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Robust H_2/H_∞ global linearization filter design for nonlinear stochastic time-varying delay systems

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Abstract One can design a robust H_∞ filter for a general nonlinear stochastic system with external disturbance by solving a second-order nonlinear stochastic partial Hamilton-Jacobi inequality (HJI), which is difficult to be solved. In this paper, the robust mixed H_2/H_∞ globally linearized filter design problem is investigated for a general nonlinear stochastic time-varying delay system with external disturbance, where the state is governed by a stochastic Itô-type equation. Based on a globally linearized model, a stochastic bounded real lemma is established by the Lyapunov–Krasovskii functional theory, and the robust H_∞ globally linearized filter is designed by solving the simultaneous linear matrix inequalities instead of solving an HJI. For a given attenuation level, the H_2 globally linearized filtering problem with the worst case disturbance in the H_∞ filter case is known as the mixed H_2/H_∞ globally linearized filtering problem, which can be formulated as a linear programming problem with simultaneous LMI constraints. Therefore, this method is applicable for state estimation in nonlinear stochastic time-varying delay systems with unknown exogenous disturbance when state variables are unavailable. A simulation example is provided to illustrate the effectiveness of the proposed method.

Keywords global linearization method, mixed H_2/H_∞ filtering, nonlinear filtering, Hamilton-Jacobi inequality, Lyapunov–Krasovskii functional, nonlinear stochastic time-varying delay system

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1 Introduction

State estimation has always been one important problem in the areas of filter design and control system design when system states are unknown [1–29]. The H_∞ filtering problem is to estimate the unavailable state variables by output measurement, which ensures the $\mathcal{L}_2$ gain to be less than a given level [1–6, 13–16, 18–26]. The advantage of H_∞ filtering is that the noise sources are arbitrary signals with bounded energy or average power instead of being Gaussian, and no exact statistics are necessary to be known [3].

General nonlinear systems exist in many real-world systems and have been studied extensively [7–19, 30–33], in which various direct and indirect methods are applied. It is well known that nonlinear control and filtering problems are associated with the solutions of the Hamilton-Jacobi inequality (HJI),

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Hamilton-Jacobi-Bellman inequality (HJBI), Hamilton-Jacobi-Isaacs equation (HJIE), and Hamilton-Jacobi equation (HJE). In [8, 9], the nonlinear H_∞ control problem for deterministic nonlinear systems with external disturbance depends on the solution of a first-order nonlinear partial differential HJIE, which has been proven to be impossibly solved analytically, even the approximate solution is still difficult to obtain. In [9], the designs of satisfaction output feedback controls for nonlinear stochastic systems under long-term tracking risk-sensitive index are related with the HJBI. Also, an integrator backstepping method is constructively applied to design an output feedback control. The infinite horizon H_∞ control problem has been solved for Itô-type stochastic systems by introducing three coupled HJEs in [10] and a single HJI in [11], respectively. In [12], a sufficient condition of the L_2-L_∞ filter for nonlinear stochastic systems has been established in terms of an HJI. For general nonlinear stochastic systems, the nonlinear stochastic H_∞ filtering problem relies on solving an HJI in [14].

It should be pointed out that the nonlinear stochastic partial differential HJBI [9], HJEs [10], and HJI [11, 12, 14] are second-order ones because of the effect of the diffusion terms. It is usually difficult to solve them. Linearization methods [15–18] are probably the alternative approaches. In [15], the globally linearized method [34] is employed to deal with the robust H_2/H_∞ filtering problem of nonlinear stochastic system under state-dependent noise and uncertain external disturbance. Under a fuzzy linearization scheme, the robust fuzzy filter design for a class of nonlinear stochastic systems has been studied in [16]. The robust H_∞ control design for nonlinear stochastic systems with external disturbance and Poisson noise has been investigated via fuzzy interpolation method instead of solving an HJI in [17]. Milstein-type discretization scheme is applied to study nonlinear filtering for stochastic time-delay systems in [18].

Practical system unavoidably involves in time delays, which may cause instability and poor performance of a control system. Therefore, much attention has been focused on the robust H_∞ filtering problem for time-delay systems [1, 5, 6, 13, 16, 18–22, 25, 26].

To the best of our knowledge, few work on mixed H_2/H_∞ filtering has been reported for an Itô-type general nonlinear stochastic time-varying delay systems. Inspired by [15–18] and based on the globally linearized scheme, a stochastic bounded real lemma (BRL) is established. The robust H_∞ globally linearized filter is designed by solving simultaneous linear matrix inequalities (LMIs) related with the filtering problem in the linear stochastic time-delay systems at vertices instead of solving the HJI associated with the H_∞ filtering problem in the nonlinear stochastic time-delay systems. When the worst case disturbance is discussed, the suboptimal mixed H_2/H_∞ globally linearized filter design problem is transformed into a convex optimization problem with simultaneous LMI constraints. A simulation example is given to verify the effectiveness of the proposed approach.

Notations. Let $|\cdot|$ denote the Euclidean vector norm. $R > 0$ means that R is a symmetric positive definite matrix. $\mathrm{Sym}(A) = A + A^{\mathrm{T}}$ and $\mathrm{TSym}(C)B = CBC^{\mathrm{T}}$. Let $(\varOmega, \mathcal{F}, P)$ be a complete probability space with an increasing family $\{\mathcal{F}\}_{t\geqslant 0}$ of σ algebras $\mathcal{F}_t \subset \mathcal{F}$ and $\mathrm{E}\{\cdot\}$ be the mathematical expectation operator as to the probability measure P. $\mathcal{L}_2[0,\infty)$ signifies the space of square integrable vector functions over $[0,\infty)$. $C([-\tau,0],\mathbb{R}^n)$ stands for the family of all continuous $\mathbb{R}^n$-valued functions ϕ on $[\tau,0]$ with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -\tau \leqslant \theta \leqslant 0\}$. Let $\mathcal{L}^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau,0],\mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\tau \leqslant \theta \leqslant 0\}$.

2 H_∞ setting for nonlinear stochastic system with interval time-varying delay

Consider the following Itô-type general nonlinear stochastic time-varying delay system:

$$\begin{cases} \mathrm{d}x(t) = (f(x(t), x(t-\tau(t)), t) + g(x(t), x(t-\tau(t)), t)v(t))\,\mathrm{d}t + h(x(t), x(t-\tau(t)), t)\mathrm{d}W(t), \\ \mathrm{d}y(t) = (q(x(t), x(t-\tau(t)), t) + k(x(t), x(t-\tau(t)), t)v(t))\,\mathrm{d}t + j(x(t), x(t-\tau(t)), t)\mathrm{d}W(t), \\ s(t) = Gx(t), \\ x(t) = \phi(t),\ t \in [-\tau_2, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^l$ is the measurement, $v(t) \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+;\mathbb{R}^{n_v})$ is the external disturbance, and $\phi(t)$ is any given initial data in $\mathcal{L}^2_{\mathcal{F}_0}([-\tau_2,0];\mathbb{R}^n)$. $W(t)$ is a standard one-dimensional

Wiener process defined on $(\Omega, \mathcal{F}, P)$, satisfying $\mathrm{E}\{W(t)\} = 0$ and $\mathrm{E}\{W^2(t)\} = t$. $s(t)$ is the signal to be estimated, and G is a combination matrix, if $x_i(t)$ is to be estimated, $G = \mathrm{diag}\{0, \dots, 1, \dots, 0\}$, $G = I$ means that the whole vector is estimated. Let $x = x(t), x_t = x(t-\tau(t))$, assume that $f(x, x_t, t), g(x, x_t, t)$, $h(x, x_t, t), q(x, x_t, t), k(x, x_t, t)$, and $j(x, x_t, t)$ satisfy the once continuously partially differentiable condition and the linear growth condition, which guarantee that the system (1) admits a unique global solution, see [35, 36]. Also, suppose that $f(0, 0, t) = h(0, 0, t) = 0$ and $q(0, 0, t) = j(0, 0, t) = 0$, then $x \equiv 0$ is an equilibrium point of (1). $\tau(t)$ is the time-delay satisfying

$$\dot{\tau}(t) \leqslant \bar{\tau} < 1, \quad 0 \leqslant \tau_1 \leqslant \tau(t) \leqslant \tau_2 < \infty. \tag{2}$$

A lemma is introduced on the globally asymptotic stability at $x \equiv 0$ of the subsequent system.

$$\begin{cases} \mathrm{d}x(t) = f(x(t), x(t-\tau(t)), t)\mathrm{d}t + h(x(t), x(t-\tau(t)), t)\mathrm{d}W(t), \\ f(0, 0, t) = h(0, 0, t) = 0, \\ x(t) = \phi(t), \ t \in [-\tau_2, 0]. \end{cases} \tag{3}$$

Lemma 1. It is assumed that there is a positive definite decrescent radially unbounded Lyapunov–Krasovskii functional $V(x, t) \in C^{2,1}(\mathbb{R}^n \times [t_0 - \tau_2, \infty); \mathbb{R}_+)$ with $V(0, 0) = 0$, satisfying

$$\frac{\partial V}{\partial t} + \frac{\partial V^{\mathrm{T}}}{\partial x} f(x, x_t, t) + \frac{1}{2} h^{\mathrm{T}}(x, x_t, t) \frac{\partial^2 V}{\partial x^2} h(x, x_t, t) < 0, \tag{4}$$

for every nonzero $x \in \mathbb{R}^n$, then system (3) is globally asymptotically stable in probability at $x \equiv 0$.

Proof. The proof can be followed from the same line of the proof for Theorem 4.2.3 in [36].

Proposition 1. If there exists a positive definite decrescent radially unbounded Lyapunov–Krasovskii functional $V(x, t) \in C^{2,1}(\mathbb{R}^n \times [t_0 - \tau_2, \infty); \mathbb{R}_+)$ with $V(0, 0) = 0$, satisfying the following HJI:

$$\frac{\partial V}{\partial t} + \frac{\partial V^{\mathrm{T}}}{\partial x} f(x, x_t, t) + \frac{1}{2}\|s(t)\|^2 + \frac{1}{2} h^{\mathrm{T}}(x, x_t, t) \frac{\partial^2 V}{\partial x^2} h(x, x_t, t) < 0, \tag{5}$$

then system (1) with $v(t) = 0$ is globally asymptotically stable in probability at $x \equiv 0$ and satisfies $\|s(t)\|^2_{L_2} \leqslant 2\mathrm{E}\{V(x(0), 0\}$.

Proof. By Lemma 1, the proof can be achieved according to the proof for Proposition 1 in [19].

