SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2017, Vol. 60 092107:1–092107:10
doi: 10.1007/s11432-016-0551-5

The propositional normal default logic and the finite/infinite injury priority method

Wei LI[1], Yuefei SUI[2] & Yuhui WANG[2,3*]

[1]*State Key Laboratory of Software Development Environment, Beihang University, Beijing* 100191*, China;*
[2]*Key Laboratory of Intelligent Information Processing, Institute of Computing Technology, Chinese Academy of Sciences, Beijing* 100190*, China;*
[3]*School of Computer and Control Engineering, University of Chinese Academy of Sciences, Beijing* 100049*, China*

Received August 17, 2016; accepted November 19, 2016; published online April 27, 2017

Abstract In propositional normal default logic, given a default theory (Δ, D) and a well-defined ordering of D, there is a method to construct an extension of (Δ, D) without any injury. To construct a strong extension of (Δ, D) given a well-defined ordering of D, there may be finite injuries for a default $\delta \in D$. With approximation deduction $\vdash_s$ in propositional logic, we will show that to construct an extension of (Δ, D) under a given well-defined ordering of D, there may be infinite injuries for some default $\delta \in D$.

Keywords default, extension, strong extension, finite/infinite injury priority method, recursively enumerable sets

Citation Li W, Sui Y F, Wang Y H. The propositional normal default logic and the finite/infinite injury priority method. Sci China Inf Sci, 2017, 60(9): 092107, doi: 10.1007/s11432-016-0551-5

1 Introduction

Finite injury priority method was firstly given by Friedberg [1] and Muchnik [2], who solved Post problem independently. To construct a recursively enumerable set, the conditions that the set should satisfy are represented by an infinite set of requirements which are decomposed into the positive ones (putting elements in the set) and the negative ones (restraining elements from entering the set). The requirements are ordered by a priority ranking, so that the satisfaction of a requirement may injure those with lower priority and cannot injure those with higher priority [3, 4].

Traditional logics are monotonic, which means that the deduction in traditional logics are monotonic, that is, for any theories Γ, Δ and formula φ, if φ is deducible from Γ and Γ is a subtheory of Δ then φ is deducible from Δ. Nonmonotonic logics [5, 6] are a class of logics in which the deductions are nonmonotonic. For example, in default logic [7–10], an extension E of a default theory (Δ, D) may not be an extension of (Δ', D), where $\Delta' \supset \Delta$, and may not be an extension of (Δ, D'), where $D' \supset D$, where Δ is a theory and D is a set of defaults.

* Corresponding author (email: yhwang_ict@qq.com)

To deduce an extension E of a default theory (Δ, D) under a well-defined ordering on D, the process is a construction with finite injuries [11–13]. Even though for propositional normal default logic, finite injuries occurs in the following two ways.

• The deduction is an approximate one $\vdash_s$, so that some default $\delta = \varphi \rightsquigarrow \psi$ with higher priority may require attention after some $\delta' = \varphi' \rightsquigarrow \neg\psi$ with lower priority does, so that $\neg\psi$ having been enumerated in an extension E may be extracted from E, to enumerate ψ in E instead.

• To construct a strong extension S of a default theory (Δ, D), receiving attention of a default $\delta = \varphi \rightsquigarrow \psi$ may result in some ψ' s extracted from E to make the constructed E be an extension. Here, a strong extension is the one with highest priority, that is, for any extension E of (Δ, D), there is the least $\delta = \varphi \rightsquigarrow \psi$ such that for any $\delta' = \varphi' \rightsquigarrow \psi' \prec \delta, \psi' \in E$ iff $\psi' \in S$ and $\psi \in S - E$.

In this paper we will give three constructions of extensions, given a propositional normal default theory (Δ, D).

◇ To construct an extension E with deduction $\vdash$ of propositional logic, where the construction is without finite injuries [14].

◇ To construct a strong extension S under an well-defined ordering $\prec$ on D, where the construction is with finite injuries.

◇ To construct an extension E with approximate deduction $\vdash_s$. Even though deduction $\vdash$ in propositional logic is recursive (computable), approximate deduction provides a deduction with which approximate complexity can be considered. Here, the construction is with infinite injuries.

The paper is organized as follows: Section 2 gives basic definitions in default logic and introduces finite injury priority method in recursion theory; Section 3 gives a construction with oracle for an extension of a default theory; Section 4 gives a construction for a strong extension of a default theory; Section 5 gives a recursive construction with approximate deduction and infinite injury priority method, and Section 6 concludes the whole paper.

2 Basic definitions in default logic

Let L be a logical language of propositional logic.

A normal default δ is any expression of form $\dfrac{\varphi : \psi}{\psi}$, denoted by $\varphi \rightsquigarrow \psi$, where φ, ψ are formulas in propositional logic. Here φ is called the prerequisite, ψ the justifications, and the consequent of δ.

A default theory is a pair (Δ, D), where Δ is a set of closed formulas and D is a set of defaults.

Given a default theory (Δ, D), an extension of (Δ, D) can be derived by applying as many defaults as consistently possible. Given a default theory (Δ, D), assume that there is a well-founded ordering $\prec$ on D and $D^{\prec} = \{\varphi_0 \rightsquigarrow \psi_0, \varphi_1 \rightsquigarrow \psi_1, \ldots\}$. Define

$$E_0 = \Delta, \quad E_{i+1} = E_i \cup \{\psi_j\}, \quad E = \bigcup_{i \in \omega} E_i,$$

where j is least such that $E_i \vdash \varphi_j$ and $E_i \nvdash \neg\psi_j$. Then, E is an extension of (Δ, D), denoted by $E = f(\Delta, D^{\prec})$.

