) = \sup_{kT\leqslant\tau\leqslant t}[|e_1(\tau)| + |e_2(\tau)|]$. Actually, by substituting the controller (30) into (27) given in Lemma 1, and combining (21), (37) we obtain that, for all $t \in [0, (k^*+1)T]$,

$$T \sup_{0\leqslant\tau\leqslant t} [|e_1(\tau)| + |e_2(\tau)|] \leqslant \sup_{0\leqslant k\leqslant k^*} T\omega_k((k+1)T) \leqslant p_\omega(T)\left(C_{\omega1}\rho_1 + \rho_2 + TC_{\omega2}\right), \tag{52}$$

where the function $p_\omega(\cdot)$ is defined by (29), $C_{\omega1} = (1+\frac{T}{2})\bar{b}\bar{\alpha}_u(p+1)$, and $C_{\omega2} = \rho_1+\rho_2+(T+\frac{T^2}{2})(M_{a0}+\sqrt{\frac{mc_4}{c_1}}M^*LT^{1/2})+L\sqrt{\frac{mc_2}{c_1}}(1+\frac{T}{2})\rho_3$. Hence, by substituting (52) into (51), it follows that $\forall t \in [0, (k^*+1)T]$,

$$\begin{aligned} T\|z(t)\|_1 &\leqslant \sqrt{mc_2/c_1}\exp\{-c_3t/(2c_2)\}T\|z(0)\|_1 + \sqrt{mc_2c_4/(c_1c_3)}\left\{TM^* + p_\omega(T)\left(C_{\omega1}\rho_1+\rho_2+TC_{\omega2}\right)\right\} \\ &= \sqrt{mc_2/c_1}\exp\{-c_3t/(2c_2)\}T\|z(0)\|_1 + p_\omega(T)(C_{\omega1}\rho_1+\rho_2) + O(T) \\ &\leqslant \sqrt{mc_2/c_1}T\|z(0)\|_1 + p_\omega(T)(3\bar{b}\bar{\alpha}_u(p+1)\rho_1/2+\rho_2) + O(T). \end{aligned} \tag{53}$$

From this, it is immediately verified that there exists $T^* \leqslant T_3$ such that, for all $T \in (0, T^*)$,

$$T\|z(k^*+1)\|_1 < \sqrt{mc_2/c_1}\rho_0 + p_\omega(1)[3\bar{b}\bar{\alpha}_u(p+1)\rho_1/2+\rho_2] \triangleq \rho_3, \tag{54}$$

contradicting $T\|z(k^*+1)\|_1 > \rho_3$.

Consequently, from Cases 1–3, it is easy to conclude that, for all $T \in (0, T^*)$, the states of the closed-loop system comprised of the controller (30) and the system (8) satisfy

$$|e_1(k)| \leqslant \rho_1,\ T|e_2(k)| \leqslant \rho_2,\ T\|z(k)\|_1 \leqslant \rho_3, \quad \forall k \geqslant 0. \tag{55}$$

Furthermore, we can get the steady-state bounds of the signals $\{e_1(k),\, e_2(k)\}$. From (41), we know that

$$\limsup_{k\to\infty} \|E_1(k+1)\| \leqslant \lim_{k\to\infty}\sum_{i=0}^{k} M_c\lambda_c^{k-i}\varrho_{10}(T) = M_c/(1-\lambda_c)\varrho_{10}(T), \tag{56}$$

where $\varrho_{10}(\cdot)$ is a class $\mathcal{K}$ function. Therefore, there exists some time $N_c(T) \in \mathbb{N}$ such that

$$|e_1(k)| \leqslant M_c/(1-\lambda_c)\varrho_{10}(T) + \varrho_{10}(T) = M_e^*\varrho_{10}(T), \quad \forall k \geqslant N_c(T), \tag{57}$$

where $M_e^* = M_c/(1-\lambda_c)+1$. For the state $e_2(k)$, based on the property of the function $\nu(\cdot)$ given by (16), that of the sampled-data controller (21), and by (48) and (57) we have that, for all $k \geqslant N_c(T)+p = N_c'(T)$, $|\nu(e_1(k+1), e_1(k), z, u(k), k+1)| \leqslant \tau_\nu\left(M_\nu^*\varrho_{10}(T)\right)$, where $M_\nu^* = [2+\bar{\alpha}_u(1+p)]M_e^*$. This together with (44) and (46) further ensures that for all $k \geqslant N_c'(T)$ and $T \in (0, T^*)$,

$$\begin{aligned} T\|E_2(k+1)\| &\leqslant TM_A^e\lambda_2^e\|E_2(k)\| + \tau_\nu(M_\nu^*\varrho_{10}(T)) + \varrho_{20}(T) \\ &\leqslant TM_A^e(\lambda_2^e)^{k+1-N_c'}\|E_2(N_c')\| + \sum_{i=N_c'}^{k} M_A^e(\lambda_2^e)^{k-i}\cdot\left[\tau_\nu\left(M_\nu^*\varrho_{10}(T)\right) + \varrho_{20}(T)\right] \\ &\leqslant TM_A^e(\lambda_2^e)^{k+1-N_c'}\|E_2(N_c')\| + M_A^e/(1-\lambda_2^e)\left[\tau_\nu(M_\nu^*\varrho_{10}(T)) + \varrho_{20}(T)\right]. \end{aligned}$$

Hence,

$$\limsup_{k\to\infty} T|e_2(k)| \leqslant M_A^e/(1-\lambda_2^e)\left[\tau_\nu(M_\nu^*\varrho_{10}(T)) + \varrho_{20}(T)\right]. \tag{58}$$

To this point, we have proved the boundedness of the sampling states $\{e_1(k), Te_2(k), Tz(k)\}$ and gave their explicit bounds (42), (50) and (54). Meanwhile, the steady-state bounds (56) and (58) of the external dynamics $\{e_1(k),\, e_2(k)\}$ were also given.

Part II. Now we further analyze the property of the trajectories of the closed-loop system composed of the system (8) and the admissible controller (30), and give an upper bound of the tracking error.

Firstly, for the internal dynamics z, based on the boundedness of the sampling signals $\{e_1(k), e_2(k), z(k)\}$ given by (55), and by a similar argument as in Eqs. (51)–(54), we obtain that, for any $T \in (0, T^*)$ and $t \geqslant 0$,

$$T\|z(t)\|_1 \leqslant p_\omega(T)\,(C_{\omega 1}\rho_1 + \rho_2) + O(T) \leqslant \rho_3, \tag{59}$$

where the function $p_\omega(\cdot)$ is given by (29), $\{C_{\omega 1}, \rho_1, \rho_2, \rho_3\}$ are constants given by (52), (42), (50) and (54) respectively. From this, it is obvious that $\sup_{t\geqslant 0}\{T\|z(t)\|_1\} \leqslant \rho_3$ holds for any $T \in (0, T^*)$.