Lemma 2. For system (1), if there is a positive definite decrescent radially unbounded Lyapunov–Krasovskii functional $V(x, t) \in C^{2,1}(\mathbb{R}^n \times [t_0 - \tau_2, \infty); \mathbb{R}_+)$ with $V(0, 0) = 0$, satisfying the HJI as below:

$$\frac{\partial V}{\partial t} + \frac{\partial V^{\mathrm{T}}}{\partial x} f(x, x_t, t) + \frac{1}{2}\gamma^{-2}\left(\frac{\partial V^{\mathrm{T}}}{\partial x} g(x, x_t, t) g^{\mathrm{T}}(x, x_t, t) \frac{\partial V}{\partial x}\right) + \frac{1}{2}\|s(t)\|^2 + \frac{1}{2} h^{\mathrm{T}}(x, x_t, t) \frac{\partial^2 V}{\partial x^2} h(x, x_t, t) < 0, \tag{6}$$

then system (1) is globally asymptotically stable in probability at $x \equiv 0$ when $v = 0$, and the inequality

$$\|s(t)\|^2_{L_2} \leqslant 2\mathrm{E}\{V(x(0), 0\} + \gamma^2 \|v(t)\|^2_{L_2}, \forall v \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R}^{n_v}), v \neq 0 \tag{7}$$

holds for some prescribed disturbance attenuation level γ, if the initial value $x(\theta) \neq 0, \theta \in [t - \tau_2, 0]$, and

$$\|s(t)\|^2_{L_2} \leqslant \gamma^2 \|v(t)\|^2_{L_2}, \forall v \in \mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R}^{n_v}), v \neq 0 \tag{8}$$

is true for some given disturbance attenuation level γ, if the initial value $x(\theta) = 0, \theta \in [t - \tau_2, 0]$.

Proof. First, by Lemma 1 and (6), it is obvious that system (1) is globally asymptotically stable in probability at the equilibrium point $x \equiv 0$. Second, by Itô formula [36], the stochastic differential $\mathrm{d}V(t, x_t)$ along any trajectory of the closed-loop system (3) can be obtained as

$$\mathrm{d}V(x, t) = \mathcal{L}_v V(x, t)\mathrm{d}t + V_x(x, t) h(x, x_t, t)\mathrm{d}W(t), \tag{9}$$

with $\mathcal{L}_v$ being the infinitesimal generator of (1), which is defined as

$$\mathcal{L}_v V(x, t) = V_t(x, t) + V_x(x, t) f_x(x, x_t, t) + \frac{1}{2} h^{\mathrm{T}}(x, x_t, t) V_{xx}(x, t) h(x, x_t, t). \tag{10}$$

According to the Hamilton-Jacoby-Isaacs condition, it follows from "completing the square" that

$$\begin{aligned}
&\mathcal{L}_v V(x,t)+\frac{1}{2}\|s(t)\|^2-\frac{1}{2}\gamma^2\|v(t)\|^2\\
&=-\left(v-\gamma^{-2}g^{\mathrm{T}}(x,x_t,t)\frac{\partial V}{\partial x}\right)^{\mathrm{T}}\frac{1}{2}\gamma^2 I\left(v-\gamma^{-2}g^{\mathrm{T}}(x,x_t,t)\frac{\partial V}{\partial x}\right)\\
&\quad+\frac{1}{2}\gamma^{-2}\left(\frac{\partial V^{\mathrm{T}}}{\partial x}g(x,x_t,t)g^{\mathrm{T}}(x,x_t,t)\frac{\partial V}{\partial x}\right)+\frac{\partial V}{\partial t}+\frac{\partial V^{\mathrm{T}}}{\partial x}f(x,x_t,t)\\
&\quad+\frac{1}{2}h^{\mathrm{T}}(x,x_t,t)\frac{\partial^2 V}{\partial x^2}h(x,x_t,t)+\frac{1}{2}\|s(t)\|^2. \qquad (11)
\end{aligned}$$

Considering (6) and (11), the following inequality is readily achieved:

$$\begin{aligned}
\mathcal{L}_v V(x,t)&\leqslant\frac{1}{2}\gamma^2\|v(t)\|^2-\left(v-\gamma^{-2}g^{\mathrm{T}}(x,x_t,t)\frac{\partial V}{\partial x}\right)^{\mathrm{T}}\frac{1}{2}\gamma^2 I\left(v-\gamma^{-2}g^{\mathrm{T}}(x,x_t,t)\frac{\partial V}{\partial x}\right)-\frac{1}{2}\|s(t)\|^2\\
&\leqslant\frac{1}{2}\gamma^2\|v(t)\|^2-\frac{1}{2}\|s(t)\|^2. \qquad (12)
\end{aligned}$$

Integrating (12) from 0 to T and taking expectation yields

$$\begin{aligned}
\mathrm{E}\{V(x(T),T)\}-\mathrm{E}\{V(x(0),0)\}&=\mathrm{E}\left\{\int_0^{\mathrm{T}}\mathrm{d}V(x(t),t)\right\}=\mathrm{E}\left\{\int_0^{\mathrm{T}}\mathcal{L}_v V(x(t),t)\mathrm{d}t\right\}\\
&\leqslant\int_0^{\mathrm{T}}\left(\frac{1}{2}\gamma^2\|v(t)\|^2-\frac{1}{2}\|s(t)\|^2\right)\mathrm{d}t, \qquad (13)
\end{aligned}$$

Eq. (13) implies that Eqs. (7) and (8) hold with initial state $x(0)=0$ and $x(0)\neq 0$, respectively.

Remark 1. Lemma 2 can be called a bounded real lemma (BRL) of nonlinear stochastic time-delay systems. A linear stochastic BRL can be found in [28]. Also, two nonlinear ones for nonlinear stochastic systems are derived in [14] and [37].

Remark 2. (1) The filter design problem satisfying the H_∞ filtering inequality in (7) or (8) with a given attenuation level γ is called the H_∞ filter design problem of (1). (2) If

$$J(v)=\frac{1}{2}\mathrm{E}\left\{\int_0^{\mathrm{T}}\left[\|s(t)\|^2-\gamma^2\|v(t)\|^2\right]\mathrm{d}t\right\}, \qquad (14)$$

then $J(v)\leqslant J(v^*)$, for any v, and $v^*\in\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+;\mathbb{R}^{n_v})\cap\bar{\Omega}$, with $\bar{\Omega}=\{v:\lim_{t\to\infty}\mathrm{E}\{V(x(t),t\}=0\}$, and

$$v^*=\gamma^{-2}g^{\mathrm{T}}\frac{\partial V}{\partial x}, \qquad (15)$$

where v^* is called the worst case disturbance, it achieves the given energy gain γ^2 from $v(t)$ to $s(t)$.

If the following nonlinear filter is employed for the estimation of $s(t)$ in (1),

$$\begin{cases}
\mathrm{d}\hat{x}(t)=\hat{f}(\hat{x}(t),\hat{x}(t-\tau(t)),t)\mathrm{d}t+\hat{L}(\hat{x}(t),\hat{x}(t-\tau(t)),t)\mathrm{d}y(t),\\
\hat{f}(0,0,t)=\hat{h}(0,0,t)=0,\\
\hat{s}(t)=G\hat{x}(t),\\
\hat{x}(t)=0,\ t\in[-\tau_2,0],
\end{cases} \qquad (16)$$

to achieve the H_∞ state estimation $\|e(t)\|^2_{L_2}\leqslant\gamma^2\|v(t)\|^2_{L_2}$, where $v\in\mathcal{L}^2_{\mathcal{F}}(\mathbb{R}_+;\mathbb{R}^{n_v})$ and $e(t)=s(t)-\hat{s}(t)=\begin{bmatrix}G & -G\end{bmatrix}\xi(t)$, with $\xi^{\mathrm{T}}(t)=\begin{bmatrix}x^{\mathrm{T}}(t) & \hat{x}^{\mathrm{T}}(t)\end{bmatrix}$, then we present the subsequent result.

Lemma 3. For a prescribed attenuation level γ, if there is a positive definite decrescent radially unbounded Lyapunov–Krasovskii functional $V(x,\hat{x},t)\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}^n\times[t_0-\tau_2,\infty);\mathbb{R}_+)$ with $V(0,0,0)=0$, solving the following HJI:

$$\frac{\partial V^{\mathrm{T}}}{\partial x}f(x,x_t,t)+\frac{\partial V^{\mathrm{T}}}{\partial\hat{x}}(\hat{f}(\hat{x},\hat{x}_t,t)+\hat{L}(\hat{x},\hat{x}_t,t)q(x,x_t,t))+\frac{1}{2}h^{\mathrm{T}}(x,x_t,t)\frac{\partial^2 V}{\partial x^2}h(x,x_t,t)$$

$$
\begin{aligned}
&+\frac{1}{2}\gamma^{-2}\left(\frac{\partial V^{\mathrm{T}}}{\partial x}g(x,x_t,t)+\frac{\partial V}{\partial \hat{x}}\hat{L}(\hat{x},\hat{x}_t,t)k(x,x_t,t)\right)\times\left(\frac{\partial V^{\mathrm{T}}}{\partial x}g(x,x_t,t)\right.\\
&\left.+\frac{\partial V}{\partial \hat{x}}\hat{L}(\hat{x},\hat{x}_t,t)k(x,x_t,t)\right)^{\mathrm{T}}+\frac{1}{2}(\hat{L}(\hat{x},\hat{x}_t,t)j(x,x_t,t))^{\mathrm{T}}\frac{\partial^2 V}{\partial \hat{x}\partial x}h(x,x_t,t)\\
&+\frac{1}{2}h^{\mathrm{T}}(x,x_t,t)\frac{\partial^2 V}{\partial x\partial \hat{x}}(\hat{L}(\hat{x},\hat{x}_t,t)j(x,x_t,t))\\
&+\frac{1}{2}(\hat{L}(\hat{x},\hat{x}_t,t)j(x,x_t,t))^{\mathrm{T}}\frac{\partial^2 V}{\partial \hat{x}^2}(\hat{L}(\hat{x},\hat{x}_t,t)j(x,x_t,t))\\
&+\frac{1}{2}\|\bar{G}\xi(t)\|^2+\frac{\partial V}{\partial t}<0, \qquad (17)
\end{aligned}
$$

for matrices $\hat{f}$ and $\hat{L}$ of appropriate dimensions, then Eq. (16) and (17) solve the stochastic H_∞ state estimation problem.

Proof. The proof is directly achieved as the same line of the proof for Theorem 1 [14].

It follows from Lemma 3 that the nonlinear filter (16) for (1) is more complicated than the filtering problem from $v(t)$ to $s(t)$ in Lemma 2. In [14], the nonlinear filtering problem relies on solving a second-order HJI, which is difficult to be solved except some special cases. There is no practicable nonlinear filter design yet. However, the global linearization method has been conveniently applied for dealing with the H_2/H_∞ filtering problem in nonlinear stochastic systems [15]. This method will be employed to study the H_2/H_∞ filtering problem in nonlinear stochastic time-varying delay systems.