A normal default theory has at least one extension.

Proposition 1. Let E be an extension of a default theory (Δ, D). Then, E is $\subseteq$-maximal, that is, there is no consistent superset $E' \supseteq E$ such that each formula $\varphi \in E'$ is produced by formulas in Δ and defaults in D.

Definition 1. Given a default theory (Δ, D), a theory Θ is a pseudo-extension of (Δ, D) if Θ is the $\subseteq$-least theory such that

(i) $\Delta \subseteq \Theta$, and

(ii) for any $\delta = \dfrac{\varphi : \psi}{\theta} \in D$, if $\Theta \vdash \varphi$ and $\Theta \nvdash \neg\psi$ imply $\Theta \vdash \theta$.

We define an ordering on the set of all the pseudo-extensions of a default theory, assuming that there is an ordering $\leqslant$ on the set $D = \{\delta_0, \delta_1, \ldots\}$ of defaults. Without loss of generality, we assume that $\delta_0 < \delta_1 < \cdots$. For simplicity, we say that $\psi_0 < \psi_1 < \cdots$.

Given two pseudo-extensions $E_1 = \{\theta_{10}, \theta_{11}, \theta_{12}, \ldots\}$ and $E_2 = \{\theta_{20}, \theta_{21}, \theta_{22}, \ldots\}$, we say that E_1 has higher priority than E_2, denoted by $E_1 \prec E_2$, if there is i such that for each $j < i, \theta_{1j} = \theta_{2j}$ and $\theta_{1i} < \theta_{2i}$.

A pseudo-extension E of a default theory (Δ, D) is of the highest priority if for any pseudo-extension E' of $(\Delta, D), E \preceq E'$.

Before giving a construction of extensions, we give the following classical construction in recursion theory by finite injury priority method.

Theorem 1 (Friedberg-Muchnik, [4], p111)**.** There is a simple set A which is low ($A' \equiv_{\mathrm{T}} \emptyset'$).

Proof. It suffices to construct a coinfinite recursively enumerable set A to meet for all e the requirements:

$$P_e : W_e \text{ infinite} \Rightarrow W_e \cap A \neq \emptyset;$$
$$N_e : \exists^{\infty} s(\{e\}_s^{A_s}(e)\downarrow) \Rightarrow \{e\}^A(e)\downarrow.$$

Let A_s consist of the elements enumerated in A by the end of stage s, and $A = \bigcup_s A_s$.

The priority ranking is assumed to be

$$N_0, P_0, N_1, P_1, \ldots.$$

The requirements $\{N_e\}_{e\in\omega}$ guarantee $A' \leqslant_{\mathrm{T}} \emptyset'$. Define recursive function g by

$$g(e,s) = \begin{cases} 1, & \text{if } \{e\}_s^{A_s}(e)\downarrow, \\ 0, & \text{otherwise.} \end{cases}$$

If requirement N_e is satisfied for all e then $\hat{g}(e) = \lim_s g(e,s)$ exists for all e, and $\hat{g} \leqslant_{\mathrm{T}} \emptyset'$. Because $\hat{g} = \chi_{A'}$, $A' \leqslant_{\mathrm{T}} \emptyset'$.

The restraint function is defined by

$$r(e,s) = u(A_s; e, e, s).$$

To meet N_e we attempt to restrain with priority N_e any elements $x \leqslant r(e,s)$ from entering A_{s+1}.

Construction of A.

Stage $s = 0$. Let $A_0 = \emptyset$.

Stage $s+1$. Given A_s we have $r(e,s)$ for all e. Choose the least $i \leqslant s$ such that

(1) $W_{i,s} \cap A_s = \emptyset$;

(2) $\exists x(x \in W_{i,s}\ \&\ x > 2i\ \&\ \forall e \leqslant i(r(e,s) < x))$.

If i exists, choose the least x satisfying (2). Enumerate x in A_{s+1}, and say that requirement P_i receives attention. Hence, $W_{i,s} \cap A_{s+1} \neq \emptyset$, so P_i is satisfied, (1) fails for stages $> s+1$, and P_i never again receives attention. If i does not exist, do nothing, so $A_{s+1} = A_s$.

Let $A = \bigcup_s A_s$. This ends the construction.

We say that x injures N_e at stage $s+1$ if $x \in A_{s+1} - A_s$ and $x \leqslant r(e,s)$.

Define the injury set for N_e as follows:

$$I_e = \{x : \exists s(x \in A_{s+1} - A_s\ \&\ x \leqslant r(e,s))\}.$$

Lemma 1. For any e, I_e is finite.

Proof. Each positive requirement P_i contributes at most one element to A by (1). By (2), N_e can be injured by P_i only if $i < e$. Hence, $|I_e| \leqslant e$.

Lemma 2. For every e, requirement N_e is met and $r(e) = \lim_s r(e,s)$ exists.

Proof. Fix e. By Lemma 1, choose stage s_e such that N_e is not injured at any stage $s > s_e$. However, if $\{e\}_s^{A_s}(e)$ converges for $s > s_e$ then by induction on $t \geqslant s, r(e,t) = r(e,s)$ and $\{e\}_t^{A_t}(e) = \{e\}_s^{A_s}(e)$ for all $t \geqslant s$, so $A_s \upharpoonright r(e,s) = A \upharpoonright r(e,s)$, and hence $\{e\}^A(e)$ is defined.

