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Quantum network coding for multi-unicast problem based on 2D and 3D cluster states

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Abstract We mainly consider quantum multi-unicast problem over directed acyclic network, where each source wishes to transmit an independent message to its target via bottleneck channel. Taking the advantage of global entanglement state 2D and 3D cluster states, these problems can be solved efficiently. At first, a universal scheme for the generation of resource states among distant communication nodes is provided. The corresponding between cluster and bigraph leads to a constant temporal resource cost. Furthermore, a new approach based on stabilizer formalism to analyze the solvability of several underlying quantum multi-unicast networks is presented. It is found that the solvability closely depends on the choice of stabilizer generators for a given cluster state. And then, with the designed measurement basis and parallel measurement on intermediate nodes, we propose optimal protocols for these multi-unicast sessions. Also, the analysis reveals that the resource consumption involving spatial resources, operational resources and temporal resources mostly reach the lower bounds.

Keywords quantum information, network coding, multi-unicast, cluster state, stabilizer formalism

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1 Introduction

Network coding [1], for some networks, can give a higher transmission rate, as well as a lower resource consumption compared to the widely used store-forward routing scheme [2]. Classical network coding, since proposed, has been widely studied both in theory [3–5] and application [6–8]. A simple instance for multicast problem that can demonstrate the network coding advantage (unit-rate transmission via network coding) is the butterfly network as shown in Figure 1. In quantum network, multicast problem reduces to multi-unicast problem due to quantum no-cloning theorem [9]. Thus, network coding for quantum multi-unicast problem is of considerable interest [10–13].

The study on quantum network coding (QNC) is still in its infancy. To analyze the solvability of specific quantum multi-unicast network and to design constructive protocols with lower communication cost are still two major issues [14]. We say that (quantum multi-unicast) network is solvable with fidelity

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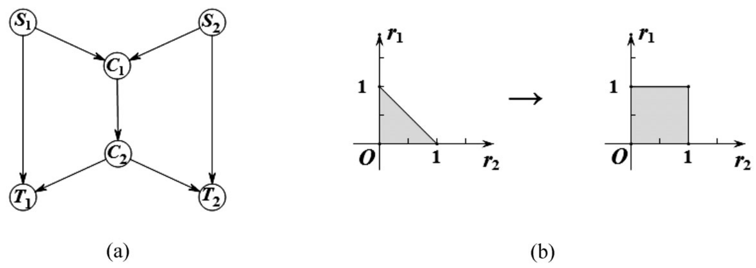

Figure 1 (a) S_1 and S_2 are source nodes. Linear coding operations are performed at intermediate nodes C_1 and C_2 as well as target nodes T_1 and T_2 by single use of the network. (b) (r_1, r_2) be the transmission rate pair from S_1 to T_1 and S_1 to T_1 respectively. Compared with store-forward routing model, the achievable transmission rate region extends to a square from a triangle.

F if there is a choice of quantum operations $\mathcal{P}$ (called a solution) such that every source message can be sent to its target with fidelity at least F. In particular, when $F = 1$, it is simply called solvable. The study on quantum network solvability got going in 2006. Hayashi et al. [10] first presented that butterfly network (two-unicast network) is solvable with fidelity less than 0.983 in the basic setting (all quantum nodes connected by noiseless quantum channels with unit capacity). In other words, butterfly network is unsolvable (with fidelity one) in this network setting. Also in above setting, Leung et al. [11] consider the asymptotic solvability with fidelity tending to one as a significantly large use of the network.

The need for perfect solvability (fidelity equals one) promoted the study. Related to these studies, multi-particle entanglement including local entanglement and global entanglement is introduced to address such imperfection and low transmission rate problems. Leung et al. [11] also explored the solvability over the butterfly and other networks with additional resource (forward, backward, two-way classical assistance) by locally pre-shared 2-particle entangled states between some pairs of nodes. Besides, Refs. [12, 15–18] considered QNC with locally pre-shared 2-particle entangled states between source-target pair or neighbor nodes. In these scenarios, teleportation achieves information transmission. Instead, global entanglement resource is also introduced to QNC [19–23]. In this scenario, remote nodes together pre-share a pure multi-particle entangled state, with local operation and classical communication (LOCC) to transmit quantum information. In Refs. [19–22], the multi-particle entangled state is pre-shared among all network nodes by introducing additional quantum systems. By constructive proof, Refs. [20, 21], give the sufficient condition that quantum network is solvable, that is, the corresponding classical network is solvable. In Ref. [23], Beaudrap demonstrated that these protocols in Refs. [20, 21] correspond in a natural way to measurement-based quantum computations [24, 25]. This observation offers a new perspective to analyze the solvability for quantum multi-unicast network. Recently, measurement-based quantum network communication [26, 27] has attracted much attention. Cluster state [28] as a universal resource for measurement-based quantum computation [24], was firstly introduced by Raussendorf et al. in 2001. It has higher persistency of entanglement compared with Bell state and GHZ state [28]. In Ref. [24], they proved that a unitary operator U can be realized up to local Pauli operators if $2n$ eigenvalue equations can be constructed successfully after proper design of measurement on intermediate nodes. This theorem plays an important role in this paper for analyzing the specific multi-unicast network. However, this theorem does not imply anything about the specific measurement models (measurement basis and measurement orders) which need to be designed properly.

The purpose of this paper is twofold. We attempt to either propose a new approach to analyze the solvability over multi-unicast network or design constructive protocols with lower resources cost. We introduce 2D and 3D cluster state to butterfly, grail and extended butterfly networks. Single qubit measurement drives the information transmission and classical communication is free. Based on stabilizer formalism, we proposed an alternative method to analyze the solvability over these networks. Eigenvalue equations are constructed according to designed measurement basis and measurement orders. Parallel measurement leads to a significant reduction of the communication complexity.

This paper is organized as follows. In Section 2 we explore how to generate cluster state efficiently

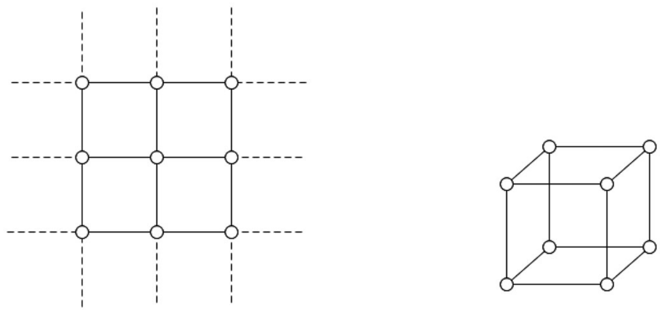

Figure 2 Examples of clusters with $d = 2$ and $d = 3$.

among distant communication nodes connected by directed edge. A universal scheme is presented with constant temporal resource cost. In Section 3 we analyze the solvability on 2-unicast session over butterfly and grail networks and design constructive protocols. A new approach is given based on stabilizer formalism. Section 4 is a generalization on extended butterfly network supporting 4-unicast session. Finally, in Section 5, we discuss the resource consumption and conclude this paper.

2 Generation of cluster state associated with a topological graph

In this section, we provide a universal scheme for the generation of cluster state among distant communication nodes connected by directed quantum channel. A usual way [24] to generate a cluster state is as follows:

- Initiate all communication nodes with quantum state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$;
- Perform unitary operation CZ on each pair of neighbor nodes.

However, for distant quantum systems, an alternative way should be introduced for efficient generation of cluster state. Before we go into the detailed scheme, in Subsection 2.1, we first present a lemma, associating a cluster to bigraph, by which we can classify the nodes into two sets. In each set, nodes can perform quantum operations simultaneously, leading to a constant temporal resource cost. In Subsection 2.2, we give our scheme in detail.