Now we explore the properties of the external dynamics $\{e_1, e_2\}$. From (26), Assumptions 1 and 2, and combining the bounded property (11) we have that, for all $k \geqslant 0$ and $t \in [kT, (k+1)T]$,

$$|e_1(t)| \leqslant |e_1(k)| + T|e_2(k)| + \int_{kT}^{t} L(t-\tau)\,[\omega_k(\tau) + \|z(\tau)\|_1]\,\mathrm{d}\tau + (T^2/2)\bar{b}|u(k)| + (T^2/2)M_{a0}, \tag{60}$$

where $\omega_k(t)$ is defined by (29), and $\bar{b}$, M_{a0} are constants given by (3) and (11), respectively. Thus, according to Lemma 1, the boundedness (55) and (59), and the property of the sampled-data controller (21), it can be further obtained that: for all $T \in (0, T^*)$ and $t \geqslant 0$,

$$|e_1(t)| \leqslant \left[1 + \bar{b}\bar{\alpha}_u(p+1)/2\right]\rho_1 + \rho_2 + O(T). \tag{61}$$

Similarly, based on Assumptions 1, 2 and Lemma 1, and combining (21), (26), (55), and (59), we also have that when $T \in (0, T^*)$ and $t \geqslant 0$,

$$T|e_2(t)| \leqslant \bar{b}\bar{\alpha}_u(p+1)\rho_1 + \rho_2 + O(T). \tag{62}$$

Then it follows from (59), (61) and (62) that $\forall T \in (0, T^*)$,

$$|e_1(t)| + T|e_2(t)| + T\|z(t)\|_1 = O(1), \quad \forall t \geqslant 0. \tag{63}$$

Consequently, this together with the boundedness of $(x_1^*(t), x_2^*(t))$ given by (7) further derives the boundedness of the trajectory $\{x_1(t),\, x_2(t),\, z(t)\}$. Hence, the boundedness (24) in Theorem 1 is proved.

Moreover, for the solution (26) of the system (8), according to Lemma 1, the steady-state property of the signals $\{e_1(k), e_2(k)\}$ given by (56) and (58), as well as the boundedness (63), it is not difficult to verify that

$$\limsup_{t\to\infty}[|e_1(t)| + T|e_2(t)|] = O(\varrho_e^*(T)), \tag{64}$$

where $\varrho_e^*(T) = \left[(1 + (3/2)\bar{b}\bar{\alpha}_u(p+1))M_c/(1-\lambda_c)\right]\varrho_{10}(T) + 2[M_A^e/(1-\lambda_2^e)][\tau_\nu(M_\nu^*\varrho_{10}(T)) + \varrho_{20}(T)] + M_sT$ is a class $\mathcal{K}$ function in which $M_s > 0$ is the sum of the O coefficients corresponding to the terms $O(T)$ in (61) and (62). Therefore, from (64), the output tracking error property (25) in Theorem 1 is verified.

Finally, when the initial states satisfy $x(0) = x^*(0)$, it is evident that $e_1(0) = 0$, $e_2(0) = 0$, and $\rho_e = 0$. Thus, from all of the above, we know by (41) that in this case $\rho_1 = [M_c/(1-\lambda_c)]\varrho_{10}(T)$. So, this together with (50) and (61) indicates that there exists a class $\mathcal{K}$ function $\varrho^{e*}(\cdot)$ such that $\sup_{t\geqslant 0}|e_1(t)| = O(\varrho^{e*}(T))$. As a result, the assertion of Theorem 1 is true.

3.2 Specific stability results

The main purpose of this subsection is to illustrate the validity of the framework for stability analysis we established. As will be seen shortly, for tracking problems in high-order minimum-phase nonlinear systems (1) of relative degree two, following our framework, we first derive an explicit characteristic model and give a sampled-data output controller based on such a characteristic model; then we develop the specific stability conditions corresponding to conditions (C1) and (C2) given in Theorem 1.

3.2.1 *Characteristic model-based golden-section adaptive control*

Here, to solve the tracking problem mentioned before, we take the golden-section adaptive control as our sampled-data output feedback control. Before pursuing further, we first present the characteristic model corresponding to the system (8).

Proposition 1. Under Assumptions 1 and 2, the characteristic model of the error system (8) can be described by the following time-varying difference equation: $\forall k \geqslant 0$,

$$\begin{cases} e_1(k+1) = f_1(k)e_1(k) + f_2(k)e_1(k-1) + (T^2/2)g_0(k)u(k) + (T^2/2)g_1(k)u(k-1), & (65) \\ Te_2(k+1) = e_1(k+1) - e_1(k) + \displaystyle\int_{kT}^{(k+1)T} (t-kT)b_e(e_1(t), e_2(t), z(t), t)u(k)\mathrm{d}t, & (66) \end{cases}$$

where

$$f_1(k) = 2 + T^2\, (\partial a_e/\partial e_1)|_{(\varsigma_1 e_1(k),0,0,k)} + T\, (\partial a_e/\partial e_2)|_{(e_1(k),\varsigma_2 e_2(k),0,k)},$$
$$f_2(k) = -1 - T\, (\partial a_e/\partial e_2)|_{(e_1(k),\varsigma_2 e_2(k),0,k)}, \quad (67)$$
$$g_0(k) = b_e(e_1(t), e_2(t), z(t), t)|_{t=(k+\varsigma_3)T}, \quad g_1(k) = b_e(e_1(t), e_2(t), z(t), t)|_{t=(k-\varsigma_4)T}, \quad (68)$$

in which the parameters $\varsigma_1, \varsigma_1 \in (0,1)$ may depend on the state variables $(e_1(k), e_2(k))$, and $\varsigma_3, \varsigma_4 \in (0,1)$. Furthermore, the time-varying parameters satisfy $(f_1(k), f_2(k), g_0(k), g_1(k))^{\mathrm{T}} \in D\, (\forall k \geqslant 0)$, where D is a closed convex set defined by

$$D \triangleq \left\{ (a_1, a_2, a_3, a_4)^{\mathrm{T}} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} a_1 \in \left[2 - TL - T^2L,\, 2 + TL + T^2L\right] \\ a_2 \in \left[-1 - TL,\, -1 + TL\right] \\ a_3 \in \left[\underline{b},\, \bar{b}\right], \quad a_4 \in \left[\underline{b},\, \bar{b}\right] \end{array} \right\}, \quad (69)$$

in which the parameters $L, \underline{b}, \bar{b}$ are given by (2) and (3), respectively.

Proof. See Appendix B.

Remark 6. It is easy to verify that Eq. (66) meets the structure requirement given by (16) and that the coefficients $\{f_1(k), f_2(k), (T^2/2)g_0(k), (T^2/2)g_1(k)\}$ satisfy (15); thus the model (65) and (66) we derived belongs to the group of characteristic models we are concerned with.