Lemma 3. For every i, requirement P_i is met.

Proof. Fix i such that W_i is infinite. By Lemma 2, choose s such that

$$\forall t \geqslant s \forall e \leqslant i(r(e,t) = r(e)).$$

Choose $s' \geqslant s$ such that no $P_j, j < i$, receives attention after stage s', and $t > s'$ such that

$$\exists x(x \in W_{i,t} \ \& \ x > 2i \ \& \ \forall e \leqslant i(r(e) < x)).$$

Now either $W_{i,t} \cap A_t \neq \emptyset$ or P_i receives attention at stage $t+1$. In either case $W_{i,t} \cap A_{t+1} \neq \emptyset$, so P_i is met by the end of stage $t+1$.

$\bar{A}$ is infinite by (2), hence A is simple and low.

3 Construction of an extension without injury

Let $D = \{\delta_0, \delta_1, \ldots\}$, where $\delta_e = \varphi_e \rightsquigarrow \psi_e$. We construct in stages a set Θ of formulas such that $\Theta_0 = \Delta$, and $\Theta = \bigcup_i \Theta_i$ is a pseudo-extension of (Δ, D).

It suffices to meet for each e the following requirements:

$$P_e : \Theta_e \vdash \varphi_e \& \Theta_e \nvdash \neg\psi_e \Rightarrow \Theta_e \vdash \psi_e,$$
$$N_e : \Theta_e \text{ is consistent.}$$

Define

$$\Theta_s \upharpoonright e = \{\psi_{e'} \in \Theta_s : e' < e\}.$$

The priority ranking of requirements is defined by

$$P_0, N_0, P_1, N_1, \ldots, P_e, N_e, \ldots.$$

A requirement P_e requires attention at stage $s+1$ if $\Theta_s \vdash \varphi_e, \Theta_s \lceil e \nvdash \neg\psi_e$ and $\Theta_s \lceil e \nvdash \psi_e$, where $\vdash$ is approximation deduction of $\vdash$.

A requirement P_e is satisfied at stage $s+1$ if $\Theta_s \vdash \varphi_e$ and $\Theta_s \nvdash \neg\psi_e$ imply $\Theta_s \vdash \psi_e$.

The construction.

Stage $s = 0$. Define $\Theta_0 = \Delta$.

Stage $s+1$. Find the least $e \leqslant s$ such that P_e requires attention. Set $\Theta_{s+1} = \Theta_s \cup \{\psi_e\}$. We say that P_e receives attention.

Define

$$\Theta = \lim_{s \to \infty} \Theta_s.$$

This ends the construction.

Lemma 4. For each e, if $\Theta \vdash \varphi_e$ and $\Theta \nvdash \psi_e$ then there is a stage s_e at which P_e is satisfied.

Proof. Assume that $\Theta \vdash \varphi_e$ and $\Theta \nvdash \neg\psi_e$. There is a stage s_e such that P_e requires attention at stage $s \geqslant s_e$, $\Theta_{s_e+1} \vdash \psi_e$, and P_e is satisfied, and for any $t \geqslant s_e, P_e$ never require attention. That is, P_e is eventually satisfied.

Lemma 5. Θ is an extension of (Δ, D).

Proof. By Lemma 4, each positive requirement P_e is satisfied. Θ is an extension of (Δ, D), because for any $\delta = \varphi \rightsquigarrow \psi \in D$, if $\Theta \vdash \varphi$ and $\Theta \nvdash \neg\psi$ then there is a stage s such that each P_e with higher priority than δ is satisfied eventually, $\Theta \vdash \varphi$, and P_δ receives attention at stage $s+1$. That is, $\psi \in \Theta_{s+1}$, and for any $t \geqslant s, \psi \in \Theta_{t+1}$, i.e., $\psi \in \Theta$.

In the following we give a Gentzen-typed deduction system for default logic: assume that for any $\varphi \rightsquigarrow \psi \in D_1$, either $\Delta \nvdash \varphi, \Delta \vdash \neg\psi$, or $\Delta \vdash \psi$.

The deduction system **L** consists of the following rules:

$$(A)\quad \frac{\Delta\vdash p\quad \Delta\nvdash\neg q}{\Delta|D_1,p\rightsquigarrow q,D_2\Rightarrow\Delta,q|D_1,D_2}$$

$$(\wedge^L)\quad \frac{\begin{array}{c}\Delta|D_1,\varphi_1\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2\\ \Delta|D_1,\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2\end{array}}{\Delta|D_1,\varphi_1\wedge\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2}\qquad (\wedge^R)\quad \frac{\begin{array}{c}\Delta|D_1,\varphi\rightsquigarrow\psi_1,D_2\Rightarrow\Delta,\psi_1|D_1,D_2\\ \Delta|D_1,\varphi\rightsquigarrow\psi_2,D_2\Rightarrow\Delta,\psi_2|D_1,D_2\end{array}}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\wedge\psi_2,D_2\Rightarrow\Delta,\psi_1\wedge\psi_2|D_1,D_2}$$

$$(\vee_1^L)\quad \frac{\Delta|D_1,\varphi_1\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2}{\Delta|D_1,\varphi_1\vee\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2}\qquad (\vee_1^R)\quad \frac{\Delta|D_1,\varphi\rightsquigarrow\psi_1,D_2\Rightarrow\Delta,\psi_1|D_1,D_2}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\vee\psi_2,D_2\Rightarrow\Delta,\psi_1\vee\psi_2|D_1,D_2}$$