Cluster state $|\Phi\rangle_{\mathcal{C}}$ [24] is a multi-particle entangled quantum state associated to a cluster $\mathcal{C}$ which is a connected subset of a simple cubic lattice Z^d in $d \geqslant 1$ dimensions. In a cluster, each site a has $2d$ neighboring sites. If occupied, these are the sites whose qubit interacts with the qubit at a. Cluster state $|\Phi\rangle_{\mathcal{C}}$ on a cluster $\mathcal{C}$ obeys the set of eigenvalue equations

$$K^a|\Phi\rangle_{\mathcal{C}} = |\Phi\rangle_{\mathcal{C}}, \tag{1}$$

with

$$K^a = \sigma_x^{(a)} \bigotimes_{b\in \mathrm{ngh}(a)} \sigma_z^{(b)}, \tag{2}$$

where $\mathrm{ngh}(a)$ is the set of neighbor nodes adjacent to $a \in \mathcal{C}$, $1 \leqslant |\ \mathrm{ngh}(a)\ | \leqslant 2d$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(we also denote σ_x as X and σ_z as Z if needed). From the perspective of stabilizer formalism (for more detailed background see [29]), cluster state $|\Phi\rangle_{\mathcal{C}}$ is a quantum state uniquely stabilized by $\mathcal{K} = \langle\{K^a\}_{a\in\mathcal{C}}\rangle$. Namely, $\mathcal{K}$ is the stabilizer of $|\Phi\rangle_{\mathcal{C}}$, $\{K^a\}_{a\in G}$ is the set of generators and a is the correlation center. In our work, we discuss the case for $d = 2$ and $d = 3$, examples are illustrated in Figure 2(a) and (b) respectively.

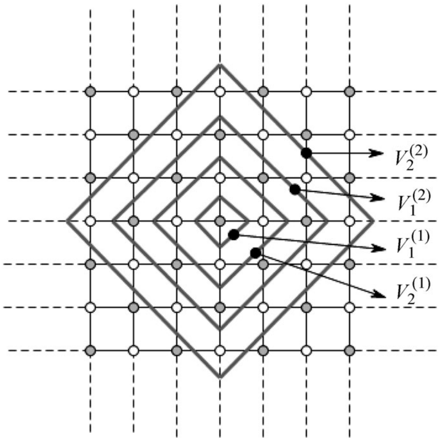

Figure 3 Partition on a cluster.

2.1 A cluster and bigraph

In this subsection, we discuss the correspondence between a cluster and bigraph. We find that a cluster is naturally a bigraph as shown in Lemma 1. This property is of great significance to the generation of cluster state over large-scale quantum network. Any finite nodes can be partitioned into two sets. All nodes in each set can perform quantum operations simultaneously. Temporal resource cost remains a constant with the increase of network scale.

A simple graph $G = (V, E)$ with V and E the set of nodes and edges in G, is called a bigraph if there exists a bipartition V_1 and V_2 for V with

(i) $V = V_1 \bigcup V_2$,

(ii) $V_1 \bigcap V_2 = \emptyset$,

(iii) V_1 and V_2 are independent set, that is, any two nodes in V_i are nonadjacent, $i = 1, 2$.

The degree $d(v)$ of node v is the amount of node in G connected with v, i.e., $d(v) = |\mathrm{ngh}(v)|$. It is easy to see, for any $v \in V_1$, $\mathrm{ngh}(v) \subset V_2$.

In fact, each site in a cluster $\mathcal{C}$ corresponds to a node, and each pair of adjacent sites corresponds to an edge. Thus a cluster $\mathcal{C}$ naturally corresponds to a simple graph $G = (V, E)$. For the dimension $d = 2$ (resp. $d = 3$), since each site s in $\mathcal{C}$ has 2×2 (resp. 2×3) neighboring sites, the amount of adjacent sites with s are no more than 4 (resp. 6). This characteristic for a cluster makes it correspond to a simple graph with degree of each nodes less than 4 (resp. 6). We claim that this graph is a bigraph.

Lemma 1. A cluster for $d = 2$ (resp. $d = 3$) is naturally corresponding to a bigraph with the degree of any node less than 4 (resp. 6), i.e., for any $v \in V$, $1 \leqslant d(v) \leqslant 4$ (resp. $1 \leqslant d(v) \leqslant 6$).

Proof. For $d = 2$, to illustrate this, we need to give a partition to all nodes. First, select one node $v \in V$ with maximum degree and let $V_1^{(1)} = \{v\}$. Also get that $\mathrm{ngh}(V_1^{(1)}) = V_2^{(1)}$ is the set of all neighbor sites of nodes in $v \in V$. Further, let $V_1^{(2)} = \bigcup_{w \in V_2^{(1)}} \mathrm{ngh}(w) - V_1^{(1)}$ and $V_2^{(2)} = \mathrm{ngh}(V_1^{(2)}) - V_2^{(1)}$. As illustrated in Figure 3, repeat finitely the above process which can cover all nodes in V (since any two sites in a cluster are connected), we get an array of sets $V_1^{(1)}, V_1^{(2)}, V_1^{(3)}, \ldots, V_2^{(1)}, V_2^{(2)}, V_2^{(3)}, \ldots$. Let

$$V_1 = V_1^{(1)} \cup V_1^{(2)} \cup V_1^{(3)} \cup \cdots, V_2 = V_2^{(1)} \cup V_2^{(2)} \cup V_2^{(3)} \cup \cdots,$$

we get two independent sets with $V = V_1 \bigcup V_2$, $V_1 \bigcap V_2 = \emptyset$ and any pair of nodes in V_1 or V_2 are nonadjacent, that is, the conditions (i), (ii) and (iii) hold. Consequently, we conclude that $G = (V, E)$ corresponding to the cluster is a bigraph. In other words, we get a partition for nodes in V. Similar partition can be constructed for cluster with $d = 3$ that makes it a bigraph.

2.2 Generation of the cluster state

The general scheme for procedures to generate a cluster state over distant communication nodes is as follows. This modified scheme based on Eq. (3) that nonlocal unitary operation $CZ_{a \to b}$ on quantum

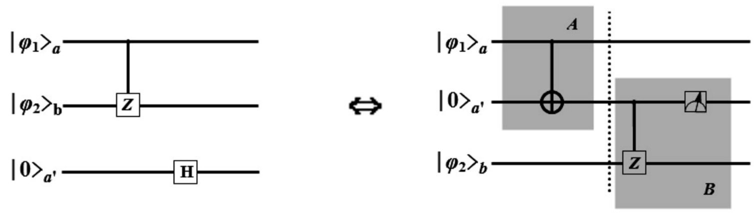

Figure 4 Simulation of nonlocal unitary operation: at A, CX performs on systems a and a' and then system a' is sent to B who performs CZ on systems a', b and measures a' in X-basis.

systems a and b can be simulated by introducing an additional quantum system a' initialized with $|0\rangle$ as illustrated in Figure 4,

$$CZ_{a\to b}\otimes H_{a'}|\varphi_1\rangle_a|\varphi_2\rangle_b|0\rangle_{a'}=\left(\frac{I+\sigma_x}{2}\right)_{a'}CZ_{a'\to b}CX_{a\to a'}|\varphi_1\rangle_a|\varphi_2\rangle_b|0\rangle_{a'},\tag{3}$$

where $CZ_{a\to b}=|0\rangle\langle 0|_a\otimes I_b+|1\rangle\langle 1|_a\otimes Z_b$, $CX_{a\to a'}=|0\rangle\langle 0|_a\otimes I_{a'}+|1\rangle\langle 1|_a\otimes X_{a'}$. Now we give this modified scheme. For brevity, the following notation would be used: Let $|V_1|=n$, $|V_2|=m$,

- v_i: node in $V_1, i=1,\ldots,n$;
- w_j: node in $V_2, j=1,\ldots,m$;
- v_{ik}: fictitious node which belongs to v_i and sents to w_k eventually.

In addition, for $\forall v_i\in V_1, w_j\in V_2$, let $|\mathrm{ngh}(v_i)|=n_i$, $|\mathrm{ngh}(w_j)|=m_j$.