Now, based on the characteristic model (65), we describe the golden-section adaptive controller. For any $k \geqslant 0$, let us denote the unknown time-varying parameter vector of the model (65) as $\theta(k) \triangleq [f_1(k), f_2(k), g_0(k), g_1(k)]^{\mathrm{T}}$, the corresponding estimation vector $\hat{\theta}(k) \triangleq [\hat{f}_1(k), \hat{f}_2(k), \hat{g}_0(k), \hat{g}_1(k)]^{\mathrm{T}}$, and the regression vector $\varphi(k) \triangleq \left[e_1(k), e_1(k-1), (T^2/2)u(k), (T^2/2)u(k-1)\right]^{\mathrm{T}}$. Then the golden-section adaptive control law is as follows:

$$u(k) = \frac{2}{T^2\hat{g}_0(k)}\left[-l_1\hat{f}_1(k)e_1(k) - l_2\hat{f}_2(k)e_1(k-1)\right], \quad (70)$$

where $l_1 = 0.382$, $l_2 = 0.618$, and $\hat{\theta}(k)$ are calculated by the following projected gradient algorithm [25]:

$$\hat{\theta}(k) = \pi_D\left\{\hat{\theta}(k-1) + \frac{\varphi(k-1)}{\mu_0 + \varphi(k-1)^{\mathrm{T}}\varphi(k-1)}\left(e_1(k) - \varphi(k-1)^{\mathrm{T}}\hat{\theta}(k-1)\right)\right\}, \quad (71)$$

in which $\mu_0 > 0$ is an adjusted parameter and $\pi_D\{x\}$ is a projection function that projects x into the set D given by (69). Obviously, our proposed controller (70) belongs to the admissible control set (20).

To this point, we have given the characteristic model (65) and (66) of the error system (8) and the characteristic model-based golden-section adaptive controller (70). Notice that the controller bears a linear structure, making it more convenient for engineering applications.

3.2.2 *Control performance*

Next, with the characteristic model (65) and (66) and the golden-section adaptive controller (70), based on the established stability analysis framework, we will explore the explicit sufficient conditions under which a closed-loop system comprised of the plant (1) and the controller (70) is stable. To achieve this, we need to verify that the characteristic model (65) and (66), along with the controller (70), satisfies the consistency condition. It is also necessary to determine under what condition the discrete-time closed-loop system consists of the characteristic model (65) and whether this controller (70) is exponentially stable.

First of all, we verify the consistency condition (22) and (23) by analyzing the modeling error of the characteristic model (65) and (66). From the error system (8), it is evident that for any given sampled-data control signals $\{u(k),\, k \geqslant 0\}$ the corresponding exact discrete-time model can be described as follows:

$$\begin{cases} e_1(k+1) = e_1(k) + Te_2(k) + \displaystyle\int_{kT}^{(k+1)T} [(k+1)T - t]\,[a_e(e_1(t), e_2(t), z(t), t) + b_e(e_1(t), e_2(t), z(t), t)u(k)]\,\mathrm{d}t \\ \qquad\qquad = e_1(k) + Te_2(k+1) - \displaystyle\int_{kT}^{(k+1)T} (t - kT) b_e(e_1(t), e_2(t), z(t), t)u(k)\mathrm{d}t + \delta_1^e(k+1), \\ e_2(k+1) = e_2(k) + \displaystyle\int_{kT}^{(k+1)T} [a_e(e_1(t), e_2(t), z(t), t) + b_e(e_1(t), e_2(t), z(t), t)u(k)]\,\mathrm{d}t \\ \qquad\qquad = e_2(k) + Ta_e(e_1(k), e_2(k), 0, k) + \displaystyle\int_{kT}^{(k+1)T} b_e(e_1(t), e_2(t), z(t), t)u(k)\mathrm{d}t + \delta_2^e(k+1), \end{cases}$$

where

$$\begin{cases} \delta_1^e(k+1) = \displaystyle\int_{kT}^{(k+1)T} (kT - t) a_e(e_1(t), e_2(t), z(t), t)\mathrm{d}t, \\ \delta_2^e(k+1) = \displaystyle\int_{kT}^{(k+1)T} [a_e(e_1(t), e_2(t), z(t), t) - a_e(e_1(k), e_2(k), 0, k)]\,\mathrm{d}t. \end{cases} \tag{72}$$

Similar to the proof of Proposition 1, we can write the above exact discrete-time model equivalently as

$$\begin{cases} e_1(k+1) = f_1(k)e_1(k) + f_2(k)e_1(k-1) + (T^2/2)g_0(k)u(k) + (T^2/2)g_1(k)u(k-1) + \Delta_1^e(k+1), \\ Te_2(k+1) = e_1(k+1) - e_1(k) + \displaystyle\int_{kT}^{(k+1)T} (t - kT) b_e(e_1(t), e_2(t), z(t), t)u(k)\mathrm{d}t - \delta_1^e(k+1), \end{cases} \tag{73}$$

where the parameters $\{f_1(k), f_2(k), g_0(k), g_1(k)\}$ have the same properties given in (67) and (68), and

$$\Delta_1^e(k+1) \triangleq \delta_1^e(k+1) - \delta_1^e(k) + T\delta_2^e(k+1) + T^2 a_e(0,0,0,k) + (T^2/2)Th(k)g_1(k)u(k-1), \tag{74}$$

in which $\delta_1^e(\cdot)$ and $\delta_2^e(\cdot)$ are defined by (72) and $h(k) = (\partial a_e/\partial e_2)|_{(e_1(k), \varsigma_2 e_2(k), 0, k)}$ with some $\varsigma_2 \in (0,1)$.

Thus, from (65), (66) and (73), it is obvious that $\Delta_1^e(k+1)$ and $\delta_1^e(k+1)$ defined by (74) and (72), respectively, are the corresponding modeling errors whose properties will be given in the lemma below.

Lemma 2. Consider the ordered pair $\{F_e^{cm}, u\}$ comprised of the characteristic model (65) and (66) and the golden-section adaptive controller (70), as well as $\{F_e^{ex}, u\}$ corresponding to the exact discrete-time model (73) and the controller (70). Suppose that Assumptions 1−3 are satisfied; then the modeling errors of the family $\{F_e^{cm}, u\}$ with respect to $\{F_e^{ex}, u\}$ satisfy the consistency condition (22) and (23). Specifically,

$$|\Delta_1^e(k+1)| \leqslant \sum_{j=0}^{2} \varrho_{1j}^e(T)|e_1(k-j)| + \sum_{j=0}^{1} \varrho_{2j}^e(T)T|e_2(k-j)| + \sum_{j=0}^{1} \varrho_{3j}^e(T)T\|z(k-j)\|_1 + \varrho_{50}^e(T), \tag{75}$$

$$|\delta_1^e(k+1)| \leqslant \sum_{j=0}^{1} \varrho_{4j}^e(T)|e_1(k-j)| + \varrho_{21}^e(T)T|e_2(k)| + \varrho_{31}^e(T)T\|z(k)\|_1 + \varrho_{51}^e(T), \tag{76}$$

where $\Delta_1^e(k+1)$ and $\delta_1^e(k+1)$ are given by (74) and (72), respectively, and $\varrho_{ij}^e(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, \ldots, 5; j \in \{0,1,2\})$ are class $\mathcal{K}$ functions satisfying

$$\varrho_{ij}^e(T) = O(T)\ (i = 1, \ldots, 4; j \in \{0,1,2\}), \quad \max\{\varrho_{50}(T), \varrho_{51}(T)\} = O(T^2), \quad \forall T \in (0, T_{\max}]. \tag{77}$$

Proof. See Appendix C.