$$(\vee_2^L)\quad \frac{\Delta|D_1,\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2}{\Delta|D_1,\varphi_1\vee\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2}\qquad (\vee_2^R)\quad \frac{\Delta|D_1,\varphi\rightsquigarrow\psi_2,D_2\Rightarrow\Delta,\psi_2|D_1,D_2}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\vee\psi_2,D_2\Rightarrow\Delta,\psi_1\vee\psi_2|D_1,D_2};$$

and

$$(A')\quad \frac{\Delta\vdash p\quad \Delta\vdash\neg q}{\Delta|D_1,p\rightsquigarrow q,D_2\Rightarrow\Delta|D_1,D_2}$$

$$(\wedge_L^1)\quad \frac{\Delta|D_1,\varphi_1\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2}{\Delta|D_1,\varphi_1\wedge\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2}\qquad (\wedge_R^1)\quad \frac{\Delta|D_1,\varphi\rightsquigarrow\psi_1,D_2\Rightarrow\Delta|D_1,D_2}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\wedge\psi_2,D_2\Rightarrow\Delta|D_1,D_2}$$

$$(\wedge_L^2)\quad \frac{\Delta|D_1,\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta,|D_1,D_2}{\Delta|D_1,\varphi_1\wedge\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2}\qquad (\wedge_R^2)\quad \frac{\Delta|D_1,\varphi\rightsquigarrow\psi_2,D_2\Rightarrow\Delta|D_1,D_2}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\wedge\psi_2,D_2\Rightarrow\Delta|D_1,D_2}$$

$$(\vee_L)\quad \frac{\begin{array}{c}\Delta|D_1,\varphi_1\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2\\ \Delta|D_1,\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2\end{array}}{\Delta|D_1,\varphi_1\vee\varphi_2\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2}\qquad (\vee_R)\quad \frac{\begin{array}{c}\Delta|D_1,\varphi\rightsquigarrow\psi_1,D_2\Rightarrow\Delta|D_1,D_2\\ \Delta|D_1,\varphi\rightsquigarrow\psi_2,D_2\Rightarrow\Delta|D_1,D_2\end{array}}{\Delta|D_1,\varphi\rightsquigarrow\psi_1\vee\psi_2,D_2\Rightarrow\Delta|D_1,D_2.}$$

Definition 2. $\Delta|D\Rightarrow\Theta$ is provable in **L**, denoted by $\vdash_{\mathbf{L}}\Delta|D\Rightarrow\Theta$, if there is a sequence $\Delta_1|D_1\Rightarrow\Delta_1'|D_1',\ldots,\Delta_n|D_n\Rightarrow\Delta_n'|D_n'$ such that $\Delta_n|D_n\Rightarrow\Delta_n'|D_n'=\Delta|D\Rightarrow\Theta$, and for each $i\leqslant n$, there is $j<i$ such that $\frac{\Delta_j|D_j\Rightarrow\Delta_j'|D_j'}{\Delta_i|D_i\Rightarrow\Delta_i'|D_i'}$ is a deduction rule.

Proposition 2. Assume that for any $\varphi\rightsquigarrow\psi\in D_1$, either $\Delta\nvdash\varphi,\Delta\vdash\neg\psi$, or $\Delta\vdash\psi$. If $\Delta\vdash\varphi$ then either

$$\vdash_{\mathbf{L}}\Delta|D_1,\varphi\rightsquigarrow\psi,D_2\Rightarrow\Delta,\psi|D_1,D_2$$

or

$$\vdash_{\mathbf{L}}\Delta|D_1,\varphi\rightsquigarrow\psi,D_2\Rightarrow\Delta|D_1,D_2.$$

Theorem 2 (The soundness theorem)**.** For any default theory (Δ,D), if there is a theory Θ such that $\Delta|D\Rightarrow\Theta$ is provable in **L** then Θ is an extension of (Δ,D).

Theorem 3 (The completeness theorem)**.** For any default theory (Δ,D) and an extension E of (Δ,D), there is an ordering $\preceq$ such that

$$\vdash_{\mathbf{L}}\Delta|D^{\preceq}\Rightarrow\Theta.$$

4 Construction of a strong extension with finite injury priority method

Consider the following example.

Example 1. Let $\Delta=\{p,r\}$ and $D=\{s\rightsquigarrow q,p\rightsquigarrow\neg q,r\rightsquigarrow s\}$. Assume that

$$s\rightsquigarrow q\ \prec\ p\rightsquigarrow\neg q\ \prec\ r\rightsquigarrow s.$$

Traditionally we have

$$\begin{aligned}&p,r|s\rightsquigarrow q,p\rightsquigarrow\neg q(*),r\rightsquigarrow s\\ &\quad\Rightarrow p,r,\neg q|s\rightsquigarrow q,p\rightsquigarrow\neg q,r\rightsquigarrow s(*)\\ &\quad\Rightarrow p,r,\neg q,s|s\rightsquigarrow q(*),p\rightsquigarrow\neg q,r\rightsquigarrow s;\end{aligned}$$

and $\{p,r,\neg q,s\}$ is a pseudo-extension, where $(*)$ marks the default active at the current stage.