Scheme 1

(1) For each node $v\in V_1$, prepare quantum state $|+\rangle$ as well as auxiliary quantum states $|0\rangle^{\otimes|\mathrm{ngh}(v)|}$ (a product state for multiple particles), and prepare $|+\rangle$ only for each node in V_2. The initial state becomes

$$|\Psi\rangle_{\text{initial}}=\bigotimes_{v_i\in V_1}|+\rangle_{v_i}|0\rangle^{\otimes n_i}_{v_{i1}\cdots v_{in_i}}\bigotimes_{w_j\in V_2}|+\rangle_{w_j}.\tag{4}$$

(2) Entangle $|+\rangle_{v_i}$ with all its auxiliary qubits by control operator $CX=\sum_{x_i\in V_1}|x_i\rangle\langle x_i|\bigotimes_{j\in\{1,\ldots,n_i\}}X^{(x_i)}_{v_{ij}}$. The resulting state becomes

$$|\Psi\rangle^{(1)}_{\text{Entangled}}=\frac{1}{2^{\frac{|V_1|}{2}}}\bigotimes_{v_i\in V_1}(|0\rangle^{\otimes(1+n_i)}+|1\rangle^{\otimes(1+n_i)})_{v_iv_{i1}\cdots v_{in_i}}\bigotimes_{w_j\in V_2}|+\rangle_{w_j}.\tag{5}$$

Then send each v_{ik} to corresponding node w_k. Note that the nodes in V_1 can perform these operations simultaneously .

(3) For each $w_j\in V_2$, let $in(w_j)$ be the received auxiliary quantum systems from its neighbor nodes, on these particles control operator $CZ_{in(w_j)\to w_j}=\sum_{\boldsymbol{y}\in\{0,1\}^{\otimes|in(w_j)|}}|\boldsymbol{y}\rangle\langle\boldsymbol{y}|\otimes\sigma_Z^{(\boldsymbol{y})}$ is performed on all quantum systems $in(w_j)$ and w_j with

$$\sigma_Z^{\boldsymbol{y}}=\begin{cases}I, & \bigoplus_{y\in\boldsymbol{y}}y=0;\\ \sigma_Z, & \bigoplus_{y\in\boldsymbol{y}}y=1.\end{cases}$$

The quantum state becomes

$$|\Psi\rangle^{(2)}_{\text{Entangled}}=\frac{1}{2^{\frac{|V_1|}{2}}}\sum_{x_{v_1},\cdots,x_{v_n}\in Z_2}|x_{v_1}\cdots x_{v_n}\rangle_{v_1\cdots v_n}\bigotimes_{w_j\in V_2,\boldsymbol{x}\subset\{x_{v_1},\ldots,x_{v_n}\}}|\boldsymbol{x}\rangle_{in(w_j)}|+_{(\boldsymbol{x})}\rangle_{w_j},\tag{6}$$

where

$$|+_{(\boldsymbol{x})}\rangle_{w_j}=\begin{cases}|+\rangle, & \bigoplus_{x\in\boldsymbol{x}}x=0;\\ |-\rangle, & \bigoplus_{x\in\boldsymbol{x}}x=1.\end{cases}$$

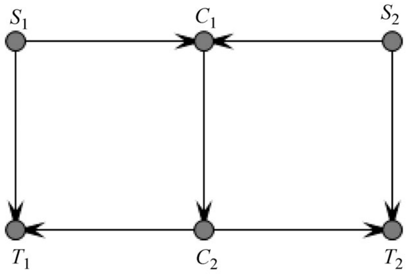

Figure 5 $G_{(B)} = (V_B, E_B)$ butterfly network nodes located on 2D cubic lattice.

(4) Measure the auxiliary particles in X-basis, thereby the cluster state associated to a graph is obtained with proper pauli operators according to classical measurement results, that is

$$|\Psi\rangle^{(2)}_{\text{Entangled}} = \frac{1}{2^{\frac{|V_1|}{2}}} \sum_{x_{v_1},\cdots,x_{v_n} \in Z_2} |x_{v_1} \cdots x_{v_n}\rangle_{v_1 \cdots v_n} \bigotimes_{w_j \in V_2, \boldsymbol{x} \subset \{x_{v_1},\ldots,x_{v_n}\}} |+_{(\boldsymbol{x})}\rangle_{w_j}. \tag{7}$$

All nodes in V_1 can perform quantum operations simultaneously as well as all nodes in V_2. Accordingly, two operation rounds are sufficient. This leads to significant reduction of the resource cost.

It should be noted that one-way quantum channel combined with free classical communication can reverse the quantum channel. Thus opposite quantum channel does not invalidate Scheme 1.

The resources, in Scheme 1, involve directed quantum communication, free classical communication and local unitary operations. Under the existing technology, these resources are available, thus, our scheme is technically feasible.

3 2-unicast problem over butterfly network and grail network

In this section, we mainly introduce the approach to analyze the solvability of multi-unicast network based on stabilizer formalism. Specific details are offered for the butterfly and grail network, both of which support 2-unicast session. The solvability can be proved by constructing the generators of the stabilizer correctly. Furthermore, we give the exact protocol according to the designed measurement model including measurement basis and measurement orders on intermediate nodes.

3.1 2-unicast problem over butterfly network

In butterfly network, two sources holding nodes S_1 and S_2 wish to communicate with their targets T_1 and T_2 respectively. Note that the butterfly network is naturally corresponding to a 2D cubic lattice of size six (as shown in Figure 5). Thus, the butterfly network graph is a bigraph denoted as $G_{(B)}$.

The exact bipartition is as follows: $V_1 = \{S_1, S_2, C_2\}$, $V_2 = \{T_1, T_2, C_1\}$ with conditions (i), (ii) and (iii). Additionally, it follows $\text{ngh}(S_1) = \{T_1, C_1\}$, $\text{ngh}(S_2) = \{T_2, C_1\}$ and $\text{ngh}(C_2) = \{T_1, T_2, C_1\}$.

Also note that, we introduce additional input systems S'_1 and S'_2 held by two sources respectively. Specifically,

- $In(G_{(B)}) = \{S'_1, S'_2\}$: the set of input quantum systems.
- $\text{Media}(G_{(B)}) = \{S_1, S_2, C_1, C_2\}$: the set of intermediate quantum systems.
- $\text{Out}(G_{(B)}) = \{T_1 = \text{out}(S'_1), T_2 = \text{out}(S'_2)\}$: the set of output quantum systems.

3.1.1 *Analysis of solvability*

There have been several papers analyzing the solvability over butterfly network [10,11]. In this subsection, an alternative proof is provided built upon stabilizer formalism. In Ref. [24], they proved the sufficient conditions for successfully simulating an n-qubit unitary operation U. They claimed that if $2n$ eigenvalue equations can be constructed successfully after proper design of measurement model on intermediate nodes over a cluster state, U can be realized up to some local pauli operators. However, this theorem does not imply anything about the specific measurement basis and measurement orders which need to be designed properly.

Lemma 2. [24] Let $G_{(\mathcal{C})}$ be a graph corresponding to a cluster $\mathcal{C}$ with

- $G_{(\mathcal{C})} = In(G_{(\mathcal{C})}) \bigcup \mathrm{Media}(G_{(\mathcal{C})}) \bigcup \mathrm{Out}(G_{(\mathcal{C})})$,
- $In(G_{(\mathcal{C})}) \bigcap \mathrm{Media}(G_{(\mathcal{C})}) = In(G_{(\mathcal{C})}) \bigcap \mathrm{Out}(G_{(\mathcal{C})}) = \mathrm{Media}(G_{(\mathcal{C})}) \bigcap \mathrm{Out}(G_{(\mathcal{C})}) = \emptyset$.

$|\varPhi\rangle_{G_{(\mathcal{C})}}$ is the cluster state on $G_{(\mathcal{C})}$ and U is an n-qubit unitary operation. If

$$|\varPsi\rangle_{G_{(\mathcal{C})}} = \mathcal{P}^{\mathrm{Media}(G_{(\mathcal{C})})}|\varPhi\rangle_{G_{(\mathcal{C})}}, \tag{8}$$

where $\mathcal{P}^{\mathrm{Media}(G_{(\mathcal{C})})}$ denotes measurement model on intermediate nodes, obeys the $2n$ eigenvalue equations

$$\begin{aligned} \sigma_x^{(i)}(U\sigma_x U^\dagger)^{\mathrm{out}(i)}|\varPsi\rangle_{G_{(\mathcal{C})}} &= (-1)^{\lambda_{x,i}}|\varPsi\rangle_{G_{(\mathcal{C})}}, \\ \sigma_z^{(i)}(U\sigma_z U^\dagger)^{\mathrm{out}(i)}|\varPsi\rangle_{G_{(\mathcal{C})}} &= (-1)^{\lambda_{z,i}}|\varPsi\rangle_{G_{(\mathcal{C})}}, \end{aligned} \tag{9}$$

where $i \in In(G_{(\mathcal{C})})$, $1 \leqslant i \leqslant n$, and $\lambda_{x,i}, \lambda_{z,i} \in \{0,1\}$. Then, on $G_{(\mathcal{C})}$ the n-qubit unitary operation U acting on an arbitrary quantum input state $|\psi_{in}\rangle$ can be realized up to some pauli operators with the measurement model in $\mathrm{Media}(G_{(\mathcal{C})})$ described by $\mathcal{P}^{\mathrm{Media}(G_{(\mathcal{C})})}$ and in $In(G_{(\mathcal{C})})$ being X-basis measurements.