Assume that the nonlinear stochastic time-delay system in (1) could globally be linearized as [34]

$$
\begin{bmatrix}
\frac{\partial f(x,y,t)}{\partial x} & \frac{\partial f(x,y,t)}{\partial y} & \frac{\partial h(x,y,t)}{\partial x} & \frac{\partial h(x,y,t)}{\partial y}\\
\frac{\partial q(x,y,t)}{\partial x} & \frac{\partial q(x,y,t)}{\partial y} & \frac{\partial j(x,y,t)}{\partial x} & \frac{\partial j(x,y,t)}{\partial y}
\end{bmatrix}\in \varOmega,\quad \forall x=x(t),\ y=x(t-\tau(t)),
$$

where the polytope $\varOmega\in\mathbb{R}^{(n+m)\times 4n}$ means the system parameters set of the globally linearized time-delay systems at vertices. Assume that $\varOmega$ could be described as the following convex hull:

$$
\varOmega\in C_0\left(\begin{bmatrix} A_{01} & A_{11} & C_{01} & C_{11}\\ Q_{01} & Q_{11} & J_{01} & J_{11}\end{bmatrix},\ldots,\begin{bmatrix} A_{0m} & A_{1m} & C_{0m} & C_{1m}\\ Q_{0m} & Q_{1m} & J_{0m} & J_{1m}\end{bmatrix}\right). \qquad (18)
$$

Actually, all the globally linearized time-delay systems in $\varOmega$ of the nonlinear stochastic time-delay system (1) can be interpolated as the subsequent m linear stochastic time-delay systems at vertices of the convex hull of $\varOmega$ [34]:

$$
\begin{cases}
\mathrm{d}x(t)=(A_{0i}x(t)+A_{1i}x(t-\tau(t))+B_iv(t))\,\mathrm{d}t+(C_{0i}x(t)+C_{1i}x(t-\tau(t)))\,\mathrm{d}W(t),\\
\mathrm{d}y(t)=(Q_{0i}x(t)+Q_{1i}x(t-\tau(t))+K_iv(t))\,\mathrm{d}t+(J_{0i}x(t)+J_{1i}x(t-\tau(t)))\,\mathrm{d}W(t),\\
s(t)=Gx(t), i=1,2,\ldots,m,\\
x(t)=\phi(t), t\in[-\tau_2,0].
\end{cases} \qquad (19)
$$

By the global linearization theory [34], it follows from (19) that each trajectory of the nonlinear stochastic time-delay system in (1) means a trajectory of the convex combination of the m linearized stochastic time-delay systems in (19), that is, the nonlinear stochastic time-delay system in (1) can be interpolated by the convex combination of m linearized stochastic time-delay systems in (19) with proper approximation errors as below:

$$\begin{cases} \mathrm{d}x(t) = (f(x(t), x(t-\tau(t)), t) + g(x(t), x(t-\tau(t)), t)v(t))\,\mathrm{d}t + h(x(t), x(t-\tau(t)), t)\mathrm{d}W(t) \\ \qquad = \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\{(A_{0i}x(t) + A_{1i}x(t-\tau(t)) + B_i v(t))\,\mathrm{d}t \\ \qquad\quad + (C_{0i}x(t) + C_{1i}x(t-\tau(t)))\,\mathrm{d}W(t)\} + \triangle f\mathrm{d}t + \triangle g v(t)\mathrm{d}t + \triangle h\mathrm{d}W(t), \\ \mathrm{d}y(t) = (q(x(t), x(t-\tau(t)), t) + k(x(t), x(t-\tau(t)), t)v(t))\,\mathrm{d}t + j(x(t), x(t-\tau(t)), t)\mathrm{d}W(t) \\ \qquad = \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\{(Q_{0i}x(t) + Q_{1i}x(t-\tau(t)) + K_i v(t))\,\mathrm{d}t \\ \qquad\quad + (J_{0i}x(t) + J_{1i}x(t-\tau(t)))\,\mathrm{d}W(t)\} + \triangle q\mathrm{d}t + \triangle k v(t)\mathrm{d}t + \triangle j\mathrm{d}W(t), \\ s(t) = Gx(t), \\ x(t) = \phi(t),\ t \in [-\tau_2, 0], \end{cases} \tag{20}$$

where

$$\triangle f = f(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\big(A_{0i}x + A_{1i}x_t\big),\ \triangle q = q(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\big(Q_{0i}x + Q_{1i}x_t\big),$$
$$\triangle h = h(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\big(C_{0i}x + C_{1i}x_t\big),\ \triangle j = j(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)\big(J_{0i}x + J_{1i}x_t\big),$$
$$\triangle g = g(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)B_i,\ \triangle k = k(x, x_t, t) - \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big)K_i,$$

with the interpolation functions $\alpha_i\big(x(t), x(t-\tau(t)), t\big)$, $i = 1, 2, \ldots, m$, satisfying $0 \leqslant \alpha_i\big(x(t), x(t-\tau(t)), t\big) \leqslant 1$ and $\sum_{i=1}^{m} \alpha_i\big(x(t), x(t-\tau(t)), t\big) = 1$.

Remark 3. It is usually difficult to solve the HJI in (6) or (17). By the global linearization method, the filter design problem for the nonlinear stochastic time-delay system can be studied by solving simultaneous LMIs related by the globally linearized systems in (20) instead of a second-order HJI.

3 H_∞ global linearization filter design

By the globally linearized model (20), the nonlinear filter (16) is replaced by the following global linearization filter to study the H_∞ filter for system in (1):

$$\mathrm{d}\hat{x}(t) = \sum_{j=1}^{m} \alpha_j\big(\hat{x}(t), \hat{x}(t-\tau(t)), t\big)\big\{\big(A_{0j}\hat{x}(t) + A_{1j}\hat{x}(t-\tau(t))\big)\mathrm{d}t + L_j\big[\mathrm{d}y(t) - \mathrm{d}\hat{y}(t)\big]\big\}, \tag{21}$$

where L_j is the gain for the jth filter and

$$\mathrm{d}\hat{y}(t) = \sum_{k=1}^{m} \alpha_k\big(\hat{x}, \hat{x}_t, t\big)\,(Q_{0i}\hat{x} + Q_{1i}\hat{x}_t)\,\mathrm{d}t.$$

In this condition, the globally linearized filter is written as follows:

$$\mathrm{d}\hat{x}(t) = \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big) \sum_{j=1}^{m} \alpha_j\big(\hat{x}, \hat{x}_t, t\big) \sum_{k=1}^{m} \alpha_k\big(\hat{x}, \hat{x}_t, t\big)\big\{\big(A_{0j}\hat{x} + A_{1j}\hat{x}_t\big)\mathrm{d}t + L_j\,\big[(Q_{0i}x + Q_{1i}x_t)\mathrm{d}t$$
$$+K_i v\mathrm{d}t + \big(J_{0i}x + J_{1i}x_t\big)\mathrm{d}W(t) + \triangle q\mathrm{d}t + \triangle k v\mathrm{d}t + \triangle j\mathrm{d}W(t) - \big(Q_{0k}\hat{x} + Q_{1k}\hat{x}_t\big)\mathrm{d}t\big]\,\big\}. \tag{22}$$

The augmented system is obtained as follows:

$$\mathrm{d}\xi(t) = \sum_{i=1}^{m} \alpha_i\big(x, x_t, t\big) \sum_{j=1}^{m} \alpha_j\big(\hat{x}, \hat{x}_t, t\big) \sum_{k=1}^{m} \alpha_k\big(\hat{x}, \hat{x}_t, t\big)\big\{\big(\bar{A}_{0ijk}\xi(t) + \bar{A}_{1ijk}\xi(t-\tau(t)) + \bar{B}_{ij}v(t)\big)\mathrm{d}t$$

$$+\big(\bar{D}_{0ij}\xi(t)+\bar{D}_{1ij}\xi(t-\tau(t))\big)\mathrm{d}W(t)\big\}+\triangle\bar{f}\mathrm{d}t+\triangle\bar{g}v(t)\mathrm{d}t+\triangle\bar{h}\mathrm{d}W(t), \tag{23}$$

where

$$\xi(t)=\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix},\ \bar{A}_{0ijk}=\begin{bmatrix}A_{0i} & 0\\ L_jQ_{0i} & A_{0j}-L_jQ_{0k}\end{bmatrix},\ \bar{A}_{1ijk}=\begin{bmatrix}A_{1i} & 0\\ L_jQ_{1i} & A_{1j}-L_jQ_{1k}\end{bmatrix},$$
$$\bar{B}_{ij}=\begin{bmatrix}B_i\\ L_jK_i\end{bmatrix},\ \bar{D}_{0ij}=\begin{bmatrix}C_{0i} & 0\\ L_jJ_{0i} & 0\end{bmatrix},\ \bar{D}_{1ij}=\begin{bmatrix}C_{1i} & 0\\ L_jJ_{1i} & 0\end{bmatrix},\ \triangle\bar{f}=\begin{bmatrix}\triangle f\\ \sum_{j=1}^{m}\alpha_j\big(\hat{x},\hat{x}_t,t\big)L_j\triangle q\end{bmatrix},$$
$$\triangle\bar{g}=\begin{bmatrix}\triangle g\\ \sum_{j=1}^{m}\alpha_j\big(\hat{x},\hat{x}_t,t\big)L_j\triangle k\end{bmatrix},\ \triangle\bar{h}=\begin{bmatrix}\triangle h\\ \sum_{j=1}^{m}\alpha_j\big(\hat{x},\hat{x}_t,t\big)L_j\triangle j\end{bmatrix}.$$

Denote $\eta(t)=\big[\xi^{\mathrm{T}}(t),\xi^{\mathrm{T}}(t-\tau(t)),\frac{1}{\tau_2}\int_{t-\tau_2}^{t}\xi^{\mathrm{T}}(s)\mathrm{d}s\big]^{\mathrm{T}}$, $\zeta(t)=\big[x^{\mathrm{T}}(t),x^{\mathrm{T}}(t-\tau(t))\big]^{\mathrm{T}}$.

Assumption 1. There are five positive bounded constants e_1,e_2,e_3,e_4, and α such that

$$\|\triangle f\|\leqslant e_1\|\zeta(t)\|,\|\triangle h\|\leqslant e_2\|\zeta(t)\|,\|\triangle q\|\leqslant e_3\|\zeta(t)\|,\|\triangle j\|\leqslant e_4\|\zeta(t)\|,\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}^{\mathrm{T}}\leqslant\alpha I. \tag{24}$$

Remark 4. The nonlinear system (1), the globally linearized model (20), the globally linearized filter (22), and the augmented system (23) are time-varying systems. Thus, the non-uniformly global stability is studied in this paper.

Let us denote

$$e(t)=s(t)-\hat{s}(t)=\big[G\quad -G\big]\xi(t)=\bar{G}\xi(t), \tag{25}$$

where $\bar{G}=\big[G\quad -G\big]$, then the nonlinear stochastic H_∞ filter design problem can be described as below. Determined the filter gains $L_j, j=1,2,\ldots,m$, such that the following hold.

(1) System (23) is globally asymptotically stable in probability at $\xi(t)\equiv 0$ when $v(t)=0$.

(2) For a prescribed $\gamma>0$, the subsequent inequality holds

$$\|e(t)\|_{L_2}^2\leqslant 2\mathrm{E}\{V(\xi(0),0\}+\gamma^2\|v(t)\|_{L_2}^2,\forall v\in\mathcal{L}_{\mathcal{F}}^2\big(\mathbb{R}_+;\mathbb{R}^{n_v}\big),v\neq 0, \tag{26}$$

where $V(\xi,t)\in C^{2,1}\big(\mathbb{R}^{2n}\times[t_0-\tau_2,\infty);\mathbb{R}_+\big)$ with $V(0,0)=0$.