We hope to have

$$\begin{aligned}&p,r|s\rightsquigarrow q,p\rightsquigarrow\neg q(*),r\rightsquigarrow s\\ &\quad\Rightarrow p,r,\neg q|s\rightsquigarrow q,p\rightsquigarrow\neg q,r\rightsquigarrow s(*)\\ &\quad\Rightarrow p,r,\neg q,s|s\rightsquigarrow q(*),p\rightsquigarrow\neg q,r\rightsquigarrow s\\ &\quad\Rightarrow p,r,q,s|s\rightsquigarrow q(*),p\rightsquigarrow\neg q,r\rightsquigarrow s,\end{aligned}$$

and $\{p, r, \neg q, s\}$ is a pseudo-extension. We think that under ordering $\prec$, $\{p, r, q, s\}$ is better than $\{p, r, \neg q, s\}$.

Definition 3. Given an ordering $\prec$ on D, an extension S of default theory (Δ, D) is strong if S is an extension of (Δ, D) and for any other extension E of (Δ, D), there is an e such that for any $e' < e$, $\psi_{e'} \in E$ iff $\psi_{e'} \in S$, and $\psi_e \in S - E$.

Let $D = \{\delta_0, \delta_1, \ldots\}$. We will construct in stages a set Θ of formulas such that $\Theta_0 = \Delta$, and $\Theta = \bigcup_i \Theta_i$ is a strong pseudo-extension of (Δ, D).

It suffices to meet for each e the following requirements:

$$P_e : \Theta \vdash \varphi_e \& \Theta \nvdash \neg\psi_e \Rightarrow \Theta \vdash \psi_e,$$
$$N_e : \Theta_e \text{ is consistent},$$

where $\delta_e = \varphi_e \rightsquigarrow \psi_e$.

The priority ranking of requirements is defined by

$$P_0, N_0, P_1, N_1, \ldots, P_e, N_e, \ldots.$$

A requirement P_e requires attention at stage $s+1$ if there are $e_1, \ldots, e_k > e$ such that

(i) $\Theta_s - \{\psi_{e_1}, \ldots, \psi_{e_k}\} \vdash \varphi_e$, $\Theta_s - \{\psi_{e_1}, \ldots, \psi_{e_k}\} \nvdash \neg\psi_e$ and $\Theta_s - \{\psi_{e_1}, \ldots, \psi_{e_k}\} \nvdash \psi_e$, and

(ii) for each $k' \leqslant k$, $\Theta_s \cup \{\psi_e\} - \{\psi_{e_1}, \ldots, \psi_{e_k}\} \vdash \neg\psi_{e_{k'}}$.

A requirement P_e is satisfied at stage $s+1$ if $\Theta_s \vdash \varphi_e$, $\Theta_s \nvdash \neg\psi_e$ and $\Theta_s \vdash \psi_e$.

The construction.

Stage $s = 0$. Define $\Theta_0 = \Delta$.

Stage $s+1$. Find the least $e \leqslant s$ such that P_e requires attention. Define $\Theta_{s+1} = (\Theta_s \cup \{\psi_e\}) - \{\psi_{e_1}, \ldots, \psi_{e_k}\}$, and we say that P_e receives attention.

Define

$$\Theta = \lim_{s \to \infty} \Theta_s.$$

This ends the construction.

We say that P_e is injured at stage $s+1$ if $\psi_e \in \Theta_s - \Theta_{s+1}$.

Define the injury set of P_e:

$$I_e = \{s : \exists i(\psi_i \in \Theta_s - \Theta_{s+1} \& \varphi_e \in \Theta_{s+1} - \Theta_s)\}.$$

Lemma 6. I_e is finite.

Proof. By definition of requiring attention, for any s, if $s \in I_e$ then there is an $i < e$ such that $\psi_i \in \Theta_s - \Theta_{s+1}$.

Lemma 7. For each e, there is a stage s_e such that for any $s \geqslant s_e$, if P_e requires attention at $s+1$ then P_e is satisfied eventually.

Proof. Assume that $\Theta \vdash \varphi_e$ and $\Theta \nvdash \neg\psi_e$. By Lemma 6, there is a stage s_e such that $P_{e'}$ for no $e' < e$ requires attention after s_e. Then, P_e requires attention at stage $s \geqslant s_e$ such that $\Theta \vdash \varphi_e$, and P_e is satisfied, and for any $t \geqslant s$, P_e never require attention. That is, P_e is eventually satisfied.

Lemma 8. Θ is an extension of (Δ, D).

Proof. By Lemma 8, each positive requirement P_e is satisfied. Θ is an extension of (Δ, D), because for any $\delta = \varphi \rightsquigarrow \psi \in D$, if $\Theta \vdash \varphi$ and $\Theta \nvdash \neg\psi$ then there is a stage s such that each P_e with higher priority than δ is satisfied eventually, $\Theta \vdash \varphi$, and P_δ receives attention at stage $s+1$. That is, $\psi \in \Theta_{s+1}$, and for any $t \geqslant s$, $\psi \in \Theta_{t+1}$, i.e., $\psi \in \Theta$.

Lemma 9. Θ has the highest priority, that is, for any pseudo-extension E of (Δ, D), $\Theta \preceq E$, that is, there is a formula φ such that $E[\prec \varphi] = \Theta[\prec \varphi]$ and $\varphi \prec \psi$ for any $\psi \in E - E[\prec \varphi]$, where $E[\prec \varphi] = \{\psi \in E : \psi \prec \varphi\}$.

Proof. By the construction.

5 Construction with infinite injury priority method

When we use approximate deduction $\vdash_s$ instead of $\vdash$, we have a construction of a pseudo-extension with infinite injury priority method.

The Gentzen deduction system for approximation reasoning.