For network topology supporting multi-unicast session, unitary operation $\widetilde{U}$ can be considered as a permutation on the input quantum systems and resulting states read from corresponding output systems. Specially, for butterfly network supporting 2-unicast session, this permutation is a swapping operation U_{swap} on two quantum systems A and B,

$$U_{\mathrm{swap}_{A\leftrightarrow B}} = \sum_{a,b\in\{0,1\}} |ab\rangle\langle ba|, \tag{10}$$

with

$$\begin{aligned} U_{\mathrm{swap}_{A\leftrightarrow B}}\sigma_x^A U^\dagger_{\mathrm{swap}_{A\leftrightarrow B}} &= \sigma_x^B, \\ U_{\mathrm{swap}_{A\leftrightarrow B}}\sigma_x^B U^\dagger_{\mathrm{swap}_{A\leftrightarrow B}} &= \sigma_x^A, \\ U_{\mathrm{swap}_{A\leftrightarrow B}}\sigma_z^A U^\dagger_{\mathrm{swap}_{A\leftrightarrow B}} &= \sigma_z^B, \\ U_{\mathrm{swap}_{A\leftrightarrow B}}\sigma_z^B U^\dagger_{\mathrm{swap}_{A\leftrightarrow B}} &= \sigma_z^A. \end{aligned} \tag{11}$$

Now, it is sufficient for us to prove the solvability of 2-unicast session over butterfly network. The key concern is to design the measurement model $\mathcal{P}^{\mathrm{Media}(G_{(\mathcal{C})})}$ on intermediate nodes making the eigenvalue equations (9) hold.

- Suppose that we have a six-particle cluster state $|\varPhi\rangle_{\mathrm{initial}}$ associated to butterfly network graph $G_{(B)}$. Sources introduce two additional quantum systems S_1' and S_2' in states $|+\rangle_{S_1'}$ and $|+\rangle_{S_2'}$ as the input respectively and perform CZ operation to obtain an eight-particle cluster state $|\varPhi\rangle_{G_{(B)}}$ as illustrated in Figure 6(a) and (b).
- Select correlation centers on which we perform X operation and its neighbor nodes perform Z operation. The choice of correlation centers decides the measurement model $\mathcal{P}^{\mathrm{Media}(G_{(B)})}$ (measurement basis and orders) and then the solvability of a given communication task. The correlation centers are shown in Figure 7(b) for each eigenvalue equation. There follow the equations:

$$\begin{aligned} |\varPhi\rangle_{G_{(B)}} &= K^{(S_1',C_1,T_2)}|\varPhi\rangle_{G_{(B)}} = \sigma_x^{(S_1',C_1,T_2)}|\varPhi\rangle_{G_{(B)}}, \\ |\varPhi\rangle_{G_{(B)}} &= K^{(S_2',C_1,T_1)}|\varPhi\rangle_{G_{(B)}} = \sigma_x^{(S_2',C_1,T_1)}|\varPhi\rangle_{G_{(B)}}, \\ |\varPhi\rangle_{G_{(B)}} &= K^{(S_1,C_2)}|\varPhi\rangle_{G_{(B)}} = \sigma_x^{(S_1,C_2)}\sigma_z^{(S_1',T_2)}|\varPhi\rangle_{G_{(B)}}, \\ |\varPhi\rangle_{G_{(B)}} &= K^{(S_1,C_2)}|\varPhi\rangle_{G_{(B)}} = \sigma_x^{(S_2,C_2)}\sigma_z^{(S_2',T_1)}|\varPhi\rangle_{G_{(B)}}. \end{aligned} \tag{12}$$

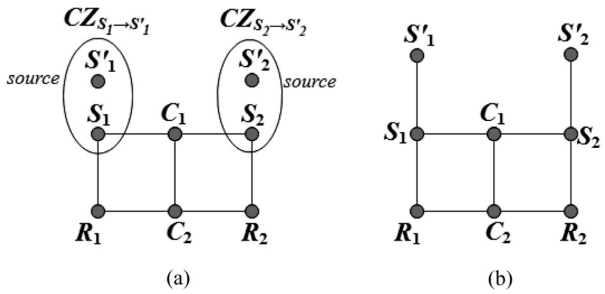

Figure 6 (a) Each source introduces an additional quantum system in state $|+\rangle$ and performs CZ operation to obtain an eight-particle cluster state (b).

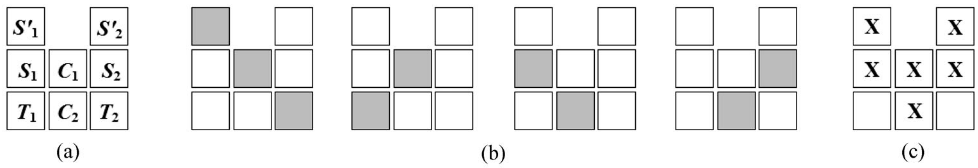

Figure 7 (a) Another graphical representation of the eight-particle cluster state $|\Phi\rangle_{G_{(B)}}$; (b) the correlation centers for the construction of eigenvalue equations (13); (c) the measurement basis on intermediate and input quantum systems.

• Measure the intermediate quantum systems S_1, S_2, C_1 and C_2 in X-basis, and get classical results s_1, s_2, c_1 and c_2. The above equations induce the following eigenvalue equations for the projected state $|\Psi\rangle_{G_{(B)}}$:

$$\begin{aligned}
\sigma_x^{(S'_1,T_2)}|\Psi\rangle_{G_{(B)}} &= \sigma_x^{(S'_1)}(U_{\text{swap}}\sigma_x^{(T_1)}U^{\dagger}_{\text{swap}})|\Psi\rangle_{G_{(B)}} = (-1)^{c_1}|\Psi\rangle_{G_{(B)}},\\
\sigma_x^{(S'_2,T_1)}|\Psi\rangle_{G_{(B)}} &= \sigma_x^{(S'_2)}(U_{\text{swap}}\sigma_x^{(T_2)}U^{\dagger}_{\text{swap}})|\Psi\rangle_{G_{(B)}} = (-1)^{c_1}|\Psi\rangle_{G_{(B)}},\\
\sigma_z^{(S'_2,T_1)}|\Psi\rangle_{G_{(B)}} &= \sigma_2^{(S'_1)}(U_{\text{swap}}\sigma_2^{(T_1)}U^{\dagger}_{\text{swap}})|\Psi\rangle_{G_{(B)}} = (-1)^{s_1+c_2}|\Psi\rangle_{G_{(B)}},\\
\sigma_z^{(S'_1,T_2)}|\Psi\rangle_{G_{(B)}} &= \sigma_z^{(S'_2)}(U_{\text{swap}}\sigma_z^{(T_2)}U^{\dagger}_{\text{swap}})|\Psi\rangle_{G_{(B)}} = (-1)^{s_2+c_2}|\Psi\rangle_{G_{(B)}}.
\end{aligned} \tag{13}$$

According to Lemma 2, swapping operation U_{swap} on butterfly network can be realized up to some local unitary operations. In other words, the 2-unicast session is solvable over butterfly network. As shown in Figure 7(c), X-basis measurement on all nodes except for target nodes can complete 2-unicast session and all measurements are parallelized. A specific protocol is presented according to this measurement model in the following subsection.