Theorem 1. For the augmented system in (23) with Assumption 1, if there exist $P>0,Q>0$, and $R>0$ solving the following matrix inequalities:

$$\begin{cases}\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_jI, & \text{with a scalar variable } \beta_j>0,\quad j=1,2,\ldots,m,\\ \Pi_1^{H_\infty}<0,\end{cases} \tag{27}$$

where

$$\Pi_1^{H_\infty}=\begin{bmatrix}\Pi_{ijk} & \tilde{D}_{ij}^{\mathrm{T}}P & \tilde{P}\bar{B}_{ij} & \tilde{P}\bar{L}_j & \tilde{P}\bar{L}_k\\ * & -\frac{P}{2} & 0 & 0 & 0\\ * & * & -\frac{\gamma^2}{2}I & 0 & 0\\ * & * & * & -\frac{\gamma^2}{\alpha}I & 0\\ * & * & * & * & -\frac{\gamma^2}{\alpha}I\end{bmatrix},$$

with

$$\Pi_{ijk}=\begin{bmatrix}\begin{matrix}P\bar{A}_{0ijk}+\bar{A}_{0ijk}^{\mathrm{T}}P+\bar{G}^{\mathrm{T}}\bar{G}+P+Q+\tau_2^2R\\ +\big(\beta_j+\beta_k\big)\big[\frac{1}{2}\big(e_1^2+e_3^2\big)I+\big(e_2^2+e_4^2\big)I\big]\end{matrix} & P\bar{A}_{1ijk} & 0\\ * & \begin{matrix}-(1-\bar{\tau})Q+\big(\beta_j+\beta_k\big)\\ \times\big[\frac{1}{2}\big(e_1^2+e_3^2\big)I+\big(e_2^2+e_4^2\big)I\big]\end{matrix} & 0\\ * & * & -\tau_2^2R\end{bmatrix},$$

and $\tilde{D}_{ij}=\begin{bmatrix}\bar{D}_{0ij} & \bar{D}_{1ij} & 0\end{bmatrix}$, $\tilde{P}=\begin{bmatrix}P & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $\bar{L}_j=\mathrm{diag}\{I,L_j\}$, $\bar{L}_k=\mathrm{diag}\{I,L_k\}$ for all $i,j,k=1,2,\ldots,m$, then the subsequent is true: (1) the augmented system (23) is globally asymptotically stable in probability at $\xi(t)\equiv 0$ when $v(t)=0$, and (2) the subsequent inequality

$$\|e(t)\|_{L_2}^2 \leqslant 2\mathrm{E}\{V(\xi(0),0\}+\gamma^2\|v(t)\|_{L_2}^2, \forall v\in\mathcal{L}_{\mathcal{F}}^2\big(\mathbb{R}_+;\mathbb{R}^{n_v}\big), v\neq 0 \tag{28}$$

holds for a prescribed disturbance attenuation level $\gamma>0$.

Proof. By Lemma 2, if there is a positive definite decrescent radially unbounded Lyapunov–Krasovskii functional $V(\xi,t)\in C^{2,1}\big(\mathbb{R}^{2n}\times[t_0-\tau_2,\infty);\mathbb{R}_+\big)$ with $V(0,0)=0$ such that

$$\begin{aligned}
&\frac{\partial V^{\mathrm{T}}}{\partial\xi}\left[\sum_{i=1}^m\alpha_i\big(\xi,\xi_t,t\big)\sum_{j=1}^m\alpha_j\big(\hat{\xi},\hat{\xi}_t,t\big)\sum_{k=1}^m\alpha_k\big(\hat{\xi},\hat{\xi}_t,t\big)\big(\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t\big)+\triangle\bar{f}\right]\\
&+\frac{1}{2}\left[\sum_{i=1}^m\alpha_i\big(\xi,\xi_t,t\big)\sum_{j=1}^m\alpha_j\big(\hat{\xi},\hat{\xi}_t,t\big)\big(\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big)+\triangle\bar{h}\right]^{\mathrm{T}}\frac{\partial^2V}{\partial\xi^2}\\
&\times\left[\sum_{l=1}^m\alpha_l\big(\xi,\xi_t,t\big)\sum_{s=1}^m\alpha_s\big(\hat{\xi},\hat{\xi}_t,t\big)\big(\bar{D}_{0ls}\xi+\bar{D}_{1ls}\xi_t\big)+\triangle\bar{h}\right]+\frac{\partial V}{\partial t}+\frac{1}{2}\|\bar{G}\xi\|^2\\
&+\frac{1}{2}\gamma^{-2}\left[\frac{\partial V^{\mathrm{T}}}{\partial\xi}\left(\sum_{i=1}^m\alpha_i\big(\xi,\xi_t,t\big)\sum_{j=1}^m\alpha_j\big(\hat{\xi},\hat{\xi}_t,t\big)\bar{B}_{ij}+\triangle\bar{g}\right)\right.\\
&\left.\times\left(\sum_{l=1}^m\alpha_l\big(\xi,\xi_t,t\big)\sum_{s=1}^m\alpha_s\big(\hat{\xi},\hat{\xi}_t,t\big)\bar{B}_{ls}+\triangle\bar{g}\right)^{\mathrm{T}}\frac{\partial V}{\partial\xi}\right]<0,
\end{aligned}\tag{29}$$

then (1) and (2) hold. Choose a Lyapunov–Krasovskii functional candidate as

$$V(\xi,t)=\frac{1}{2}\xi^{\mathrm{T}}(t)P\xi(t)+\frac{1}{2}\int_{t-\tau(t)}^t\xi^{\mathrm{T}}(s)Q\xi(s)\mathrm{d}s+\frac{\tau_2}{2}\int_{-\tau_2}^0\int_{t+\theta}^t\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\mathrm{d}\theta. \tag{30}$$

For simplicity, denote $\alpha_i\big(\xi,\xi_t,t\big)=\alpha_i$, $\alpha_l\big(\xi,\xi_t,t\big)=\alpha_l$, $\alpha_j\big(\hat{\xi},\hat{\xi}_t,t\big)=\hat{\alpha}_j$, $\alpha_k\big(\hat{\xi},\hat{\xi}_t,t\big)=\hat{\alpha}_k$, $\alpha_s\big(\hat{\xi},\hat{\xi}_t,t\big)=\hat{\alpha}_s$ with all the variables omitted.

By (30), Eq. (29) can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^m\alpha_i\sum_{j=1}^m\hat{\alpha}_j\sum_{k=1}^m\hat{\alpha}_k\sum_{l=1}^m\alpha_l\sum_{s=1}^m\hat{\alpha}_s\times\big\{\mathrm{Sym}\,\big(\xi^{\mathrm{T}}P\big[\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t\big]\big)+\gamma^{-2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\bar{B}_{ij}\big)\\
&\quad+\xi^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{G}\xi+\mathrm{TSym}\big(\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]^{\mathrm{T}}\big)P\ \big\}+\frac{1}{2}\mathrm{Sym}\big(\xi^{\mathrm{T}}P\triangle\bar{f}\big)+\frac{1}{2}\gamma^{-2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\triangle\bar{g}\big)\\
&\quad+\frac{1}{2}\gamma^{-2}\left[\xi^{\mathrm{T}}P\sum_{i=1}^m\alpha_i\sum_{j=1}^m\hat{\alpha}_j\bar{B}_{ij}\triangle\bar{g}^{\mathrm{T}}P\xi+\xi^{\mathrm{T}}P\triangle\bar{g}\sum_{l=1}^m\alpha_l\sum_{s=1}^m\hat{\alpha}_s\bar{B}_{ls}^{\mathrm{T}}P\xi\right]\\
&\quad+\frac{1}{2}\left[\sum_{i=1}^m\alpha_i\sum_{j=1}^m\hat{\alpha}_j\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]^{\mathrm{T}}P\triangle\bar{h}+\triangle\bar{h}^{\mathrm{T}}P\sum_{l=1}^m\alpha_l\sum_{s=1}^m\hat{\alpha}_s\big[\bar{D}_{0ls}\xi+\bar{D}_{1ls}\xi_t\big]\right]\\
&\quad+\frac{1}{2}\triangle\bar{h}^{\mathrm{T}}P\triangle\bar{h}+\frac{1}{2}\left[\xi^{\mathrm{T}}Q\xi-\big(1-\dot{\tau}(t)\big)\xi_t^{\mathrm{T}}Q\xi_t+\tau_2^2\xi^{\mathrm{T}}R\xi-\tau_2\int_{t-\tau_2}^t\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\right]\\
&\leqslant\frac{1}{2}\sum_{i=1}^m\alpha_i\sum_{j=1}^m\hat{\alpha}_j\sum_{k=1}^m\hat{\alpha}_k\sum_{l=1}^m\alpha_l\sum_{s=1}^m\hat{\alpha}_s\times\big\{\mathrm{Sym}\,\big(\xi^{\mathrm{T}}P\big[\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t\big]\big)+\gamma^{-2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\bar{B}_{ij}\big)\\
&\quad+\xi^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{G}\xi+\mathrm{TSym}\big(\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]^{\mathrm{T}}\big)P\big\}+\frac{1}{2}\big[\xi^{\mathrm{T}}P\xi+\triangle\bar{f}^{\mathrm{T}}P\triangle\bar{f}\big]+\frac{1}{2}\gamma^{-2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\triangle\bar{g}\big)\\
&\quad+\frac{1}{2}\gamma^{-2}\left[\xi^{\mathrm{T}}P\left[\sum_{i=1}^m\alpha_i\sum_{j=1}^m\hat{\alpha}_j\frac{1}{2}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}+\sum_{l=1}^m\alpha_l\sum_{s=1}^m\hat{\alpha}_s\frac{1}{2}\bar{B}_{ls}\bar{B}_{ls}^{\mathrm{T}}\right]P\xi\right.
\end{aligned}$$

Table 1 Procedure for H_∞ global linearization filtering

Steps	Design procedures
Step 1:	Establish the globally linearized model (19);
Step 2:	Choose the matrix G;
Step 3:	Obtain the five positive bounded constants e_1, e_2, e_3, e_4, and α in (24) along the entire trajectory of the nonlinear system (1) that comes from the simulation of the system (1);
Step 4:	Solve the LP in (35) to get $\gamma_{\min}, P_1, P_2, Q_1, Q_2, Q_3, R_1, R_2, R_3, \beta_j$, and Y_j, then $L_j = P_2^{-1} Y_j$;
Step 5:	Establish the globally linearized filter in (21).