Next, we explore the stability of the discrete-time closed-loop system comprised of the characteristic model (65) and the golden-section adaptive controller (70). By introducing the following constant matrix:

$$A_c \triangleq \begin{bmatrix} 2(1-l_1) & -(1-l_2+2l_1) & l_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{78}$$

the stability of the above closed-loop system is given in the following lemma.

Lemma 3. Consider the characteristic model (65) and the golden-section adaptive controller (70). If Assumptions 1 and 2 hold, then there exist a sampling period $T_*^e > 0$ and a positive number $C_b^e > 1$ that depend on the control parameters l_1, l_2 and the matrix A_c, such that when $T \in (0, T_*^e)$ and

$$\bar{b}/\underline{b} < C_b^e, \tag{79}$$

the discrete-time closed-loop system comprised of (65) and (70) is exponentially stable.

Proof. See Appendix D.

Consequently, from the above two lemmas, and based on our framework, we can derive the stability of the hybrid closed-loop system consisting of the plant (1) and the golden-section adaptive controller (70), where the specific stability condition and the control performance are given below.

Theorem 2. Consider the high-order nonlinear plant (1), the characteristic model (65) and (66), and the golden-section adaptive controller (70). Suppose that Assumptions 1−3 are satisfied, and the parameters $\{\bar{b}, \underline{b}\}$ satisfy the condition given by (79), then for any $\rho_0^e > 0$ and any $\|(x_1(0), x_2(0), z(0))\|_1 \leqslant \rho_0^e$, there exists a sampling period $T^{e*} > 0$ such that, when $T \in (0, T^{e*})$, the trajectories of the closed-loop system comprised of the controller (70) and the plant (1) have the following properties:

(1) Boundedness:

$$\sup_{t \geqslant 0} \{|x_1(t)| + T|x_2(t)| + T\|z(t)\|_1\} = O(1); \tag{80}$$

(2) The tracking error satisfies:

$$\limsup_{t \to \infty} |y(t) - y^*(t)| = O(T). \tag{81}$$

Furthermore, if the initial state satisfies $x(0) = x^*(0)$, then $\sup_{t \geqslant 0} |y(t) - y^*(t)| = O(T)$.

Remark 7. Notice that the above stability results are obtained based on the golden-section adaptive controller (70) with the control parameters $l_1 = 0.382, l_2 = 0.618$. In fact, the control parameters l_1, l_2 that can achieve these results are not unique. It is easy to verify by following our proof that, $\{l_1, l_2\}$ is feasible provided that the roots of $z^3 - 2(1-l_1)z^2 + (1+2l_1-l_2)z - l_2 = 0$ are located in the unit circle.

4 Simulation

We consider attitude control for a three-axis stabilized satellite with flexible solar arrays. As given in [26], the corresponding pitch axis dynamic model is

$$I_p\ddot{\theta}_p + C_q^{\mathrm{T}}\ddot{q} = T_c + T_d, \quad \ddot{q} + 2\Xi\Omega_q\dot{q} + \Omega_q^2 q + C_q\ddot{\theta}_p = 0, \quad y = \theta_p,$$

where θ_p is the pitch angle, I_p is the moment of inertia of the pitch axis, and T_c, T_d are the control moment and the disturbance moment, respectively. $C_q = [d_1, \ldots, d_n]^{\mathrm{T}}$, $q = [q_1, \ldots, q_n]^{\mathrm{T}}$, $\Xi = \mathrm{diag}\{\xi_1, \ldots, \xi_n\}$, and $\Omega_q = \mathrm{diag}\{\omega_{q1}, \ldots, \omega_{qn}\}$, where $d_i\,(i = 1, \ldots, n)$ is the ith flexible coupling coefficient between the solar arrays vibrations and the spacecraft motion, q_i, $\xi_i, \omega_{qi}\,(i = 1, \ldots, n)$ are the ith flexible modal coordinate, modal damping coefficient, and modal frequency, respectively. It is evident that the q-subsystem with input $(\theta, \dot{\theta})$ is ISS stable as given in Remark 2. We take $I_p = 2000$ kg $\cdot$ m^2, $T_d = 10 + 3\sin(0.01t)$, $n = 2$, $\xi_1 = \xi_2 = 0.005$, $\omega_{q1} = 0.25$, $\omega_{q2} = 0.11$, and $d_1, d_2 \in (0, 1]$. Our goal is

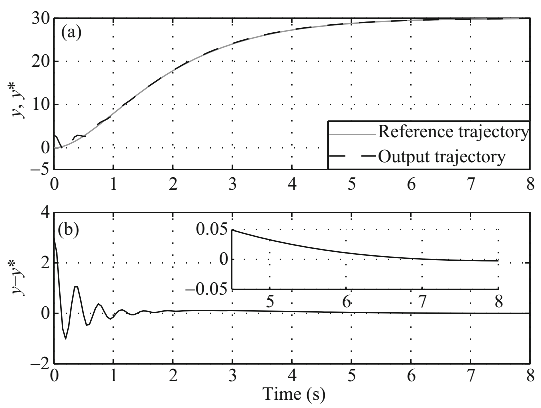

Figure 1 (a) System output, reference trajectory, and (b) corresponding tracking error ($\theta_p(0) = 3$).

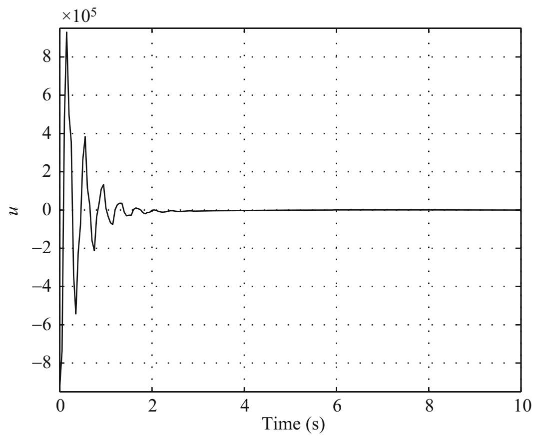

Figure 2 Control signal ($\theta_p(0) = 3$).