3.1.2 *Our protocol*

The procedure to realize swapping operation on butterfly network with $In(G_{(B)}) = \{S'_1, S'_2\}, \text{Out}(G_{(B)}) = \{T_1 = \text{out}(S'_1), T_2 = \text{out}(S'_2)\}$ is as follows.

• Prepare six-particle cluster state $|\Phi\rangle_{(\text{initial})}$ related to butterfly network according to Scheme 1.

• Each source introduces an arbitrary input quantum system and entangles it with its quantum system belonging to the cluster state.

• Measure all qubits in the cluster state as well as the input quantum systems except for the output qubits in X-basis simultaneously (the choice of measurement basis is based on the analysis in Subsection 3.1.1).

• Recover the original states with the help of classical communication and local pauli operators.

a) Generation of eight particles entangled state associated to butterfly network

Let the initial states of quantum systems S'_1 and S'_2 be

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle, \quad |\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle.$$

(1) Prepare $|+\rangle$ for all nodes in $V_{G_{(B)}}$ and auxiliary states $|0\rangle$ for each node $v \in V_1$ according to the amount of neighbor nodes $|\mathrm{ngh}(v)|$. That is, for $S_1 \in V_1$, prepare $|0\rangle^{\otimes 2}_{\{C_1',T_1'\}}$, and so on. After that, the initial state is prepared as follows:

$$|\Phi\rangle_{\text{initial}} = |+\rangle_{S_1}|0^{\otimes 2}\rangle_{\{C_1',T_1'\}}|+\rangle_{S_2}|0^{\otimes 2}\rangle_{\{C_1'',T_2'',R_2'\}}|+\rangle_{C_2}|0^{\otimes 3}\rangle_{\{C_1''',T_1'',T_2''\}}|+^{\otimes 3}\rangle_{\{C_1,T_1,T_2\}}. \tag{14}$$

(2) Select proper nodes to be the control nodes for the generating cluster state. This selection is of great importance in reducing the communication cost. For each node in V_1, perform CX on its qubit and auxiliary qubits. The resulting state becomes

$$|\Phi\rangle^{(1)} = \frac{1}{\sqrt{2}}(|0^{\otimes 3}\rangle + |1^{\otimes 3}\rangle)_{\{S_1,C_1',T_1'\}}(|0^{\otimes 3}\rangle + |1^{\otimes 3}\rangle)_{\{S_2,C_1'',T_2'\}}(|0^{\otimes 4}\rangle + |1^{\otimes 4}\rangle)_{\{C_2,C_1''',T_1'',T_2''\}}|+^{\otimes 3}\rangle_{\{C_1,T_1,T_2\}}. \tag{15}$$

Then auxiliary states are then sent to their correlative neighboring nodes.

(3) At each node $w \in V_2$, $in(w)$ is the auxiliary qubits received by nodes w. Clearly, $in(C_1) = \{C_1', C_1'', C_1'''\}$, $in(T_1) = \{T_1', T_1''\}$, $in(T_2) = \{T_2', T_2''\}$. Control operation $CZ_{in(w)\to w}$ is performed on $in(w)$ and w. The resulting state becomes

$$|\Phi\rangle^{(2)} = \sum_{x,y,z\in\{0,1\}} |xyz\rangle_{S_1S_2C_2}|xyz+_{(xyz)}\rangle_{C_1'C_1''C_1'''C_1}|xz+_{(xz)}\rangle_{T_1'T_1''T_1}|yz+_{(yz)}\rangle_{T_2'T_2''T_2}. \tag{16}$$

(4) Measure $in(w)$ in X-basis, a highly entangled state

$$|\Phi\rangle^{(3)} = \sum_{x,y,z\in\{0,1\}} |xyz\rangle_{S_1S_2C_2}|+_{(xyz)}+_{(xz)}+_{(yz)}\rangle_{C_1T_1T_2} \tag{17}$$

is generated among nodes in $V_{G_{(B)}}$.

(5) Further, entangle $|\psi_1\rangle$ and $|\psi_2\rangle$ with this six-particle cluster state in Eq. (17). Finally, we get an eight-particle entangled state

$$\begin{aligned}|\Phi\rangle^{(4)} = {} & [(|000+++\psi_1\psi_2\rangle + |001---\psi_1\psi_2\rangle) + (|010+--\psi_1\overline{\psi_2}\rangle + |011-++\psi_1\overline{\psi_2}\rangle) + (|100\\ & -+-\overline{\psi_1}\psi_2\rangle + |101+-+\overline{\psi_1}\psi_2\rangle) + (|110--+\overline{\psi_1\psi_2}\rangle + |111++-\overline{\psi_1\psi_2}\rangle)]_{S_1S_2C_2T_1T_2C_1S_1'S_2'}.\end{aligned} \tag{18}$$

b) Transmitting information crossly over butterfly network

We rewrite Eq. (18) with X-basis in a compact way as

$$|\Phi\rangle^{(4)} = \sum_{m,n,p,q\in\{0,1\}} |+_{(m)}+_{(n)}+_{(p)}+_{(q)}\rangle_{S_1,S_2,C_2,C_1}(Z^qX^{n+p})_{T_1}\otimes(Z^qX^{m+p})_{T_2}|\Phi\rangle^{(5)}, \tag{19}$$

where $|\Phi\rangle^{(5)} = (|++\psi_1\psi_2\rangle + |-+\psi_1\overline{\psi_2}\rangle + |+-\overline{\psi_1}\psi_2\rangle + |--\overline{\psi_1\psi_2}\rangle)_{T_1T_2S_1'S_2'}$ with $|+_{(0)}\rangle = |+\rangle$, $|+_{(1)}\rangle = |-\rangle$, $|\overline{\varphi}\rangle = Z|\varphi\rangle$.

- Measurements on S_1, S_2, C_2, C_1 lead to classical results s_1, s_2, c_2, c_1 and the resulting state

$$|\Phi\rangle^{(6)} = (Z^{c_2}X^{s_2+c_2})_{T_1}\otimes(Z^{c_2}X^{s_1+c_2})_{T_2}|\Phi\rangle^{(5)}, \tag{20}$$

s_2, c_2, c_1 and s_1, c_2, c_1 are sent to T_1 and T_2 with free classical communication, respectively.

- Next, we measure particles S_1', S_2' in X-basis and get that

$$|\Phi\rangle^{(7)} = (Z_{T_1}^{s_2'}\otimes Z_{T_2}^{s_1'})\left(\frac{I+(-1)^{s_1'}X}{2}\right)_{S_1'}\left(\frac{I+(-1)^{s_2'}X}{2}\right)_{S_2'}|\Phi\rangle^{(6)}. \tag{21}$$

Note that $(\frac{I+(-1)^{s_1'}X}{2})_{S_1'}(\frac{I+(-1)^{s_2'}X}{2})_{S_2'}$ are commuted with $(Z^{c_2}X^{s_2+c_2})_{T_1}\otimes(Z^{c_2}X^{s_1+c_2})_{T_2}$ because of acting on different quantum systems. Thus all the measurements can be performed simultaneously which will lead to a significant reduction of the communication complexity. We rewrite Eq. (21) as

$$|\Phi\rangle^{(7)} = (Z^{c_2+s_2'}X^{s_2+c_2})_{T_1}\otimes(Z^{c_2+s_1'}X^{s_1+c_2})_{T_2}\left(\frac{I+(-1)^{s_1'}X}{2}\right)_{S_1'}\left(\frac{I+(-1)^{s_2'}X}{2}\right)_{S_2'}|\Phi\rangle^{(5)}. \tag{22}$$

Table 1 Pauli operators where $s_1, s_2, c_1, c_2, s_1', s_2' \in \{0,1\}$ to attain $|\psi_2\psi_1\rangle_{T_1T_2}$

Measurement results	Local pauli operators
$s_1s_2c_1c_2s_1's_2'$	$Z_{T_1}^{c_1+s_2'}X_{T_1}^{s_2+c_2} \otimes Z_{T_2}^{c_1+s_1'}X_{T_2}^{s_1+c_2}$

- After proper pauli operators as illustrated in Table 1, we have

$$|\psi\rangle_{\text{out}} = (\alpha_2|0\rangle + \beta_2|1\rangle)_{T_1}(\alpha_1|0\rangle + \beta_1|1\rangle)_{T_2}.$$

Consequently, it accomplishes the 2-unicast session over butterfly network with the final state denoted as

$$|\psi\rangle_{\text{out}} = U_{\text{swap}}|\psi_1\psi_2\rangle_{T_1T_2} = |\psi_2\psi_1\rangle_{T_1T_2}.$$

We can conclude that swapping operator U_{swap} can be realized with inputs $\{S_1', S_2'\}$ and outputs $\{T_1, T_2\}$. In other words, 2-unicast session can be completed by X-basis measurement on all intermediate nodes $\{S_1, S_2, C_1, C_2\}$ and also on the input nodes $\{S_1', S_2'\}$ over the butterfly network with bottleneck channel (C_1, C_2).