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\triangle\bar{g}\big)+\frac{1}{2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\triangle\bar{g}\big)\Bigg]+\frac{1}{2}\Bigg[\sum_{i=1}^{m}\alpha_i\sum_{j=1}^{m}\hat{\alpha}_j\frac{1}{2}\mathrm{TSym}\big(\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]^{\mathrm{T}}\big)P\\
&+\sum_{l=1}^{m}\alpha_l\sum_{s=1}^{m}\hat{\alpha}_s\frac{1}{2}\mathrm{TSym}\big(\big[\bar{D}_{0ls}\xi+\bar{D}_{1ls}\xi_t\big]^{\mathrm{T}}\big)P+\frac{1}{2}\triangle\bar{h}^{\mathrm{T}}P\triangle\bar{h}+\frac{1}{2}\triangle\bar{h}^{\mathrm{T}}P\triangle\bar{h}\Bigg]+\frac{1}{2}\triangle\bar{h}^{\mathrm{T}}P\triangle\bar{h}\\
&+\frac{1}{2}\left[\xi^{\mathrm{T}}Q\xi-\big(1-\bar{\tau}\big)\xi_t^{\mathrm{T}}Q\xi_t+\tau_2^2\xi^{\mathrm{T}}R\xi-\tau_2\int_{t-\tau_2}^{t}\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\right],\ \{\text{by Lemma 4 in [15]}\}\\
&\leqslant\frac{1}{2}\sum_{i=1}^{m}\alpha_i\sum_{j=1}^{m}\hat{\alpha}_j\sum_{k=1}^{m}\hat{\alpha}_k\times\Big\{\mathrm{Sym}\left(\xi^{\mathrm{T}}P\big[\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t\big]\right)+2\gamma^{-2}\mathrm{TSym}\big(\xi^{\mathrm{T}}P\bar{B}_{ij}\big)+\xi^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{G}\xi\\
&+2\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]^{\mathrm{T}}P\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t\big]+\xi^{\mathrm{T}}\big(P+Q+\tau_2^2R\big)\xi-\big(1-\bar{\tau}\big)\xi_t^{\mathrm{T}}Q\xi_t\\
&-\left[\frac{1}{\tau_2}\int_{t-\tau_2}^{t}\xi(s)\mathrm{d}s\right]^{\mathrm{T}}(\tau_2^2R)\frac{1}{\tau_2}\int_{t-\tau_2}^{t}\xi(s)\mathrm{d}s+\gamma^{-2}\xi^{\mathrm{T}}P\bar{L}_j\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}^{\mathrm{T}}\bar{L}_j^{\mathrm{T}}P\xi\\
&+\gamma^{-2}\xi^{\mathrm{T}}P\bar{L}_k\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}\begin{bmatrix}\triangle g\\ \triangle k\end{bmatrix}^{\mathrm{T}}\bar{L}_k^{\mathrm{T}}P\xi+\frac{1}{2}\begin{bmatrix}\triangle f\\ \triangle q\end{bmatrix}^{\mathrm{T}}\bar{L}_j^{\mathrm{T}}P\bar{L}_j\begin{bmatrix}\triangle f\\ \triangle q\end{bmatrix}\\
&+\frac{1}{2}\begin{bmatrix}\triangle f\\ \triangle q\end{bmatrix}^{\mathrm{T}}\bar{L}_k^{\mathrm{T}}P\bar{L}_k\begin{bmatrix}\triangle f\\ \triangle q\end{bmatrix}+\begin{bmatrix}\triangle h\\ \triangle j\end{bmatrix}^{\mathrm{T}}\bar{L}_j^{\mathrm{T}}P\bar{L}_j\begin{bmatrix}\triangle h\\ \triangle j\end{bmatrix}+\begin{bmatrix}\triangle h\\ \triangle j\end{bmatrix}^{\mathrm{T}}\bar{L}_k^{\mathrm{T}}P\bar{L}_k\begin{bmatrix}\triangle h\\ \triangle j\end{bmatrix}\Bigg\}\\
&\{\text{by Lemma 4 in [15] and Proposition B.8 (Jensen inequality) in [38]}\}\\
&\leqslant\frac{1}{2}\sum_{i=1}^{m}\alpha_i\sum_{j=1}^{m}\hat{\alpha}_j\sum_{k=1}^{m}\hat{\alpha}_k\Big\{\eta^{\mathrm{T}}(t)\Big[\Pi_{ijk}+2\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}+2\tilde{D}_{ij}^{\mathrm{T}}P\tilde{D}_{ij}+\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}\\
&+\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}\Big]\eta(t)\Big\}<0.\qquad\{\text{by (24) and }\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_jI\}
\end{aligned}\tag{31}
$$

The aforementioned inequality holds if the following inequalities:

$$
\Pi_{ijk}+2\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}+2\tilde{D}_{ij}^{\mathrm{T}}P\tilde{D}_{ij}+\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}+\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}<0 \tag{32}
$$

are true for all $i, j, k = 1, 2, \ldots, m$, with Π_{ijk}, $\tilde{D}_{ij}$, $\tilde{P}, \bar{L}_j$, and $\bar{L}_k$ being defined in Theorem 1. By the Schur complement [34], Eq. (32) is equivalent to (27) under $\bar{L}_j^{\mathrm{T}}P\bar{L}_j < \beta_j I$. It follows from the inequalities in (32) and $\bar{L}_j^{\mathrm{T}}P\bar{L}_j < \beta_j I$ that (1) and (2) of the theorem hold. The proof is completed accordingly.

Remark 5. By the declaration of (33), the matrix inequalities in (27) can be formulated as in (34) to solve $P_1 > 0, P_2 > 0, Q_1 > 0, Q_2, Q_3 > 0$ and $R_1 > 0, R_2, R_3 > 0$. For ease of design, let

$$
P=\begin{bmatrix}P_1 & P_2\\ P_2 & P_2\end{bmatrix},\quad Q=\begin{bmatrix}Q_1 & Q_2\\ Q_2 & Q_3\end{bmatrix},\quad R=\begin{bmatrix}R_1 & R_2\\ R_2 & R_3\end{bmatrix}, \tag{33}
$$

and Eq. (27) should be modified as the LMIs in (34):

$$\Pi_2^{H_\infty} = \begin{bmatrix} (1,1) & (1,2) \\ * & (2,2)_{\Pi_2^{H_\infty}} \end{bmatrix} < 0, \ \ \forall\ i,\ j,\ k = 1,2,\ldots,m,$$
$$\Theta = \begin{bmatrix} -\beta_j I & 0 & P_1 & P_2 \\ * & -\beta_j I & Y_j^{\mathrm{T}} & Y_j^{\mathrm{T}} \\ * & * & -P_1 & -P_2 \\ * & * & * & -P_2 \end{bmatrix} < 0 \ \ \beta_j > 0, \ \ j = 1,2,\ldots,m, \tag{34}$$

where

$$(1,1) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ * & * & \Gamma_{33} & \Gamma_{34} \\ * & * & * & \Gamma_{44} \end{bmatrix}, \ (1,2) = \begin{bmatrix} 0 & 0 & \Gamma_{17} & \Gamma_{18} & \Gamma_{19} & P_1 & Y_j & P_1 & Y_k \\ 0 & 0 & 0 & 0 & \Gamma_{29} & P_2 & Y_j & P_2 & Y_k \\ 0 & 0 & \Gamma_{37} & \Gamma_{38} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$(2,2)_{\Pi_2^{H_\infty}} = \mathrm{diag}\left\{ \begin{bmatrix} -\tau_2^2 R_1 & -\tau_2^2 R_2 \\ * & -\tau_2^2 R_3 \end{bmatrix}, \begin{bmatrix} -\frac{P_1}{2} & -\frac{P_2}{2} \\ * & -\frac{P_2}{2} \end{bmatrix}, \ -\frac{\gamma^2 I}{2}, \ -\frac{\gamma^2 I}{\alpha}, \ -\frac{\gamma^2 I}{\alpha}, \ -\frac{\gamma^2 I}{\alpha}, \ -\frac{\gamma^2 I}{\alpha} \right\},$$
$$Y_j = P_2 L_j, \ \sigma = \tfrac{1}{2}\big(e_1^2 + e_3^2\big) + \big(e_2^2 + e_4^2\big),$$
$$\Gamma_{11} = P_1 A_{0i} + A_{0i}^{\mathrm{T}} P_1 + Y_j Q_{0i} + Q_{0i}^{\mathrm{T}} Y_j^{\mathrm{T}} + G^{\mathrm{T}} G + P_1 + Q_1 + \tau_2^2 R_1 + \big(\beta_j + \beta_k\big)\sigma I,$$
$$\Gamma_{12} = A_{0i}^{\mathrm{T}} P_2 + P_2 A_{0j} + Q_{0i}^{\mathrm{T}} Y_j^{\mathrm{T}} - Y_j Q_{0k} - G^{\mathrm{T}} G + P_2 + Q_2 + \tau_2^2 R_2, \ \Gamma_{13} = P_1 A_{1i} + Y_j Q_{1i},$$
$$\Gamma_{14} = P_2 A_{1j} - Y_j Q_{1k}, \ \Gamma_{17} = C_{0i}^{\mathrm{T}} P_1 + J_{0i}^{\mathrm{T}} Y_j^{\mathrm{T}}, \ \Gamma_{18} = C_{0i}^{\mathrm{T}} P_2 + J_{0i}^{\mathrm{T}} Y_j^{\mathrm{T}}, \ \Gamma_{19} = P_1 B_i + Y_j K_i,$$
$$\Gamma_{22} = P_2 A_{0j} - Y_j Q_{0k} + A_{0j}^{\mathrm{T}} P_2 - Q_{0k}^{\mathrm{T}} Y_j^{\mathrm{T}} + G^{\mathrm{T}} G + P_2 + Q_3 + \tau_2^2 R_3 + \big(\beta_j + \beta_k\big)\sigma I,$$
$$\Gamma_{23} = P_2 A_{1i} + Y_j Q_{1i}, \ \Gamma_{24} = P_2 A_{1j} - Y_j Q_{1k}, \ \Gamma_{29} = P_2 B_i + Y_j K_i,$$
$$\Gamma_{33} = -\big(1-\bar{\tau}\big) Q_1 + \big(\beta_j + \beta_k\big)\sigma I, \ \Gamma_{44} = -\big(1-\bar{\tau}\big) Q_3 + \big(\beta_j + \beta_k\big)\big(e_2^2 + e_4^2\big)\sigma I,$$
$$\Gamma_{34} = -\big(1-\bar{\tau}\big) Q_2, \ \Gamma_{37} = C_{1i}^{\mathrm{T}} P_1 + J_{1i}^{\mathrm{T}} Y_j^{\mathrm{T}}, \ \Gamma_{38} = C_{1i}^{\mathrm{T}} P_2 + J_{1i}^{\mathrm{T}} Y_j^{\mathrm{T}}.$$

By the previous discussion and the choices of P, Q, and R in (33), the optimal H_∞ globally linearized filter design is formulated as the subsequent linear programming problem (LP),

$$\min_{\{P_1,P_2,Q_1,Q_2,Q_3,R_1,R_2,R_3,\beta_j,Y_j\}} \rho, \quad \text{s.t.} \quad (34) \text{ with } \rho = \gamma^2, \ P = P^{\mathrm{T}} > 0, \ Q = Q^{\mathrm{T}} > 0, R = R^{\mathrm{T}} > 0. \tag{35}$$

For clarity, a design procedure for the H_∞ globally linearized filter design is given in Table 1.

Remark 6. (1) In the globally linearized filter design, the HJI in (17) of the nonlinear state filter design in (16) is replaced with the matrix inequalities in (27), which can be efficiently solved by the LMI toolbox in Matlab. (2) By Remark 2, one can obtain a corresponding v^* as

$$v^* = \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{m} \hat{\alpha}_j \gamma^{-2} \left[\bar{B}_{ij}^{\mathrm{T}} \frac{\partial V}{\partial \xi} \right] + \gamma^{-2} \triangle \bar{g}^{\mathrm{T}} \frac{\partial V}{\partial \xi} = \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{m} \hat{\alpha}_j \gamma^{-2} \left[\bar{B}_{ij}^{\mathrm{T}} P \xi \right] + \gamma^{-2} \triangle \bar{g}^{\mathrm{T}} P \xi, \tag{36}$$

where v^* can be taken as the worst case disturbance achieving the desired energy gain γ^2.

4 Suboptimal mixed H_2/H_∞ global linearization filter design

If a desired attenuation level γ is chosen, we can obtain an H_∞ filter in (21) under $v^*(t)$ in (36) for the nonlinear stochastic time-delay system. The H_∞ filter design that minimizes the norm of the estimation error is called the mixed H_2/H_∞ filter design, and the worst case disturbance $v^*(t)$ in the H_∞ filter case is considered for $v(t)$ in (1).