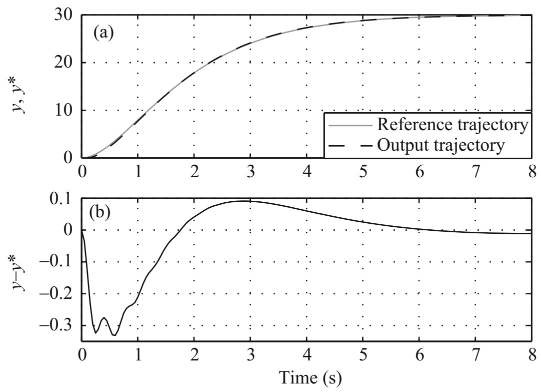

Figure 3 (a) System output, reference trajectory, and (b) corresponding tracking error ($\theta_p(0) = 0$).

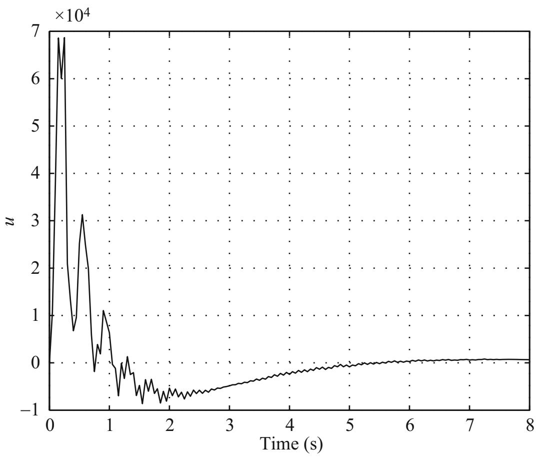

Figure 4 Control signal ($\theta_p(0) = 0$).

to have the pitch angle θ_p asymptotically track the reference signal generated from the target system $\dot{x}_1^* = x_2^*$, $\dot{x}_2^* = -2w_0x_2^* - w_0^2x_1^* + r(t)$, $y^* = x_1^*$, where the parameters $w_0 = 1$, $r(t) = 30$ and the initial condition is $x_1^*(0) = x_2^*(0) = 0$. Then the corresponding characteristic model is $y_e(k+1) = 2y_e(k) - y_e(k-1) + (T^2/2)[b(k)u(k) + b(k-1)u(k-1)]$, where $y_e(k) = y(k) - y^*(k)$, $b(k) = 1/(I_p - C_q^{\mathrm{T}}C_q) \in [1/2000, 1/1900]$. Also, the golden-section adaptive controller is $u(k) = -2[2l_1((y(k) - y^*(k)) - l_2(y(k-1) - y^*(k-1))]/(T^2\hat{b}(k))$, where $l_1 = 0.382, l_2 = 0.618$, and $\hat{b}(k)$ is calculated by the projected gradient algorithm. Thus, by taking the sampling period $T = 0.1$, the tracking performance and the control signals for the initial values $\theta_p(0) = 3$ and $\theta_p(0) = 0$ are given in Figures 1, 2 and Figures 3, 4, respectively.

As expected, in both cases the tracking performance of closed-loop systems can meet requirements. Moreover, the transient performance is improved as the initial error decreases, which coincides with our theoretical results.

5 Conclusion

This paper presents a framework for stability analysis of high-order minimum-phase nonlinear systems with a relative degree of two based on the CMAC method, trying to answer the basic question: if the characteristic model corresponding to the plant can be stabilized by a characteristic model-based adaptive controller, then under what condition can such a controller stabilize the plant? The consistency condition, an essential ingredient of this framework, is proposed for describing quantitatively the modeling error of this characteristic model, corresponding to a class of second-order characteristic models as well as to admissible sampled-data output feedback controllers. With this condition and the exponential stability of the discrete-time closed-loop system consisting of the characteristic model and the sampled-data controller, we proved the stability of the closed-loop system comprised of this controller and of the above high-order plants. Moreover, a group of detailed sufficient conditions corresponding to a specific characteristic model of the above high-order nonlinear systems and to the golden-section adaptive controller was proposed to illustrate the validity of our framework. Our results provide a new perspective for exploring the stability of high-order nonlinear plants under the CMAC. To the best of our knowledge, the group of stability conditions we developed seems to be the weakest for dealing with high-order minimum-phase systems of relative degree two by the CMAC method. Of course, there are still many problems remaining to be solved concerning more general nonlinear systems; these belong to further, future investigations.

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A Proof of Lemma 1

Firstly, it follows from Assumption 3 and (7) that, $\forall t \in [kT,(k+1)T]$,

$$\|z(t)\|_1 \leqslant \sqrt{\frac{mc_2}{c_1}}\exp\{-c_3(t-kT)/(2c_2)\}\|z(k)\|_1+\sqrt{\frac{mc_4}{c_1}}M^*(t-kT)^{1/2}+\sqrt{\frac{mc_4}{c_1}}(t-kT)^{1/2}\sup_{kT\leqslant\tau\leqslant t}[|e_1(\tau)|+|e_2(\tau)|], \quad \text{(A1)}$$

where $c_i\,(i=1,\ldots,4)$ are constants given by (4), m is the dimension of the internal dynamics z, and the constant M^* is given by (7). Thus, the inequality (28) is correct. In addition, for the solution (26) of the error system (8), under Assumptions 1 and 2, and from (9), (10) and (11), we know that $\forall t \in [kT,(k+1)T]$,

$$\begin{aligned}|e_1(t)|+|e_2(t)| \leqslant & |e_1(k)|+|e_2(k)|+(t-kT)|e_2(k)|+\left[(t-kT)^2/2+(t-kT)\right](\bar{b}|u(k)|+M_{a0})\\ &+\int_{kT}^{t}L(t-\tau+1)[|e_1(\tau)|+|e_2(\tau)|+\|z(\tau)\|_1]\mathrm{d}\tau, \end{aligned}\quad \text{(A2)}$$

where the parameters L, $\bar{b}$, M_{a0} are given by (2), (3), and (11), respectively. Therefore, substituting (A1) into (A2), and by simple calculations, we can obtain

$$\begin{aligned}|e_1(t)|+|e_2(t)| \leqslant & |e_1(k)|+|e_2(k)|+(t-kT)|e_2(k)|+\int_{kT}^{t}L(t-\tau+1)[|e_1(\tau)|+|e_2(\tau)|]\mathrm{d}\tau+\left[(t-kT)+(t-kT)^2/2\right]\\ &\cdot\left[\bar{b}|u(k)|+M_{a0}+\sqrt{\frac{mc_2}{c_1}}L\|z(k)\|_1+\sqrt{\frac{mc_4}{c_1}}M^*L(t-kT)^{1/2}\right]\\ &+\int_{kT}^{t}L(t-\tau+1)\sqrt{\frac{mc_4}{c_1}}(\tau-kT)^{1/2}\sup_{kT\leqslant s\leqslant\tau}[|e_1(s)|+|e_2(s)|]\mathrm{d}\tau.\end{aligned}$$