3.2 2-unicast session over grail network

In this subsection, we analyze the solvability of the grail network supporting 2-unicast session. Ref. [18] has discussed the solvability on the 2-unicast session in grail network. In their protocol, EPR pairs pre-shared between neighbor nodes are considered as the fundamental resources which are local entanglement resource. Now we provide an alternative way for the solvability over this network according to Lemma 2 and the designed measurement model. The grail network (Figure 8(a)) naturally relates to a 2D cubic lattice (Figure 8(b)) over which we can pre-share an eight-particle cluster state. In grail network, two sources holding nodes S_1 and S_2 wish to communicate with their targets T_1 and T_2 respectively.

- Firstly, suppose that we pre-share an eight-particle cluster state $|\Phi\rangle_{Gr}$ associated to the grail network (Figure 8(b)). The bipartition to this graph is given by $V_1 = \{S_1, S_2, M_2, M_4\}$ and $V_2 = \{M_1, M_3, T_1, T_2\}$. According to Scheme 1, we get the eight-particle cluster state,

$$|\phi\rangle = \sum_{x_1x_2x_3x_4\in\{0,1\}} |x_1x_2x_3x_4\rangle_{S_1S_2M_2M_4}|+_{(\{x_1,x_2,x_3\})}\rangle_{M_1}|+_{(\{x_1,x_3,x_4\})}\rangle_{M_3}|+_{(\{x_3,x_4\})}\rangle_{T_1}|+_{(\{x_4\})}\rangle_{T_2}.$$

- Sources introduce input quantum systems S_1' and S_2', respectively. Each source performs CZ on S_i and S_i', $i = 1, 2$ (as illustrated in Figure 8(c)), we then get a ten-particle cluster state in Figure 8(d),

$$\begin{aligned}|\phi\rangle = \sum_{x_1x_2x_3x_4\in\{0,1\}} & |x_1x_2x_3x_4\rangle_{S_1S_2M_2M_4}|+_{(\{x_1\})}\rangle_{S_1'}|+_{(\{x_2\})}\rangle_{S_2'} \\ & |+_{(\{x_1,x_2,x_3\})}\rangle_{M_1}|+_{(\{x_1,x_3,x_4\})}\rangle_{M_3}|+_{(\{x_3,x_4\})}\rangle_{T_1}|+_{(\{x_4\})}\rangle_{T_2}. \end{aligned} \tag{23}$$

- Select correlation centers to perform unitary operation of Eq. (2). The choice of correlation centers decides the measurement model $\mathcal{P}^{\text{Media}(G_r)}$(measurement basis and orders) and then the solvability of the 2-unicast session over grail network. We get that

$$\begin{aligned}
|\Phi\rangle_{Gr} &= K^{(S_1',M_1,T_2,M_3)}|\Phi\rangle_{Gr} = \sigma_x^{(S_1',M_1,T_2,M_3)}|\Phi\rangle_{Gr}, \\
|\Phi\rangle_{Gr} &= K^{(S_2',T_1,M_3)}|\Phi\rangle_{Gr} = \sigma_x^{(S_2',T_1,M_3)}|\Phi\rangle_{Gr}, \\
|\Phi\rangle_{Gr} &= K^{(S_1,M_2,M_4)}|\Phi\rangle_{Gr} = \sigma_z^{(S_1',T_2)}\sigma_x^{(S_1,M_2,M_4)}|\Phi\rangle_{Gr}, \\
|\Phi\rangle_{Gr} &= K^{(S_2,M_2)}|\Phi\rangle_{Gr} = \sigma_z^{(S_2',T_1)}\sigma_x^{(S_2,M_2)}|\Phi\rangle_{Gr}.
\end{aligned} \tag{24}$$

X-basis measurement on intermediate nodes gets that

$$\begin{aligned}
\sigma_x^{(S_1',T_2)}|\Psi\rangle_{Gr} &= \sigma_x^{(S_1')}(U_{\text{swap}}\sigma_x^{(T_1)}U_{\text{swap}}^{\dagger})|\Psi\rangle_{Gr} = (-1)^{m_1+m_3}|\Psi\rangle_{Gr}, \\
\sigma_x^{(S_2',T_1)}|\Psi\rangle_{Gr} &= \sigma_x^{(S_2')}(U_{\text{swap}}\sigma_x^{(T_2)}U_{\text{swap}}^{\dagger})|\Psi\rangle_{Gr} = (-1)^{m_3}|\Psi\rangle_{Gr},
\end{aligned}$$

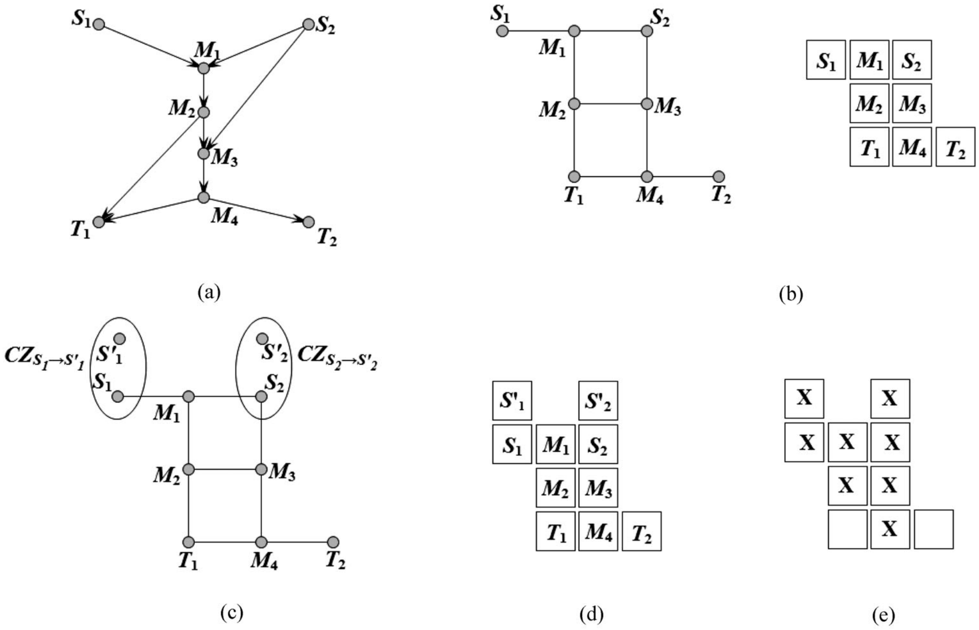

Figure 8 (a) Grail Network G_r; (b) Grail Network associated to 2D cubic lattice; (c) each source introduces an additional quantum system in state $|+\rangle$ and performs CZ operation to obtain a ten-particle cluster(d); (e) the measurement basis on intermediate nodes.

$$\sigma_z^{(S_2',T_1)}|\Psi\rangle_{Gr} = \sigma_2^{(S_1')}(U_{\rm swap}\sigma_2^{(T_1)}U_{\rm swap}^{\dagger})|\Psi\rangle_{Gr} = (-1)^{s_1+m_2+m_4}|\Psi\rangle_{Gr},$$
$$\sigma_z^{(S_1',T_2)}|\Psi\rangle_{Gr} = \sigma_z^{(S_2')}(U_{\rm swap}\sigma_z^{(T_2)}U_{\rm swap}^{\dagger})|\Psi\rangle_{Gr} = (-1)^{s_2+m_2}|\Psi\rangle_{Gr}. \quad (25)$$

According to Lemma 2, swapping operation on arbitrary 2-qubit input state can be realized up to some local unitary operations. As shown in Figure 8(e), X-basis measurement on all nodes except for target nodes can complete 2-unicast session and all measurements are parallelized.