In view of (36), the augmented system in (23) is rewritten as follows:

$$\begin{aligned}\mathrm{d}\xi(t) = &\sum_{i=1}^{m}\alpha_i(x,x_t,t)\sum_{j=1}^{m}\alpha_j(\hat{x},\hat{x}_t,t)\sum_{k=1}^{m}\alpha_k(\hat{x},\hat{x}_t,t)\sum_{l=1}^{m}\alpha_l(x,x_t,t)\sum_{s=1}^{m}\alpha_s(\hat{x},\hat{x}_t,t)\times\big\{\big[\bar{A}_{0ijk}\xi(t)\\&+\bar{A}_{1ijk}\xi(t-\tau(t))+\gamma^{-2}\bar{B}_{ij}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\bar{B}_{ij}\triangle\bar{g}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\triangle\bar{g}^{\mathrm{T}}P\xi\big]\mathrm{d}t\\&+\triangle\bar{f}\mathrm{d}t+\big[\bar{D}_{0ij}\xi(t)+\bar{D}_{1ij}\xi(t-\tau(t))+\triangle\bar{h}\big]\mathrm{d}W(t)\big\}.\end{aligned}\tag{37}$$

It follows from Proposition 1 and Theorem 1 that we obtain the subsequent result.

Theorem 2. For the augmented system in (23) with Assumption 1, if there are $P>0, Q>0$, and $R>0$ satisfying the following matrix inequalities:

$$\begin{cases}\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_j I, & \text{with a scalar variable } \beta_j>0,\quad j=1,2,\ldots,m,\\ \Pi_1^{H_2/H_\infty}<0,\end{cases}\tag{38}$$

where

$$\Pi_1^{H_2/H_\infty}=\begin{bmatrix}\Pi_{ijk} & \tilde{D}_{ij}^{\mathrm{T}}P & \tilde{P}\bar{B}_{ij} & \tilde{P}\bar{L}_j & \tilde{P}\bar{L}_k\\ * & -\frac{P}{2} & 0 & 0 & 0\\ * & * & -\frac{\gamma^2}{4}I & 0 & 0\\ * & * & * & -\frac{\gamma^2}{2\alpha}I & 0\\ * & * & * & * & -\frac{\gamma^2}{2\alpha}I\end{bmatrix},$$

with $\Pi_{ijk}, \tilde{D}_{ij}$, and $\tilde{P}$ being defined in Theorem 1, $\bar{L}_j=\mathrm{diag}\{I,L_j\}, \bar{L}_k=\mathrm{diag}\{I,L_k\}$ for all $i,j,k=1,2,\ldots,m$, then the subsequent hold: (1) the augmented system (23) is globally asymptotically stable in probability at $\xi(t)\equiv 0$ when $v(t)=0$, (2) the estimation error satisfies the H_2 norm property as

$$\begin{aligned}\|e(t)\|_{L_2}^2 &\leqslant 2\mathrm{E}\{V(\xi(0),0\}\\&=\mathrm{E}\{\xi^{\mathrm{T}}(0)P\xi(0)\}+\mathrm{E}\left\{\int_{-\tau(0)}^{0}\xi^{\mathrm{T}}(s)Q\xi(s)\mathrm{d}s+\tau_2\int_{-\tau_2}^{0}(s+\tau_2)\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\right\},\end{aligned}\tag{39}$$

that is, $\mathrm{E}\{\xi^{\mathrm{T}}(0)P\xi(0)\}+\mathrm{E}\big\{\int_{-\tau(0)}^{0}\xi^{\mathrm{T}}(s)Q\xi(s)\mathrm{d}s+\tau_2\int_{-\tau_2}^{0}(s+\tau_2)\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\big\}$ is the upper bound of $\|e(t)\|_{L_2}^2$ and is to be minimized in the suboptimal H_2 filtering case; and (3) the global linearization filter also satisfies the filtering performance in (28).

Proof. By Proposition 1, if there exists a Lyapunov–Krasovskii functional $V(\xi,t)\in C^{2,1}\big(\mathbb{R}^{2n}\times[t_0-\tau_2,\infty);\mathbb{R}_+\big)$ with $V(0,0)=0$ such that the equation as below is derived, then (1) and (2) hold.

$$\begin{aligned}&\sum_{i=1}^{m}\alpha_i(x,x_t,t)\sum_{j=1}^{m}\alpha_j(\hat{x},\hat{x}_t,t)\sum_{k=1}^{m}\alpha_k(\hat{x},\hat{x}_t,t)\sum_{l=1}^{m}\alpha_l(x,x_t,t)\sum_{s=1}^{m}\alpha_s(\hat{x},\hat{x}_t,t)\times\left\{\frac{\partial V^{\mathrm{T}}}{\partial\xi}\big[\bar{A}_{0ijk}\xi\right.\\&\quad+\bar{A}_{1ijk}\xi_t+\gamma^{-2}\bar{B}_{ij}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\bar{B}_{ij}\triangle\bar{g}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\triangle\bar{g}^{\mathrm{T}}P\xi+\triangle\bar{f}\big]\\&\quad\left.+\frac{1}{2}\|\bar{G}\xi\|^2+\frac{1}{2}\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t+\triangle\bar{h}\big]^{\mathrm{T}}\frac{\partial^2V^{\mathrm{T}}}{\partial\xi^2}\big[\bar{D}_{0ls}\xi+\bar{D}_{1ls}\xi_t+\triangle\bar{h}\big]+\frac{\partial V}{\partial t}\right\}<0.\end{aligned}\tag{40}$$

By substituting (30) into (40), it follows from the same lines as the proof of Theorem 1 that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{m}\alpha_i(x,x_t,t)\sum_{j=1}^{m}\alpha_j(\hat{x},\hat{x}_t,t)\sum_{k=1}^{m}\alpha_k(\hat{x},\hat{x}_t,t)\sum_{l=1}^{m}\alpha_l(x,x_t,t)\sum_{s=1}^{m}\alpha_s(\hat{x},\hat{x}_t,t)\\&\quad\times\Big\{\xi^{\mathrm{T}}P\big[\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t+\gamma^{-2}\bar{B}_{ij}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\bar{B}_{ij}\triangle\bar{g}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\bar{B}_{ls}^{\mathrm{T}}P\xi\big]\\&\quad+\big[\bar{A}_{0ijk}\xi+\bar{A}_{1ijk}\xi_t+\gamma^{-2}\bar{B}_{ij}\bar{B}_{ls}^{\mathrm{T}}P\xi+\gamma^{-2}\bar{B}_{ij}\triangle\bar{g}^{\mathrm{T}}P\xi+\gamma^{-2}\triangle\bar{g}\bar{B}_{ls}^{\mathrm{T}}P\xi\big]^{\mathrm{T}}P\xi\\&\quad+2\gamma^{-2}\xi^{\mathrm{T}}P\triangle\bar{g}\triangle\bar{g}^{\mathrm{T}}P\xi+\xi^{\mathrm{T}}P\triangle\bar{f}+\triangle\bar{f}^{\mathrm{T}}P\xi+\xi^{\mathrm{T}}\bar{G}^{\mathrm{T}}\bar{G}\xi+\xi^{\mathrm{T}}Q\xi-\big(1-\dot{\tau}(t)\big)\xi_t^{\mathrm{T}}Q\xi_t\end{aligned}$$

$$
\begin{aligned}
&+\tau_2^2\xi^{\mathrm{T}}R\xi-\tau_2\int_{t-\tau_2}^{t}\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s+\big[\bar{D}_{0ij}\xi+\bar{D}_{1ij}\xi_t+\triangle\bar{h}\big]^{\mathrm{T}}P\big[\bar{D}_{0ls}\xi+\bar{D}_{1ls}\xi_t+\triangle\bar{h}\big]\Big\}\\
&\leqslant\frac{1}{2}\sum_{i=1}^{m}\alpha_i\sum_{j=1}^{m}\hat{\alpha}_j\sum_{k=1}^{m}\hat{\alpha}_k\Big\{\eta^{\mathrm{T}}(t)\Big[\Pi_{ijk}+4\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}+2\tilde{D}_{ij}^{\mathrm{T}}P\tilde{D}_{ij}\\
&+2\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}+2\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}\Big]\eta(t)\Big\}<0.\qquad\{\text{by (24) and }\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_jI\}
\end{aligned}\tag{41}
$$

From the previous analysis, if the subsequent inequalities

$$
\Pi_{ijk}+4\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}+2\tilde{D}_{ij}^{\mathrm{T}}P\tilde{D}_{ij}+2\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}+2\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}<0,\tag{42}
$$

hold for all $i,j,k=1,2,\ldots,m$, then both (40) and (41) hold. By the Schur complement [35], Eq. (42) is equivalent to (38) under $\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_jI$. It follows from matrix inequalities in (38) and $\bar{L}_j^{\mathrm{T}}P\bar{L}_j<\beta_jI$ that (1) and (2) of the theorem hold. By (42), the following inequality can be obtained:

$$
\begin{aligned}
&\Pi_{ijk}+\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}+2\tilde{D}_{ij}^{\mathrm{T}}P\tilde{D}_{ij}+\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}+\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}\\
&\quad<-2\gamma^{-2}\tilde{P}\bar{B}_{ij}\bar{B}_{ij}^{\mathrm{T}}\tilde{P}-\alpha\gamma^{-2}\tilde{P}\bar{L}_j\bar{L}_j^{\mathrm{T}}\tilde{P}-\alpha\gamma^{-2}\tilde{P}\bar{L}_k\bar{L}_k^{\mathrm{T}}\tilde{P}\leqslant0.
\end{aligned}\tag{43}
$$

It follows from (43) and the Schur complement that Eq. (27) is satisfied. Then, (3) of Theorem 2 holds from Theorem 1. The proof is complete accordingly.

By the previous discussion, the suboptimal mixed H_2/H_∞ filter design is obtained by the following minimization problem:

$$
\begin{aligned}
&\min_{\{P,Q,R,\beta_j,Y_j\}}\mathrm{E}\{\xi^{\mathrm{T}}(0)P\xi(0)\}+\mathrm{E}\left\{\int_{-\tau(0)}^{0}\xi^{\mathrm{T}}(s)Q\xi(s)\mathrm{d}s+\tau_2\int_{-\tau_2}^{0}\big(s+\tau_2\big)\xi^{\mathrm{T}}(s)R\xi(s)\mathrm{d}s\right\},\\
&\text{s.t.}\qquad(38)\text{ with }P=P^{\mathrm{T}}>0,\ Q=Q^{\mathrm{T}}>0,\ \text{and }R=R^{\mathrm{T}}>0.
\end{aligned}\tag{44}
$$

Similarly, with P,Q,R in (33), the matrix inequality constraints in (38) should be modified as (45), with (1,1), (1,2), and Γ_{ij} being defined in Remark 5.

$$
\begin{aligned}
&\Pi_2^{H_2/H_\infty}=\begin{bmatrix}(1,1)&(1,2)\\ *&(2,2)_{\Pi_2^{H_2/H_\infty}}\end{bmatrix}<0,\quad\forall\ i,\ j,\ k=1,2,\ldots,m,\\
&\Theta=\begin{bmatrix}-\beta_jI&0&P_1&P_2\\ *&-\beta_jI&Y_j^{\mathrm{T}}&Y_j^{\mathrm{T}}\\ *&*&-P_1&-P_2\\ *&*&*&-P_2\end{bmatrix}<0\quad\beta_j>0,\quad j=1,2,\ldots,m,
\end{aligned}\tag{45}
$$

where

$$
(2,2)_{\Pi_2^{H_2/H_\infty}}=\mathrm{diag}\left\{\begin{bmatrix}-\tau_2^2R_1&-\tau_2^2R_2\\ *&-\tau_2^2R_3\end{bmatrix},\begin{bmatrix}-\frac{P_1}{2}&-\frac{P_2}{2}\\ *&-\frac{P_2}{2}\end{bmatrix},\ -\frac{\gamma^2I}{4},\ -\frac{\gamma^2I}{2\alpha},\ -\frac{\gamma^2I}{2\alpha},\ -\frac{\gamma^2I}{2\alpha},\ -\frac{\gamma^2I}{2\alpha}\right\}.
$$