Furthermore, by taking the maximum value in the interval $[kT,t]$ on both sides of the above inequality, we have

$$\begin{aligned}\omega_k(t) \leqslant & |e_1(k)|+|e_2(k)|+(t-kT)|e_2(k)|+\int_{kT}^{t}L(t-\tau+1)\omega_k(\tau)\mathrm{d}\tau\\ &+\left[(t-kT)+(t-kT)^2/2\right]\left[M_{a0}+L\sqrt{mc_2/c_1}\|z(k)\|_1+\bar{b}|u(k)|+\sqrt{mc_4/c_1}M^*L(t-kT)^{1/2}\right]\\ &+\int_{kT}^{t}L(t-\tau+1)\sqrt{mc_4/c_1}(\tau-kT)^{1/2}\omega_k(\tau)\mathrm{d}\tau,\ \forall t\in[kT,(k+1)T].\end{aligned}\quad \text{(A3)}$$

From this, by the Gronwall-Bellman inequality, it is easy to verify the inequality (27). Hence, Lemma 1 is true.

Appendix B Proof of Proposition 1

For any given sampling period $T>0$, based on the core idea of the characteristic modeling that does not lose the characteristics of the plant (8), we consider the following approximate model:

$$\begin{cases} e_1(k+1)=e_1(k)+Te_2(k+1)-\displaystyle\int_{kT}^{(k+1)T}(t-kT)b_e(e_1(t),e_2(t),z(t),t)u(k)\mathrm{d}t, & \text{(B1)}\\ e_2(k+1)=e_2(k)+Ta_e(e_1(k),e_2(k),0,k)+\displaystyle\int_{kT}^{(k+1)T}b_e(e_1(t),e_2(t),z(t),t)u(k)\mathrm{d}t. & \text{(B2)}\end{cases}$$

By simple calculations, the above equation (B1) is equivalent to the following one:

$$\begin{aligned}e_1(k+1)=\ & 2e_1(k)-e_1(k-1)+T^2a_e(e_1(k),e_2(k),0,k)+\int_{kT}^{(k+1)T}[(k+1)T-t]b_e(e_1(t),e_2(t),z(t),t)u(k)\mathrm{d}t\\ &+\int_{(k-1)T}^{kT}[t-(k-1)T]b_e(e_1(t),e_2(t),z(t),t)u(k-1)\mathrm{d}t.\end{aligned}\quad \text{(B3)}$$

For the function $a_e(\cdot)$, by the differential mean value theorem, we obtain

$$a_e(e_1(k),e_2(k),0,k)=a_e(0,0,0,k)+(\partial a_e/\partial e_1)|_{(\varsigma_1 e_1(k),0,0,k)}\cdot e_1(k)+(\partial a_e/\partial e_2)|_{(e_1(k),\varsigma_2 e_2(k),0,k)}\cdot e_2(k), \quad \text{(B4)}$$

where $\varsigma_i \in (0,1)\,(i=1,2)$ are constants depending on the states $e_1(k), e_2(k)$. On the other hand, for (B3), it follows from the integral mean value theorem that

$$\begin{cases} \displaystyle\int_{kT}^{(k+1)T} [(k+1)T - t] b_e(e_1(t), e_2(t), z(t), t)\mathrm{d}t = (T^2/2) b_e(e_1(t), e_2(t), z(t), t)|_{t=(k+\varsigma_3)T}, \\ \displaystyle\int_{(k-1)T}^{kT} [t-(k-1)T] b_e(e_1(t), e_2(t), z(t), t)\mathrm{d}t = (T^2/2) b_e(e_1(t), e_2(t), z(t), t)|_{t=(k-\varsigma_4)T}, \end{cases} \tag{B5}$$

where $\varsigma_i \in (0,1)\,(i=3,4)$. Hence, by substituting (B4) and (B5) into (B3), and noticing our appointment $e_1(-1) = e_1(0) - Te_2(0)$ and $u(-1) = 0$, we can further derive that $\forall\, k \geqslant 0$,

$$\begin{aligned} e_1(k+1) &= f_1(k)e_1(k) + f_2(k)e_1(k-1) + (T^2/2)g_0(k)u(k) + (T^2/2)g_1(k)u(k-1) \\ &\quad + T^2 a_e(0,0,0,k) + (T^2/2)h(k)g_1(k)u(k-1), \end{aligned} \tag{B6}$$

where the time-varying parameters $f_1(k), f_2(k), g_0(k), g_1(k)$ satisfy (67) and (68), and $h(k) \triangleq T\ (\partial a_e/\partial e_2)|_{(e_1(k),\varsigma_2 e_2(k),0,k)}$. Also, noting that the last two terms in (B6) have the properties $T^2 a_e(0,0,0,k) = O(T^2)$, $h(k) = O(T)$, we omit these two terms. Consequently, based on the above discussion, by (B6) and (B1), we obtain the characteristic model of the system (8) that can be described by (65) and (66). Furthermore, from Assumption 1 and Eq. (9), it is easy to verify that $|\partial a_e/\partial e_i| = |\partial a/\partial x_i| \leqslant L$, $i=1,2$, which, together with Assumption 2, further derives that the coefficients of model (65) are uniformly bounded and belong to the compact set D. Therefore, the proof is complete.

Appendix C Proof of Lemma 2

We first calculate upper bounds of $|\delta_1^e(k+1)|$ and $|\delta_2^e(k+1)|$. From (72), according to Assumption 1 and (11), we obtain

$$|\delta_1^e(k+1)| \leqslant \int_{kT}^{(k+1)T} (t-kT)[L(|e_1(t)| + |e_2(t)| + \|z(t)\|_1) + M_{a0}]\mathrm{d}t, \tag{C1}$$

where L, M_{a0} are given by (2) and (11). So, from Lemma 1, and substituting (27) and (28) into (C1), it follows that

$$\begin{aligned} |\delta_1^e(k+1)| &\leqslant LT^2 p_\omega(T)[1/2 + (2/5)\sqrt{mc_4/c_1}T^{1/2}]|e_1(k)| + LT^2 p_\omega(T)[1/2 + T/3 + \sqrt{mc_4/c_1}((2/5)T^{1/2} + (2/7)T^{3/2})] \\ &\quad \cdot |e_2(k)| + LT^2\sqrt{mc_2/c_1}[1/2 + Lp_\omega(T)(T/3 + T^2/8 + \sqrt{mc_4/c_1}((2/7)T^{3/2} + (1/9)T^{5/2}))]\|z(k)\|_1 \\ &\quad + LT^3 p_\omega(T)[1/3 + T/8 + \sqrt{mc_4/c_1}((2/7)T^{1/2} + (1/9)T^{3/2})](\bar{b}|u(k)| + M_{a0}) + T^2 M_{a0}/2 \\ &\quad + LT^{5/2}\sqrt{mc_4/c_1}M^*[2/5 + Lp_\omega(T)((2/7)T + (1/9)T^2 + \sqrt{mc_4/c_1}((1/4)T^{3/2} + (1/10)T^{5/2}))] \\ &\triangleq c_{11}^e(T)|e_1(k)| + c_{12}^e(T)T|e_2(k)| + c_{13}^e(T)T\|z(k)\|_1 + c_{14}^e(T)T^2|u(k)| + c_{15}^e(T). \end{aligned} \tag{C2}$$