We claim that swapping operator $U_{\rm swap}$ can be realized with inputs $In(G_r) = \{S_1', S_2'\}$ and outputs $\text{Out}(G_r) = \{T_1, T_2\}$. In other words, 2-unicast session can be completed by X-basis measurement on all intermediate nodes $\text{Media}(G_r) = \{S_1, S_2, M_1, M_2, M_3, M_4\}$ and also on the input nodes $\{S_1', S_2'\}$ over the grail network with bottleneck channel (M_1, M_2), (M_3, M_4).

4 Extended butterfly network

In this section, we attempt to generalize the method discussed above to a more general network and accordingly, provide a solution to the problems of transmission congestion and the lower transmission rate on upcoming large-scale quantum communication. We concentrate on a general network supporting 4-unicast session. It is worth noting that we restrict entanglement resource on 2D and 3D cluster state in which the neighbor sites of each node are no more than four or six. Thus we generalize our study to 4-unicast session. For the general k-unicast session, cluster state on higher dimension would be introduced.

As state in Lemma 2, a solvable network should satisfy two conditions

- The network graph corresponds to a cluster,
- Constructing eigenvalue equations that satisfy the condition of Eq. (9).

Now, we prove the solvability of extended butterfly network from the above two aspects.

A general network topology supporting 4-unicast session can be seen in Figure 9(a). Each source aims to communicate with its corresponding target and all source-target pairs share a single bottleneck channel. Unlike butterfly and grail networks, the extended butterfly network fails to correspond to a cluster naturally. Therefore, we introduce eight additional nodes making the network graph to be 3D cubic lattice of size 18-particle as illustrated in Figure 9(b). In this modified graph, links (S_1, T_2) and

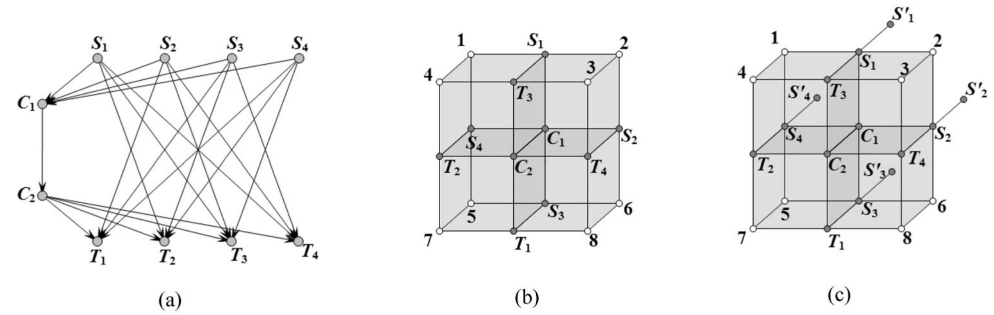

Figure 9 (a) Extended butterfly network G_4; (b) modified graph related to G_4; (c) each source introduces an additional quantum system in state $|+\rangle$ and perform CZ operation to obtain a 22-particle cluster.

(S_4, T_3) share a common channel $(1,4)$ as well as (S_1, T_4) and (S_2, T_3) share $(2,3)$, (S_4, T_1) and (S_3, T_2) share $(5,7)$, (S_3, T_4) and (S_2, T_1) share $(6,8)$. We will show that even in this restricted and modified network, 4-unicast session can be realized by single qubit measurement and free classical communication.

According to Scheme 1, a bipartition on Figure 9(b) is given by

$$V_1 = \{S_1, S_2, S_3, S_4, C_2, 3, 4, 7, 8\}, \quad V_2 = \{C_1, 1, 2, 5, 6, T_1, T_2, T_3, T_4\}.$$

Similarly, sources introduce four additional input quantum systems S_1', S_2', S_3', S_4' and we have

- $In(G_4) = \{S_1', S_2', S_3', S_4'\}$,
- $\mathrm{Media}(G_4) = \{S_1, S_2, S_3, S_4, C_1, C_2, 1, \ldots, 8\}$,
- $\mathrm{Out}(G_4) = \{T_1 = \mathrm{out}(S_1'), T_2 = \mathrm{out}(S_2'), T_3 = \mathrm{out}(S_3'), T_4 = \mathrm{out}(S_4')\}$.

Besides, the unitary operations to complete this 4-unicast session can be considered as identity transformation I with

$$I\sigma_x^{S_i'}I^\dagger = \sigma_x^{S_i'},\ I\sigma_z^{S_i'}I^\dagger = \sigma_z^{S_i'}, i = 1, \ldots, 4, \tag{26}$$

on the input quantum systems $In(G_4)$ and resulting states are output from corresponding target quantum systems $\mathrm{Out}(G_4)$.

Now we construct eigenvalue equations that satisfy the condition of Eq. (9).

- The cluster state over network G_4 shown in Figure 9(b) can be represented as follows:

$$\begin{aligned}\sum_{x_1\cdots x_9\in\{0,1\}} & |x_1\cdots x_9\rangle_{S_1,S_2,S_3,S_4,C_2,3,4,7,8}|+_{(x_1,x_2,x_3,x_4)}\rangle_{C_1}|+_{(x_1,x_4,x_7)}\rangle_1|+_{(x_1,x_2,x_6)}\rangle_2 \\ & |+_{(x_3,x_4,x_8)}\rangle_5|+_{(x_2,x_3,x_9)}\rangle_6|+_{(x_3,x_5,x_8,x_9)}\rangle_{T_1}|+_{(x_4,x_5,x_7,x_8)}\rangle_{T_2}|+_{(x_1,x_5,x_6,x_7)}\rangle_{T_3}|+_{(x_2,x_5,x_6,x_9)}\rangle_{T_4}. \end{aligned}\tag{27}$$

- Entangle the source inputs quantum system in state $|+\rangle_{S_i'}$, $i = 1, \ldots, 4$ with the cluster state generated above by CZ, we get the cluster state of size 22 (as shown in Figure 9(c)). The new cluster state can be represented as

$$\begin{aligned}\sum_{x_1,\cdots,x_9\in\{0,1\}} & |x_1\cdots x_9\rangle_{S_1,S_2,S_3,S_4,C_2,3,4,7,8}|+_{(x_1)}\rangle_{S_1'}|+_{(x_2)}\rangle_{S_2'}|+_{(x_3)}\rangle_{S_3'}|+_{(x_4)}\rangle_{S_4'} \\ & |+_{(x_1,x_2,x_3,x_4)}\rangle_{C_1}|+_{(x_1,x_4,x_7)}\rangle_1|+_{(x_1,x_2,x_6)}\rangle_2|+_{(x_3,x_4,x_8)}\rangle_5|+_{(x_2,x_3,x_9)}\rangle_6 \\ & |+_{(x_3,x_5,x_8,x_9)}\rangle_{T_1}|+_{(x_4,x_5,x_7,x_8)}\rangle_{T_2}|+_{(x_1,x_5,x_6,x_7)}\rangle_{T_3}|+_{(x_2,x_5,x_6,x_9)}\rangle_{T_4}. \end{aligned}\tag{28}$$