Remark 7. It is assumed that $\mathrm{E}\{\int_{-\tau(0)}^{0}\xi(s)\xi^{\mathrm{T}}(s)\mathrm{d}s\}=NN^{\mathrm{T}}$, $\mathrm{E}\{\int_{-\tau_2}^{0}\big(s+\tau_2\big)\xi(s)\xi^{\mathrm{T}}(s)\mathrm{d}s\}=LL^{\mathrm{T}}$ and $\mathrm{E}\{\xi(\theta)\xi^{\mathrm{T}}(\theta)\}=M=\mathrm{diag}\{M_{11},0\}$, $\theta\in[-\tau_2,0]$, if $\hat{x}(\theta)=0$ is always assumed. Then, the suboptimal mixed H_2/H_∞ globally linearized filtering design problem in (44) is formulated as follows:

$$
\begin{aligned}
&\min_{\{P,Q,R,\beta_j,Y_j\}}\mathrm{tr}\{MP\}+\mathrm{tr}\{N^{\mathrm{T}}QN\}+\mathrm{tr}\{L^{\mathrm{T}}\big(\tau_2R\big)L\},\\
&\text{s.t.}\qquad(45)\text{ with }P=P^{\mathrm{T}}>0,\ Q=Q^{\mathrm{T}}>0,\ \text{and }R=R^{\mathrm{T}}>0.
\end{aligned}\tag{46}
$$

Remark 8. By the Schur complement, $\Pi_2^{H_2/H_\infty}<0$ implies $\Pi_2^{H_\infty}<0$. Therefore, the sufficient condition (34) in the mixed H_2/H_∞ case has been removed.

A design procedure for the suboptimal mixed H_2/H_∞ globally linearized filter is presented in Table 2.

Table 2 Procedure for suboptimal mixed H_2/H_∞ global linearization filtering

Steps	Design procedures
Step 1:	Establish the globally linearized model (19);
Step 2:	Choose an attenuation level γ and a G;
Step 3:	Obtain the five positive constants e_1, e_2, e_3, e_4, and α in (24) along the entire trajectory of the nonlinear system (1) that comes from the simulation of the system (1);
Step 4:	Calculate $NN^{\mathrm{T}}, LL^{\mathrm{T}}$, and M in Remark 7 as the expectations of the initial state in $\mathcal{L}_2$ sense and mean-square sense, respectively;
Step 5:	Solve the LP in (46) to get $P_1, P_2, Q_1, Q_2, Q_3, R_1, R_2, R_3, \beta_j$, and Y_j, then $L_j = P_2^{-1} Y_j$;
Step 6:	Establish the globally linearized filter in (21).

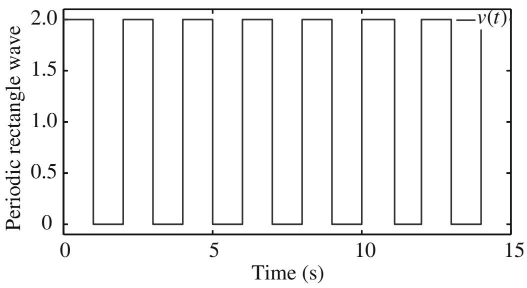

Figure 1 Exogenous disturbance $v(t)$.

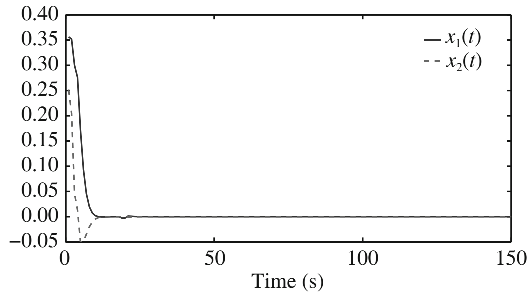

Figure 2 Trajectories of the states with $v(t) = 0$.

5 Simulation example

Consider the following Itô-type general nonlinear stochastic time-varying delay system:

$$\begin{cases} \mathrm{d}x_1(t) = \left[-4x_1(t) - 2x_1^3(t) + 5x_2(t) + 0.1x_1(t-\tau(t)) - 0.5\cos(x_1(t))v(t)\right]\mathrm{d}t \\ \qquad\qquad + \left[0.1x_1^2(t) + 0.1x_1(t-\tau(t))\right]\mathrm{d}W(t), \\ \mathrm{d}x_2(t) = \left[-4x_1(t) - 10x_2(t) + 0.1x_2(t-\tau(t)) - \sin(x_1(t))v(t)\right]\mathrm{d}t \\ \qquad\qquad + \left[0.1x_2(t) + 0.1x_2(t-\tau(t))\right]\mathrm{d}W(t), \\ \mathrm{d}y(t) = \left[9x_1(t) - x_1^2(t) - 9x_2(t) + 0.1x_1(t-\tau(t)) + 0.1x_2(t-\tau(t)) + 12x_1(t)v(t)\right]\mathrm{d}t \\ \qquad\qquad + \left[0.1x_1^2(t) + 0.1x_2(t) + 0.1x_1(t-\tau(t)) + 0.1x_2(t-\tau(t))\right]\mathrm{d}W(t), \\ s(t) = x(t), \\ x(\theta) = \left[x_1(\theta), x_2(\theta)\right]^{\mathrm{T}}, \ \theta \in [-\tau_2, 0], \end{cases} \tag{47}$$

where $x^{\mathrm{T}}(t) = \left[x_1(t), x_2(t)\right]$. The external disturbance $v(t)$ is described by the periodic rectangle wave with amplitude 2 in Figure 1, and the trajectories of the states are displayed in Figure 2 when $v(t) = 0$. $\tau(t) = \exp(-(1+t)^{-1})$, then $\tau_2 = 1$, $\bar{\tau} = 0.3679$.

(1) Design 1: optimal H_∞ globally linearized filter design.

Step (a). Establish the three vertices of the globally linearized model at $x_1(t) = -0.5, -0.1$, and 1.5 as in (19) with $A_{0i}, A_{1i}, C_{0i}, C_{1i}, Q_{0i}, Q_{1i}, J_{0i}, J_{1i}, i = 1, 2, 3$ are shown in Appendix A.

Step (b). Choose the matrix $G = I$.

Step (c). Obtain the five positive bounding constants in (24) with the Simulink in Matlab as follows: $e_1 = 0.0411, e_2 = 0.0066, e_3 = 0.0914, e_4 = 0.0066$, and $\alpha = 0.4751$.

Step (d). Solve the LP in (35) with the LMI optimization toolbox in Matlab. From Table 1, one can obtain $\gamma^2_{\min} = 7.8745$, with $P_1, P_2, Q_1, Q_2, Q_3, R_1, R_2, R_3, \beta_j$, and Y_j being displayed in Appendix B.

Then, the H_∞ globally linearized filter gains are proposed as follows:

$$L_1 = \begin{bmatrix} -0.1278 \\ 0.1871 \end{bmatrix}, \ L_2 = \begin{bmatrix} -0.0758 \\ 0.0288 \end{bmatrix}, \ L_3 = \begin{bmatrix} -0.0704 \\ 0.0477 \end{bmatrix}.$$

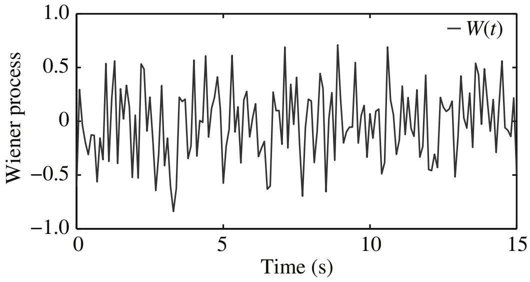

Figure 3 Wiener process $W(t)$ in Design 1.

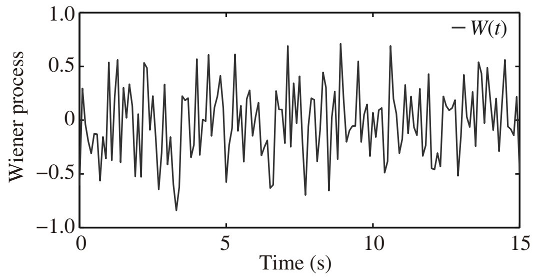

Figure 4 Wiener process $W(t)$ in Design 2.

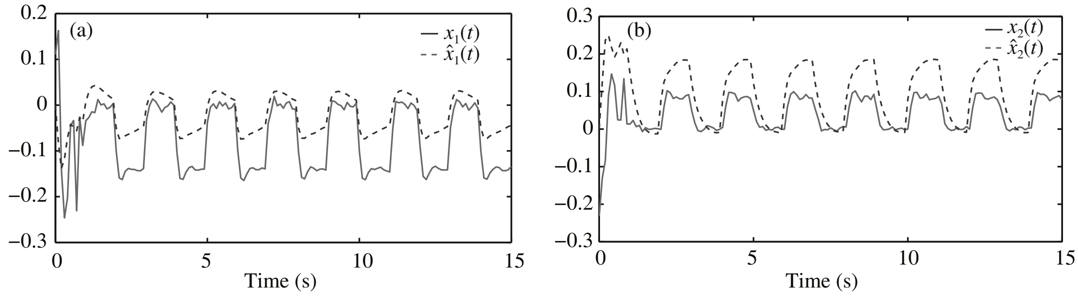

Figure 5 Trajectories for the proposed optimal H_∞ global linearization filter. (a) Trajectories for x_1 and $\hat{x}_1$; (b) trajectories for x_2 and $\hat{x}_2$.

Step (e). Establish the globally linearized filter as in (21) with the subsequent interpolation functions

$$\alpha_1 = \frac{\frac{1}{(\hat{x}_1+0.5)^2}}{\frac{1}{(\hat{x}_1+0.5)^2}+\frac{1}{(\hat{x}_1+0.1)^2}+\frac{1}{(\hat{x}_1-1.5)^2}},\ \alpha_2 = \frac{\frac{1}{(\hat{x}_1+0.1)^2}}{\frac{1}{(\hat{x}_1+0.5)^2}+\frac{1}{(\hat{x}_1+0.1)^2}+\frac{1}{(\hat{x}_1-1.5)^2}},\ \alpha_3 = \frac{\frac{1}{(\hat{x}_1-1.5)^2}}{\frac{1}{(\hat{x}_1+0.5)^2}+\frac{1}{(\hat{x}_1+0.1)^2}+\frac{1}{(\hat{x}_1-1.5)^2}},$$

for $x_1 = -0.5$, $x_1 = -0.1$, $x_1 = 1.5$, respectively.

Let $W(t)$ be a standard one-dimensional Wiener process with $\mathrm{E}\{W(t)\} = 0$ and $\mathrm{E}\{W^2(t)\} = t$. The corresponding Wiener processes are shown in Figures 3 and 4. Figure 5 shows the trajectories of $x_1(t), x_2(t), \hat{x}_1(t)$, and $\hat{x}_2(t)$, respectively, by the proposed H_∞ globally linearized filter. It follows from Figures 1 and 5 that the estimation error mainly results from the sharp change of exogenous disturbance before finishing estimating the upward (downward) signal. In other words, the filter is forced to estimate the downward (upward) signal before finishing estimating the upward (downward) signal so that the peak and valley of the signal cannot be estimated accurately.