From this, it is not difficult to see that the functions $c_{1i}^e(\cdot) \in \mathcal{K}\,(i=1,\ldots,5)$ and satisfy $\max\{c_{11}^e(T), c_{15}^e(T)\} = O(T^2)$ and $\max_{j=2,3,4}\{c_{1j}^e(T)\} = O(T)$, $\forall T \in (0, T_{\max}]$. Furthermore, substituting the golden-section adaptive controller (70) into Eq. (C2), we have

$$\begin{aligned} |\delta_1^e(k+1)| &\leqslant \left[c_{11}^e(T) + 2c_{14}^e(T)l_1(2 + TL + T^2L)/\underline{b}\right]|e_1(k)| + [2c_{14}^e(T)l_2(1+TL)/\underline{b}]|e_1(k-1)| \\ &\quad + c_{12}^e(T)T|e_2(k)| + c_{13}^e(T)T\|z(k)\|_1 + c_{15}^e(T). \end{aligned} \tag{C3}$$

Hence, it is immediately verified from (C3) and the fact that the functions $c_{1i}^e(\cdot)\,(i=1,\ldots,5)$ belong to class $\mathcal{K}$ that the error $\delta_1^e(k+1)$ satisfies Eq. (76), and the corresponding coefficients satisfy the property (77). In the same way, for $\delta_2^e(k+1)$, by combining Assumption 1, the property (12), as well as (27) and (28), it can be obtained that

$$\begin{aligned} T|\delta_2^e(k+1)| &\leqslant LT^2[1 + p_\omega(T)(1 + (2/3)\sqrt{mc_4/c_1}T^{1/2})]|e_1(k)| + LT^2[1 + p_\omega(T)(1 + T/2 + \sqrt{mc_4/c_1}((2/3)T^{1/2} \\ &\quad + (2/5)T^{3/2}))]|e_2(k)| + LT^2\sqrt{mc_2/c_1}[1 + Lp_\omega(T)(T/2 + T^2/6 + \sqrt{mc_4/c_1}((2/5)T^{3/2} \\ &\quad + (1/7)T^{5/2}))]\|z(k)\|_1 + LT^3 p_\omega(T)[1/2 + T/6 + \sqrt{mc_4/c_1}((2/5)T^{1/2} + (1/7)T^{3/2})](\bar{b}|u(k)| + M_{a0}) \\ &\quad + T^2 M_{a1} + LT^{5/2}\sqrt{mc_4/c_1}M^* \cdot [2/3 + Lp_\omega(T)((2/5)T + (1/7)T^2 + \sqrt{mc_4/c_1}((1/3)T^{3/2} + (1/8)T^{5/2}))] \\ &\triangleq c_{21}^e(T)|e_1(k)| + c_{22}^e(T)T|e_2(k)| + c_{23}^e(T)T\|z(k)\|_1 + c_{24}^e(T)T^2|u(k)| + c_{25}^e(T), \end{aligned} \tag{C4}$$

where $c_{2i}^e(\cdot) \in \mathcal{K}\,(i=1,\ldots,5)$, and $\max\{c_{21}^e(T), c_{25}^e(T)\} = O(T^2)$, $\max_{j=2,3,4}\{c_{2j}^e(T)\} = O(T)$ hold for all $T \in (0, T_{\max}]$. Again, substituting the golden-section adaptive control (70) into (C4), we have

$$\begin{aligned} T|\delta_2^e(k+1)| &\leqslant \left[c_{21}^e(T) + 2c_{24}^e(T)l_1(2 + TL + T^2L)/\underline{b}\right]|e_1(k)| + [2c_{24}^e(T)l_2(1+TL)/\underline{b}]|e_1(k-1)| \\ &\quad + c_{22}^e(T)T|e_2(k)| + c_{23}^e(T)T\|z(k)\|_1 + c_{25}^e(T). \end{aligned} \tag{C5}$$

So, substituting (C3) and (C5) into (74), and combining (11), (68) and (70), we can further obtain that the upper bound of $\Delta_1^e(k+1)$ satisfies (75), in which the coefficients satisfy (77). As a result, we verify that the family $\{F_{cm}^e, u\}$ is consistent with $\{F_{ex}^e, u\}$, i.e., the corresponding modeling errors satisfy the consistency condition. Hence, Lemma 2 is proved.

Appendix D Proof of Lemma 3

First of all, by putting the golden-section adaptive controller (70) into the characteristic model (65), we can obtain the discrete-time closed-loop system as follows: $\forall k \geqslant 0$,

$$\begin{aligned} e_1(k+1) = &[f_1(k) - l_1\hat{f}_1(k)]e_1(k) + [f_2(k) - l_2\hat{f}_2(k) - l_1\hat{f}_1(k-1)]e_1(k-1) \\ &- l_2\hat{f}_2(k-1)e_1(k-2) + \tilde{g}_0(k)u(k) + \tilde{g}_1(k)u(k-1), \end{aligned} \tag{D1}$$

where $\tilde{g}_0(k) \triangleq T^2[g_0(k) - \hat{g_0}(k)]/2$ and $\tilde{g}_1(k) \triangleq T^2[g_1(k) - \hat{g_0}(k-1)]/2$. To analyze its stability, let us introduce the following notations: $\forall k \geqslant 0$,

$$E_1^*(k+1) \triangleq \begin{bmatrix} e_1(k+1) \\ e_1(k) \\ e_1(k-1) \end{bmatrix}, \quad B_1 \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_e^*(k) \triangleq \begin{bmatrix} \alpha_1(k) & \alpha_2(k) & \alpha_3(k) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{D2}$$

where $\alpha_1(k) = f_1(k) - l_1\hat{f}_1(k)$, $\alpha_2(k) = f_2(k) - l_2\hat{f}_2(k) - l_1\hat{f}_1(k-1)$, $\alpha_3(k) = -l_2\hat{f}_2(k-1)$, in which $f_i(k)$, $\hat{f}_i(k)$ $(i = 1, 2)$ are given by (67) and (71), respectively. Then the system (D1) can be rewritten as

$$E_1^*(k+1) = A_e^*(k)E_1^*(k) + B_1\left[\tilde{g}_0(k)u(k) + \tilde{g}_1(k)u(k-1)\right] = A_e^*(k)E_1^*(k) + B_1\varepsilon^{\mathrm{T}}(k)E_1^*(k), \quad \forall k \geqslant 0, \tag{D3}$$

where $A_e^*(k)$, B_1 are given by (D2), and

$$\varepsilon(k) \triangleq \left[\left(1 - \frac{g_0(k)}{\hat{g}_0(k)}\right) l_1\hat{f}_1(k), \left(1 - \frac{g_0(k)}{\hat{g}_0(k)}\right) l_2\hat{f}_2(k) + \left(1 - \frac{g_1(k)}{\hat{g}_0(k-1)}\right) l_1\hat{f}_1(k-1), \left(1 - \frac{g_1(k)}{\hat{g}_0(k-1)}\right) l_2\hat{f}_2(k-1)\right]. \tag{D4}$$