• Axiom:

$$\Gamma, \varphi \Rightarrow_0 \varphi, \Delta.$$

• Logical rules:

$$\frac{\Gamma, \varphi \Rightarrow_s \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow_{s+1} \Delta} (\wedge^{L_1}) \quad \frac{\Gamma \Rightarrow_s \varphi, \Delta \quad \Gamma \Rightarrow_s \psi, \Delta}{\Gamma \Rightarrow_{s+1} \varphi \wedge \psi, \Delta} (\wedge^{R})$$

$$\frac{\Gamma, \psi \Rightarrow_s \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow_{s+1} \Delta} (\wedge^{L_2})$$

$$\frac{\Gamma, \varphi \Rightarrow_s \Delta \quad \Gamma, \psi \Rightarrow_s \Delta}{\Gamma, \varphi \vee \psi \Rightarrow_{s+1} \Delta} (\vee^{L}) \quad \frac{\Gamma \Rightarrow_s \psi, \Delta}{\Gamma \Rightarrow_{s+1} \varphi \vee \psi, \Delta} (\vee^{R_2})$$

$$\frac{\Gamma \Rightarrow_s \varphi, \Delta}{\Gamma \Rightarrow_{s+1} \varphi \vee \psi, \Delta} (\vee^{R_1})$$

$$\frac{\Gamma \Rightarrow_s \varphi, \Delta \quad \Gamma, \psi \Rightarrow_s \Delta}{\Gamma, \varphi \to \psi \Rightarrow_{s+1} \Delta} (\to^{L}) \quad \frac{\Gamma, \varphi \Rightarrow_s \psi, \Delta}{\Gamma \Rightarrow_{s+1} \varphi \to \psi, \Delta} (\to^{R})$$

$$\frac{\Gamma \Rightarrow_s \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow_{s+1} \Delta} (\neg^{L}) \quad \frac{\Gamma, \varphi \Rightarrow_s \Delta}{\Gamma \Rightarrow_{s+1} \neg\varphi, \Delta} (\neg^{R}).$$

Definition 4. A sequent $\Gamma \Rightarrow \Delta$ is s-deducible, denoted by $\vdash_s \Gamma \Rightarrow \Delta$, if there is a sequence $\Gamma \Rightarrow_{i_0} \Delta, \ldots, \Gamma_n \Rightarrow_{i_n} \Delta_n$ which is a proof and $\Gamma_n = \Gamma, \Delta_n = \Delta$, and for each $i \leqslant n, i_n \leqslant s$.

Intuitively, a sequent $\Gamma \Rightarrow \Delta$ is s-deducible if there is a deduction tree for $\Gamma \Rightarrow \Delta$ with depth $\leqslant s$.

Proposition 3. (i) For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_s \Gamma \Rightarrow \Delta$ then $\vdash \Gamma \Rightarrow \Delta$.

(ii) For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash \Gamma \Rightarrow \Delta$ then there is an $s \in \omega$ such that

$$\vdash_s \Gamma \Rightarrow \Delta.$$

Therefore, $\vdash$ is the limit of $\{\vdash_s : s \in \omega\}$, i.e., $\vdash = \lim_{s\to\infty} \vdash_s$.

Assume that at stage $s+1, \Theta_s \vdash_s \neg\varphi_e, \Theta_s \nvdash_s \neg\psi_e$, and P_e receives attention by putting ψ_e in Θ_{s+1}. At a stage $t+1 > s+1$, we find that $\Theta_s \vdash_t \neg\psi_e$, and then we need extract ψ_e out of Θ_{t+1} to ensure that the constructed Θ is consistent. In this case, we say that P_e is injured by itself.

There is the following case in which P_e is injured infinitely often.

At some stage s_0+1, P_e requires attention and ψ_e is put in Θ_{s_0+1}, because $\Theta_{s_0} \vdash_{s_0} \varphi_e$ and $\Theta_{s_0} \nvdash_{s_0} \neg\psi_e$.

At some stage $t_0+1 > s_0+1$, some ψ in Θ_{s_0+1} is extracted out of Θ_{s_0+1} so that $\Theta_{s_0+1} \nvdash_{t_0+1} \varphi_e$, and ψ_e is extracted out of Θ_{t_0+1} at stage t_0+1.

At some stage $s_1+1 > t_0+1, P_e$ requires attention again and ψ_e is put in Θ_{s_1+1}, because $\Theta_{s_1} \vdash_{s_1} \varphi_e$ and $\Theta_{s_1} \nvdash_{s_1} \neg\psi_e$.

At some stage $t_1+1 > s_1+1$, some ψ in Θ_{s_1+1} is extracted out of Θ_{s_1+1} so that $\Theta_{s_1+1} \nvdash_{t_1+1} \varphi_e$, and ψ_e is extracted out of Θ_{t_1+1} at stage t_1+1.

And so on.

In this way, ψ_e is put in and extracted out of Θ infinitely often. In this case, P_e is injured infinitely often by other requirements.

Define

$$\Theta = \lim_{s\to\infty} \Theta_s = \{\psi_e : \exists s \forall t > s(\psi_e \in \Theta_t)\}.$$

Hence, $\psi_e \notin \Theta$ iff $\forall s \exists t > s(\psi_e \notin \Theta_t)$.