Assume that $[x_1(\theta), x_2(\theta), \hat{x}_1(\theta), \hat{x}_2(\theta)]^{\mathrm{T}} = [x_1(\theta), x_2(\theta), 0, 0]^{\mathrm{T}}$, where $x_1(\theta), x_2(\theta)$ are random initial values with $\mathrm{E}\{[x_1(\theta), x_2(\theta)]\} = 0$, $\mathrm{E}\{[x_1(\theta), x_2(\theta)]^{\mathrm{T}}[x_1(\theta), x_2(\theta)]\} = I$. Thus, $M = \mathrm{diag}\,\{I, 0\}$, and then, in Remark 7, $N = \mathrm{diag}\left\{\frac{I}{\sqrt{\exp(1)}}, 0\right\}$, $L = \mathrm{diag}\left\{\frac{\tau_2 I}{\sqrt{2}}, 0\right\}$.

(2) Design 2: suboptimal mixed H_2/H_∞ globally linearized filter design. Solve the LP in (46) with the LMI optimization toolbox in Matlab. In this case, for a prescribed $\gamma^2 = (4.16)^2$, by Table 2, one can obtain $P_1, P_2, Q_1, Q_2, Q_3, R_1, R_2, R_3, \beta_j$, and Y_j, which are shown in Appendix C, the optimal H_2 cost $J_2^* = 48.0289$. Then, the H_2/H_∞ globally linearized filter gains are presented as follows:

$$L_1 = \begin{bmatrix} -0.0955 \\ 0.1206 \end{bmatrix},\ L_2 = \begin{bmatrix} -0.0790 \\ 0.1589 \end{bmatrix},\ L_3 = \begin{bmatrix} -0.0418 \\ 0.0548 \end{bmatrix}.$$

By the proposed suboptimal mixed H_2/H_∞ globally linearized filter, the trajectories of $x_1(t), x_2(t), \hat{x}_1(t)$, and $\hat{x}_2(t)$ are displayed in Figure 6, respectively. One can obtain the average estimation errors by the proposed filters as in Table 3. By the extra consideration of the H_2 suboptimal filtering in (44), it follows from Figures 7–10 that the mixed H_2/H_∞ filter substantially improves the estimation performance compared with the H_∞ filter.

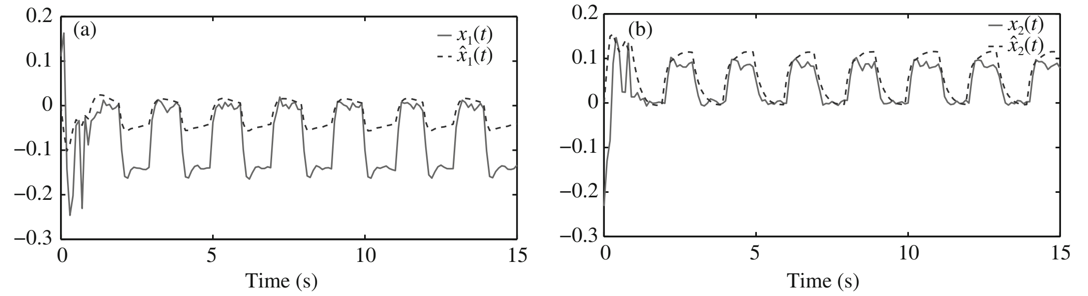

Figure 6 Trajectories for the proposed optimal H_2/H_∞ global linearization filter. (a) Trajectories for x_1 and $\hat{x}_1$; (b) trajectories for x_2 and $\hat{x}_2$.

Table 3 Performance of the proposed filters

Filters	$\mathrm{E}\{\int_0^{15} \|e(t)\|^2 \mathrm{d}t\}$
The proposed optimal H_∞ global linearization filter	0.1668
The proposed suboptimal mixed H_2/H_∞ global linearization filter	0.1073

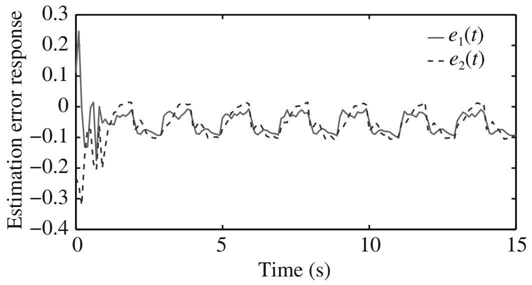

Figure 7 Estimation error response for H_∞ filter.

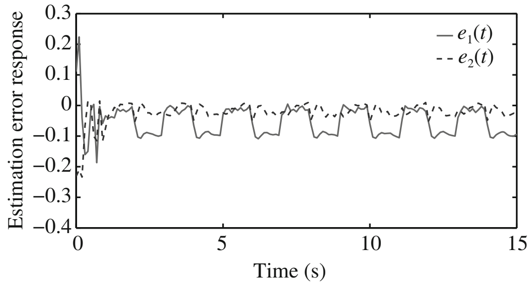

Figure 8 Estimation error response for H_2/H_∞ filter.

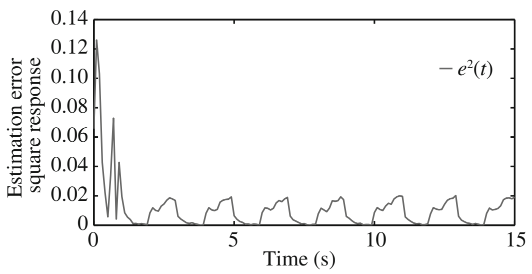

Figure 9 Estimation error response in mean-square sense for H_∞ global linearization filter.

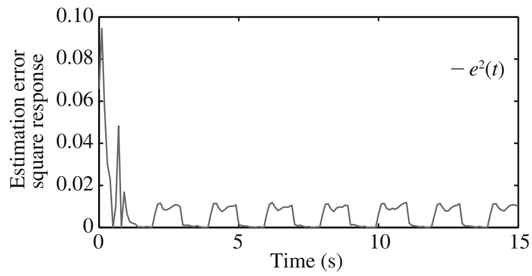

Figure 10 Estimation error response in mean-square sense for suboptimal H_2/H_∞ global linearization filter.

6 Conclusion

In this paper, based on a globally linearized model, a stochastic BRL is established to design the H_∞ globally linearized filter for a nonlinear stochastic time-varying delay system by solving simultaneous LMIs instead of a second-order HJI. The mixed H_2/H_∞ globally linearized filter design problem is formulated as a LP with a desired attenuation level when the worst case disturbance is considered. A simulation example is presented to demonstrate the proposed method. This method is applicable for state estimation in filtering problem and state-estimator-based control designs in nonlinear stochastic time-varying delay systems when state variables are unavailable. Undoubtedly, how to solve the HJI is a very valuable research topic and deserves further study.

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A Coefficient matrices of the linearized systems in Example 1.

$$A_{01} = \begin{bmatrix} -5.5 & 5 \\ -4.0 & -10 \end{bmatrix},\ A_{02} = \begin{bmatrix} -4.06 & 5 \\ -4.00 & -10 \end{bmatrix},\ A_{03} = \begin{bmatrix} -17.5 & 5 \\ -4.0 & -10 \end{bmatrix},\ C_{01} = \begin{bmatrix} -0.1 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},$$
$$C_{02} = \begin{bmatrix} -0.02 & 0.0 \\ 0.00 & 0.1 \end{bmatrix},\ C_{03} = \begin{bmatrix} 0.3 & 0.0 \\ 0.0 & 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} -0.4388 \\ 0.4794 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.4975 \\ 0.0998 \end{bmatrix},\ B_3 = \begin{bmatrix} -0.0353 \\ -0.9975 \end{bmatrix},$$
$$Q_{01} = \begin{bmatrix} 10 & -9 \end{bmatrix},\ Q_{02} = \begin{bmatrix} 9.2 & -9 \end{bmatrix},\ Q_{03} = \begin{bmatrix} 6 & -9 \end{bmatrix},\ J_{01} = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix},\ J_{02} = \begin{bmatrix} -0.02 & 0.1 \end{bmatrix},$$
$$J_{03} = \begin{bmatrix} 0.3 & 0.1 \end{bmatrix},\ A_{1i} = C_{1i} = 0.1I,\ J_{1i} = Q_{1i} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, i = 1,2,3.\ K_1 = -6,\ K_2 = -1.2,\ K_3 = 18.$$

Appendix B The solutions for the Design 1.

$$P_1 = \begin{bmatrix} 98.8078 & -2.0187 \\ -2.0187 & 58.8091 \end{bmatrix},\ P_2 = \begin{bmatrix} 90.4942 & 0.9774 \\ 0.9774 & 57.3793 \end{bmatrix},\ Q_1 = \begin{bmatrix} 202.8815 & -126.6186 \\ -126.6186 & 405.5264 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 134.3078 & -67.1257 \\ -31.3462 & 318.8367 \end{bmatrix},\ Q_3 = \begin{bmatrix} 109.2999 & -3.9626 \\ -3.9626 & 267.6041 \end{bmatrix},\ R_1 = \begin{bmatrix} 166.0323 & -78.6251 \\ -78.6251 & 275.3162 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 115.8016 & -32.5824 \\ -14.7431 & 218.8133 \end{bmatrix},\ R_3 = \begin{bmatrix} 96.0875 & 6.0596 \\ 6.0596 & 187.9714 \end{bmatrix},\ Y_1 = \begin{bmatrix} -11.3778 \\ 10.6131 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -6.8297 \\ 1.5799 \end{bmatrix},\ Y_3 = \begin{bmatrix} -6.3267 \\ 2.6685 \end{bmatrix},\ \beta_1 = 133.5388,\ \beta_2 = 302.5023,\ \beta_3 = 305.4660.$$

Appendix C The solutions for the Design 2.

$$P_1 = \begin{bmatrix} 36.4424 & 0.9653 \\ 0.9653 & 26.4368 \end{bmatrix},\ P_2 = \begin{bmatrix} 33.7293 & 2.0105 \\ 2.0105 & 26.0207 \end{bmatrix},\ Q_1 = \begin{bmatrix} 10.4481 & -3.8656 \\ -3.8656 & 16.1123 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 6.6629 & -1.2627 \\ -0.4789 & 12.0140 \end{bmatrix},\ Q_3 = \begin{bmatrix} 6.6007 & 0.5101 \\ 0.5101 & 10.9685 \end{bmatrix},\ R_1 = \begin{bmatrix} 6.2920 & -2.7815 \\ -2.7815 & 14.6409 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 4.6083 & -0.9929 \\ -0.3309 & 12.4857 \end{bmatrix},\ R_3 = \begin{bmatrix} 3.7606 & 0.6977 \\ 0.6977 & 11.1249 \end{bmatrix},\ Y_1 = \begin{bmatrix} -2.9799 \\ 2.9471 \end{bmatrix},$$
$$Y_2 = \begin{bmatrix} -2.3460 \\ 3.9747 \end{bmatrix},\ Y_3 = \begin{bmatrix} -1.2982 \\ 1.3412 \end{bmatrix},\ \beta_1 = 37.1182,\ \beta_2 = 40.1896,\ \beta_3 = 40.1227.$$