Now we proceed to analyze the stability of this closed-loop system. Rewrite $A_e^*(k) = A_c + B_1\zeta(k)$, where A_c, B_1 are given by (78) and (D2) respectively, and $\zeta(k)$ is defined by

$$\zeta(k) \triangleq \left[\big(f_1(k) - 2\big) + l_1\big(2 - \hat{f}_1(k)\big), \ \big(1 + f_2(k)\big) - l_2\big(1 + \hat{f}_2(k)\big) + l_1\big(2 - \hat{f}_1(k-1)\big), \ -l_2\big(1 + \hat{f}_2(k-1)\big)\right]. \tag{D5}$$

We now take a close look at the matrix A_c and the vector $\zeta(k)$. For the constant matrix A_c with the parameters $l_1 = 0.381, l_2 = 0.618$, by the Jury stability criterion [1)], we know that the eigenvalues of A_c are all located in the unit circle. Therefore, the spectral radius of the matrix A_c satisfies $\rho(A_c) = \max_{1\leqslant i\leqslant 3}\{|\lambda_i(A_c)|\} < 1$. In addition, for the vector $\zeta(k)$, it follows from (D5) as well as the properties (67) and (71) that there exists a positive number $c_\zeta > 0$ that only depends on the control parameters l_1, l_2, such that $\|\zeta(k)\| \leqslant c_\zeta TL$, $\forall k \geqslant 0$. Thus, (i) by the continuous dependence of eigenvalues of the matrix on its elements (see, e.g., Ostrowski theorem [24]), we have that for any $\rho_1^e \in (\rho(A_c), 1)$, there exists $T_1^e \in (0, T_{\max}]$ such that for all $T \in (0, T_1^e)$, $\rho(A_e^*(k)) = \max_{1\leqslant i\leqslant 3}\{|\lambda_i(A_e^*(k))|\} < \rho_1^e < 1$, $\forall k \geqslant 0$, and from this it can be immediately obtained that $\limsup_{k\to\infty}\rho(A_e^*(k)) \leqslant \rho_1^e < 1$. (ii) The variation of the matrix sequence $\{A_e^*(k), k \geqslant 0\}$ satisfies $\limsup_{k\to\infty}\|A_e^*(k+1) - A_e^*(k)\| = \limsup_{k\to\infty}\|B_1[\zeta(k+1) - \zeta(k)]\| \leqslant 2c_\zeta TL$. (iii) From (D2), and by the boundedness given in (69) and (71), we have that there exists a positive number $M_{e0} > 0$ depending on the parameters $\{l_1, l_2, L, T_{\max}\}$ such that $\sup_{k\geqslant 0}\|A_e^*(k)\| = M_{e0} < \infty$. Hence, combining (i)–(iii), and based on the stability theory of slowly time-varying linear systems (see Theorem 2.4.1 in [25]), we obtain that there exists a sampling period $T_2^e \leqslant T_1^e$ that only depends on the parameters ρ_1^e, M_{e0}, c_ζ, and L, such that when $T \in (0, T_2^e]$,

$$\|\Phi^e(k+1, i)\| \leqslant M_{e1}\lambda_{e1}^{k+1-i}, \quad \forall k \geqslant i \geqslant 0, \tag{D6}$$

where $M_{e1} > 0$, $\lambda_{e1} \in (0, 1)$ are constants, and the matrix sequence $\{\Phi^e(k, i), \forall k \geqslant i \geqslant 0\}$ is defined by $\Phi^e(k+1, i) = A_e^*(k)\Phi^e(k, i)$, $\Phi^e(i, i) = I_3$, $\forall k \geqslant i \geqslant 0$. Moreover, for $\varepsilon(k)$ given by (D4), it follows from the properties (69) and (71) that

$$\sup_{k\geqslant 0}\|B_1\varepsilon^{\mathrm{T}}(k)\| = \sup_{k\geqslant 0}\|\varepsilon(k)\| \leqslant \delta_0^e(l_1, l_2)\left(\bar{b} - \underline{b}\right)/\underline{b} + O(T), \tag{D7}$$

where $\delta_0^e(l_1, l_2) = \sqrt{8l_1^2 + 2l_2^2 + 4l_1l_2}$. Again, from (D3), we know that

$$E_1^*(k+1) = \Phi^e(k+1, 0)E_1^*(0) + \sum_{i=0}^{k}\Phi^e(k+1, i+1)B_1\varepsilon^{\mathrm{T}}(i)E_1^*(i). \tag{D8}$$

Substituting (D6) and (D7) into (D8), we further have

$$\|E_1^*(k+1)\| \leqslant M_{e1}\lambda_{e1}^{k+1}\|E_1^*(0)\| + \sum_{i=0}^{k}M_{e1}\lambda_{e1}^{k+i}[\delta_0^e\left(\bar{b} - \underline{b}\right)/\underline{b} + O(T)] \cdot \|E_1^*(i)\|.$$

From this and by the Gronwall-Bellman inequality, it is easy to verify that

$$\|E_1^*(k+1)\| \leqslant M_{e1}\left(\lambda_{e1} + M_{e1}\delta_0^e(\bar{b} - \underline{b})/\underline{b} + O(T)\right)^{k+1}\|E_1^*(0)\|. \tag{D9}$$

Thus, by taking $C_b^e \triangleq 1 + (1 - \lambda_{e1})/(M_{e1}\delta_0^e)$, and under the condition (79), it follows that $\lambda_{e1} + M_{e1}\delta_0^e(\bar{b} - \underline{b})/\underline{b} < 1$. Therefore, for any $\lambda_{e1} + M_{e1}\delta_0^e(\bar{b} - \underline{b})/\underline{b} < \lambda_*^e < 1$, there exists $T_*^e \leqslant T_2^e$ such that $\lambda_{e1} + M_{e1}\delta_0^e(\bar{b} - \underline{b})/\underline{b} + O(T) < \lambda_*^e < 1$ holds for any $T \in (0, T_*^e)$. By this and (D9), we finally obtain that the discrete-time closed-loop system (D3) is exponentially stable. Equivalently, the closed-loop characteristic model (D1) is exponentially stable. Therefore, the assertion of this lemma is correct.

1) Hu S S. Automatic Control (in Chinese). 5th ed. Beijing: Science Press, 2007.