Example 2. Let

$$\Delta = \{r_1, r_2, \ldots, s_1, s_2, \ldots\}, \quad D = \{p \rightsquigarrow q, r_1 \rightsquigarrow p \wedge q_1, r_2 \rightsquigarrow p \wedge q_2, \ldots\},$$

where

$$s_1 = r_1 \to \neg q_1, \quad s_2 = r_1 \to (r_2 \to (r_3 \to \neg q_2)), \quad s_i = r_1 \to (\cdots(r_{2i-2} \to (r_{2i-1} \to \neg q_i))\cdots).$$

Then,

$$\begin{aligned}
&\Delta \vdash_0 r_1, \Delta \nvdash_0 \neg(p \wedge q_1)\\
&\Delta \vdash_1 \neg q_1;\\
&\Delta \vdash_2 r_2, \Delta \nvdash_1 \neg(p \wedge q_2)\\
&\Delta \vdash_3 \neg q_2;\\
&\Delta \vdash_{2i} r_i, \Delta \nvdash_{2i-1} \neg(p \wedge q_i)\\
&\Delta \vdash_{2i+1} \neg q_i;\\
&\dots
\end{aligned}$$

Let

$$[\varphi] : \varphi \text{ is enumerated in } \Theta, \quad \langle\varphi\rangle : \varphi \text{ is extracted from } \Theta.$$

At stage $s = 0, \Theta_0 = \Delta$; at stage $s = 1, \Theta_0 \vdash_1 r_1, \Theta_0 \nvdash_1 \neg(p \wedge q_1)$, and $[p \wedge q_1], [q]$, i.e., $\Theta_1 = \Theta_0 \cup \{p \wedge q_1, q\}$; at stage $s = 2, \Theta_1 \vdash_2 \neg q_1$, and $\langle p \wedge q_1\rangle, \langle q\rangle$, i.e., $\Theta_2 = \Theta_1 - \{p \wedge q_1, q\}$; at stage $s = 3, \Theta_2 \vdash_3 r_2, \Theta_2 \nvdash_3 \neg(p \wedge q_2)$, and $[p \wedge q_2], [q]$, i.e., $\Theta_3 = \Theta_2 \cup \{p \wedge q_2, q\}$; at stage $s = 4, \Theta_3 \vdash_4 \neg q_2$, and $\langle p \wedge q_2\rangle, \langle q\rangle$, i.e., $\Theta_4 = \Theta_3 - \{p \wedge q_1, q\}$; at stage $s = 2i - 1, \Theta_{s-1} \vdash_s r_i, \Theta_{s-1} \nvdash_s \neg(p \wedge q_i)$, and $[p \wedge q_i], [q]$, i.e., $\Theta_s = \Theta_{s-1} \cup \{p \wedge q_i, q\}$; at stage $s = 2i, \Theta_{s-1} \vdash_s \neg q_i$, and $\langle p \wedge q_i\rangle, \langle q\rangle$, i.e., $\Theta_s = \Theta_{s-1} - \{p \wedge q_i, q\}$.

Then, $\Theta = \lim_{s\to\infty} \Theta_s = \Delta$, that is, p is enumerated in Θ and extracted out of Θ infinitely often.

Given a default theory (Δ, D), we will construct a theory Θ in stages such that for any default $\varphi \rightsquigarrow \psi \in D$, if $\Theta \vdash \varphi$ and $\Theta \nvdash \neg\psi$ then $\psi \in \Theta$.

The construction is in stages with approximation deductions $\Theta_s \vdash_s \varphi_i$ and $\Theta_s \nvdash_s \neg\psi_i$.

It suffices to meet for each e the following requirements:

$$\begin{aligned}
P_e &: \Theta \vdash \varphi_e \& \Theta \nvdash \neg\psi_e \Rightarrow \varphi_e \in \Theta;\\
N_e &: \Theta \vdash \varphi_e \& \Theta \vdash \neg\psi_e \Rightarrow \varphi_e \neg \in \Theta.
\end{aligned}$$

The priority ranking of requirements is defined by

$$P_0, N_0, P_1, N_1, \dots, P_e, N_e, \dots$$

If $\{\Theta_s : s \in \omega\}$ is a sequence satisfying all the requirements then $\Theta = \bigcup_s \Theta_s$ is an extension of (Δ, D).

A requirement P_e requires attention at stage $s + 1$ if $\Theta_s \vdash_{s+1} \varphi_e, \Theta_s \nvdash_{s+1} \neg\psi_e$ and $\psi_e \notin \Theta_s$.

By the condition of P_e requiring attention, for any i with $\psi_i \in \Theta_s, \Theta_s \cup \{\psi_e\} \nvdash \neg\psi_i$, otherwise, $\Theta_s \vdash \neg\psi_e$, a contradiction.

A requirement P_e is satisfied at stage $s + 1$ if $\Theta_s \vdash_{s+1} \varphi_e$ and $\psi_e \in \Theta_s$; and a requirement N_e requires attention at stage $s + 1$ if $\Theta_s \vdash_{s+1} \neg\psi_e$.

The construction.

Stage $s = 0$. Define $\Theta_0 = \Delta$.

Stage $s + 1$. Find the least e such that P_e or N_e requires attention.

If P_e requires attention then define $\Theta_{s+1} = \Theta_s \cup \{\psi_e\}$, and we say that P_e receives attention.

If N_e requires attention then let s_e be the stage at which ψ_e is enumerated in Θ_{s_e+1}, and $\{\psi_{i_1}, \dots, \psi_{i_n}\}$ be the set of all the ψs enumerated in Θ between s_e and s, define

$$\Theta_{s+1} = \Theta_s - \{\psi_{i_1}, \dots, \psi_{i_n}\},$$

and we say that N_e receives attention.