- Select correlation centers to construct 2×4 eigenvalue equations as

$$\begin{aligned} |\Phi\rangle_{G_4} &= K^{(S_1,T_1,C_1,5,7)}|\Phi\rangle_{G_4} = \sigma_x^{(S_1,T_1,C_1,5,7)}|\Phi\rangle_{G_4}, \\ |\Phi\rangle_{G_4} &= K^{(S_2,T_2,C_1,1,5)}|\Phi\rangle_{G_4} = \sigma_x^{(S_2,T_2,C_1,1,5)}|\Phi\rangle_{G_4}, \\ |\Phi\rangle_{G_4} &= K^{(S_3,T_3,C_1,1,3)}|\Phi\rangle_{G_4} = \sigma_x^{(S_3,T_3,C_1,1,3)}|\Phi\rangle_{G_4}, \\ |\Phi\rangle_{G_4} &= K^{(S_4,T_4,C_1,3,7)}|\Phi\rangle_{G_4} = \sigma_x^{(S_4,T_4,C_1,3,7)}|\Phi\rangle_{G_4}. \end{aligned}\tag{29}$$

$$\begin{aligned}
|\Phi\rangle_{G_4} &= K^{(S_1',C_2,2,4)}|\Phi\rangle_{G_4} = \sigma_z^{(S_1,T_1)}\sigma_x^{(S_1',C_2,2,4)}|\Phi\rangle_{G_4},\\
|\Phi\rangle_{G_4} &= K^{(S_2',C_2,4,8)}|\Phi\rangle_{G_4} = \sigma_z^{(S_2,T_2)}\sigma_x^{(S_2',C_2,4,8)}|\Phi\rangle_{G_4},\\
|\Phi\rangle_{G_4} &= K^{(S_3',C_2,6,8)}|\Phi\rangle_{G_4} = \sigma_z^{(S_3,T_3)}\sigma_x^{(S_3',C_2,6,8)}|\Phi\rangle_{G_4},\\
|\Phi\rangle_{G_4} &= K^{(S_4',C_2,2,6)}|\Phi\rangle_{G_4} = \sigma_z^{(S_4,T_4)}\sigma_x^{(S_4',C_2,2,6)}|\Phi\rangle_{G_4}.
\end{aligned} \tag{30}$$

• We measure all intermediate nodes in X-basis, getting the result state $|\psi\rangle_{(G_4)}$ with the following eigenvalue equations that satisfy Eq. (9):

$$\begin{aligned}
\sigma_x^{(S_1',T_1)}|\psi\rangle_{(G_4)} &= \sigma_x^{(S_1')}(I\sigma_x^{(T_1)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{c_1}+r_5+r_7}|\psi\rangle_{G_4},\\
\sigma_x^{(S_2',T_2)}|\psi\rangle_{(G_4)} &= \sigma_x^{(S_2')}(I\sigma_x^{(T_2)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{c_1}+r_1+r_5}|\psi\rangle_{G_4},\\
\sigma_x^{(S_3',T_3)}|\psi\rangle_{(G_4)} &= \sigma_x^{(S_3')}(I\sigma_x^{(T_3)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{c_1}+r_1+r_3}|\psi\rangle_{G_4},\\
\sigma_x^{(S_4',T_4)}|\psi\rangle_{(G_4)} &= \sigma_x^{(S_4')}(I\sigma_x^{(T_4)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{c_1}+r_3+r_7}|\psi\rangle_{G_4}.
\end{aligned} \tag{31}$$

$$\begin{aligned}
\sigma_z^{(S_1',T_1)}|\psi\rangle_{(G_4)} &= \sigma_z^{(S_1')}(I\sigma_z^{(T_1)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{s_1}+r_{c_2}+r_2+r_4}|\psi\rangle_{G_4},\\
\sigma_z^{(S_2',T_2)}|\psi\rangle_{(G_4)} &= \sigma_z^{(S_2')}(I\sigma_z^{(T_2)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{s_2}+r_{c_2}+r_4+r_8}|\psi\rangle_{G_4},\\
\sigma_z^{(S_2',T_2)}|\psi\rangle_{(G_4)} &= \sigma_z^{(S_2')}(I\sigma_z^{(T_2)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{s_3}+r_{c_2}+r_6+r_8}|\psi\rangle_{G_4},\\
\sigma_z^{(S_2',T_2)}|\psi\rangle_{(G_4)} &= \sigma_z^{(S_2')}(I\sigma_z^{(T_2)}I^\dagger)|\psi\rangle_{(G_4)} = (-1)^{r_{s_4}+r_{c_2}+r_2+r_6}|\psi\rangle_{G_4}.
\end{aligned} \tag{32}$$

We can conclude that identity operator I can be realized with inputs $\{S_1', S_2', S_3', S_4'\}$ and outputs $\{T_1, T_2, T_3, T_4\}$. In other words, 4-unicast session can be completed by X-basis measurement on all intermediate nodes $\{S_1, S_2, S_3, S_4, 1, 2, 3, 4, 5, 6, 7, 8\}$ and also on the input nodes $\{S_1', S_2', S_3', S_4'\}$ over the network G_4.

5 Discussion and conclusion

5.1 Resource consumption

Raussendorf et al. [24] introduce three parameters as the metrics $\mathcal{M}_{(G)} = (\mathcal{S}, \mathcal{O}, \mathcal{T})$ of the resource consumption with spatial resources $\mathcal{S}$, operational resources $\mathcal{O}$ and temporal resources $\mathcal{T}$. Spatial resources $\mathcal{S}$ are defined as the number of particles in the required cluster state associated to a given network G. In this paper, $\mathcal{S} \geqslant |V_G|$ since additional quantum systems are probably introduced to complete the given communication task. The computation is driven by one-qubit measurement only. Thus, a single one-qubit measurement is one unit of operational resources, and the operational resources $\mathcal{O}$ are defined as the total number of one-qubit measurements involved. Since each cluster qubit is measured at most once, we have $\mathcal{O} \leqslant \mathcal{S}$. As for the temporal resources, specified by the logical depth $\mathcal{T}$ is the minimum number of measurement rounds to which the measurements can be parallelized. In this paper, all the measurement can be parallelized. Thus the measurement round equals one.

Specially, $\mathcal{M}_{(G_{(B)})} = (|S_1, S_2, C_1, C_2, T_1, T_2|, |In(G_{(B)}) \bigcup \mathrm{Media}(G_{(B)})|, 1) = (6, 6, 1)$. For our protocol over butterfly network, the spatial resources $\mathcal{S}$ reaches the lower bound, i.e., $\mathcal{S} = 6 = |V_{G_{(B)}}|$. The operational resources consumption is 6 and all the measurement can be performed simultaneously, thus logical depth $\mathcal{T}$ equals 1.

For the grail network, also no additional quantum system is introduced except for the input systems. We have $\mathcal{S} = |\{S_1, S_2, C_1, C_2, C_3, C_4, T_1, T_2\}| = 8$. All nodes except for the target nodes are measured in X-basis simultaneously, we have $\mathcal{O} = |In(Gr) \bigcup \mathrm{Media}(Gr)| = 8$ and $\mathcal{T} = 1$. Thus we have $\mathcal{M}_{(Gr)} = (8, 8, 1)$.

For the extended butterfly network G_4, apart from the input systems, eight more additional systems were introduced to form a 3D cubic lattice. Thus we have $\mathcal{S} = |\{S_1, \ldots, S_4, T_1, \ldots, T_4, 1, \ldots, 8, C_1, C_2\}| = 18$. All nodes except for the target nodes are measured in X-basis simultaneously, we have $\mathcal{O}$= 18 and

$\mathcal{T}= 1$. Thus $\mathcal{M}_{(G_4)} = (18, 18, 1)$. Because of the additional quantum systems 1 to 8, the spatial resources $\mathcal{S} = 18$ is greater than $|V_{G_4}| = 10$.

In conclusion, the spatial resources are no less than the network size, but the operational resources are always less than the spatial resources. It should be noted that the temporal resources independent of the network size are always a constant. The analysis reveals that resources consumption mostly reaches the lower bounds.

5.2 Conclusion

The problems of transmission congestion and higher resources consumption are the first concern in large-scale quantum network communication. Efficient protocols for given communication tasks should be presented. The quantum multi-unicast network coding based on global entanglement state is considered. In this paper, we first solve how to pre-share cluster state which is a basic resource for our protocols over distant communication nodes. With the bigraph property of a cluster, we give a constant-step scheme as the network scale increases. Parallel operations lead to the significant reduction of the temporal resource cost. Further, we study the solvability of several quantum network with bottleneck channels. Local measurement and free classical communication drive the information transmission. Specific measurement basis and orders are designed. Based on the stabilizer formalism, we confirm, under the above measurement model, the solvability of butterfly and grail networks which are two classical networks in network coding. Because no additional nodes are introduced to these network graphs, the resource consumption reaches the lower bounds. Also, we give a generalization to a more general network supporting four-unicast session. Eight additional nodes are introduced making this graph corresponding to a cluster. Even in this restricted and modified network, 4-unicast session is solvable with lower resource cost.

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