Define

$$\Theta = \lim_{s\to\infty} \Theta_s = \{\psi : \exists s \forall t \geqslant s(\psi \in \Theta_t)\}.$$

Table 1 **R**-calculus and default logic

	Condition	Consequence
R-calculus	$\Delta \vdash \neg\varphi$	$\Delta\vert\varphi, \Gamma \Rightarrow \Delta\vert\Gamma$
	$\Delta \nvdash \neg\varphi$	$\Delta\vert\varphi, \Gamma \Rightarrow \Delta, \varphi\vert\Gamma$
Default logic	$E \vdash \varphi \& E \vdash \neg\psi$	$E\vert\varphi \rightsquigarrow \psi \Rightarrow E$
	$E \vdash \varphi \& E \nvdash \neg\psi$	$E\vert\varphi \rightsquigarrow \psi \Rightarrow E, \psi$

This ends the construction.

We say that P_e is injured at stage $s+1$ if $\varphi_e \in \Theta_s - \Theta_{s+1}$. Define the injury set for P_e as follows:

$$I_e = \{s + 1 : \varphi_e \in \Theta_s - \Theta_{s+1})\}.$$

Lemma 10. I_e may be infinite.

Proof. By Example 2.

Lemma 11. $\Theta = \lim_{s\to\infty} \Theta_{s+1}$ exists, and requirement P_e is met.

Proof. Fix e. Either ψ_e is enumerated in Θ_{s+1} eventually or finitely often, or infinitely often. In the first case, $\psi_e \in \Theta$, and $\Theta \vdash \varphi_e$; in the second case, P_e requires attention only finitely often, and either $\Theta \nvdash \varphi_e$ or $\Theta \vdash \neg\psi_e$; and in the third case, $\Theta \nvdash \varphi_e$.

Lemma 12. For every e, requirement N_e is met.

Proof. Fix e. Either ψ_e is enumerated in Θ_{s+1} eventually, or finitely often, or infinitely often. In the first case, $\psi_e \in \Theta$, $\Theta \vdash \varphi_e$, and N_e is satisfied eventually; in the second case, P_e requires attention only finitely often, and either $\Theta \nvdash \varphi_e$ or $\Theta \vdash \neg\psi_e$, and N_e is satisfied eventually; and in the third case, $\Theta \nvdash \varphi_e$, and N_e is satisfied eventually.

6 Conclusion

In this paper we use finite injury priority method and infinite injury priority method in recursion theory to construct extensions, pseudo-extensions and approximate extensions of default theories to show the versatility of the famous methods.

The similarity of **R**-calculus and default logic consists in that both are non-monotonic; and differences between them are shown in Table 1.

In **R**-calculus, φ being put in Δ does not interfere the satisfaction of other formulas in Γ; and in default logic, ψ being put in E may make the satisfaction of other defaults $\varphi' \rightsquigarrow \psi'$ unsatisfied, because $E \nvdash \neg\psi'$ becomes $E \vdash \neg\psi'$.

Therefore, in the propositional case, deductions in **R**-calculus have no injury; and ones in default logic may have injuries. We conjecture that in each nonmonotonic logic, there is a corresponding finite injury priority method to construct deduction in that logic.

Acknowledgements This work was supported by Open Fund of the State Key Laboratory of Software Development Environment (Grant No. SKLSDE-2010KF-06), Beihang University, and National Basic Research Program of China (973 Program) (Grant No. 2005CB321901).

Conflict of interest The authors declare that they have no conflict of interest.

References

1 Friedberg R M. Two recursively enumerable sets of incomparable degrees of unsolvability. Proc Natl Acad Sci, 1957, 43: 236–238
2 Muchnik A A. On the separability of recursively enumerable sets (in Russian). Dokl Akad Nauk SSSR, 1956, 109: 29–32
3 Rogers H. Theory of Recursive Functions and Effective Computability. Cambridge: The MIT Press, 1967
4 Soare R I. Recursively Enumerable Sets and Degrees, a Study of Computable Functions and Computably Generated Sets. Berlin: Springer-Verlag, 1987

5 Marek W, Truszczynski M. Nonmonotonic Logics: Context-Dependent Reasoning. Berlin: Springer. 1993

6 Nicolas P, Saubion F, Stéphan I. Heuristics for a default logic reasoning system. Int J Artif Intell Tools, 2001, 10: 503–523

7 Antoniou G. A tutorial on default logics. ACM Comput Surv, 1999, 31: 337–359

8 Delgrande J P, Schaub T, Jackson W K. Alternative approaches to default logic. Artif Intell, 1994, 70: 167–237

9 Lukaszewicz W. Considerations on default logic: an alternative approach. Comput Intell, 1988, 4: 1–16

10 Reiter R. A logic for default reasoning. Artif Intell, 1980, 13: 81–132

11 Avron A, Lev I. Canonical propositional Gentzen-type systems. In: Proceedings of the 1st International Joint Conference on Automated Reasoning. London: Springer, 2001. 529–544

12 Li W. Mathematical Logic, Foundations for Information Science. Basel: Birkhäuser. 2010

13 Li W, Sui Y, Sun M. The sound and complete R-calculus for revising propositional theories. Sci China Inf Sci, 2015, 58: 092101

14 Li W, Sui Y. The **R**-calculus and the finite injury priority method. In: Proceedings of the 2nd International Conference on Artificial Intelligence, Vancouver, 2015