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# Global practical tracking via adaptive output-feedback for uncertain nonlinear systems with generalized control coefficients

Shaoli JIN & Yungang LIU*

*School of Control Science and Engineering, Shandong University, Jinan* 250061*, China*

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**Abstract** This paper investigates the global practical tracking via adaptive output-feedback for a class of uncertain nonlinear systems with generalized control coefficients. Notably, the system in question has the function-of-output control coefficients and the serious unknowns in the system and the reference signal, and hence is essentially different from the existing closely related literature. To solve the global practical tracking, a high-gain observer is first introduced to reconstruct the unmeasurable system states, and then an adaptive output-feedback controller is designed. It is worth emphasizing that the gains in the designed observer and controller are functions of time and output, for which a novel updating law of the high-gain is introduced to overcome the additional system nonlinearities and the serious unknowns mentioned above. The designed controller is shown such that all the states of the closed-loop system are globally bounded, and furthermore, tracking error will be ultimately prescribed sufficiently small. A numerical simulation is provided to demonstrate the effectiveness of the proposed approach.

**Keywords** uncertain nonlinear systems, function-of-output control coefficients, global practical tracking, dynamical high-gain, adaptive output-feedback

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## 1 Introduction and problem formulation

In this paper, we consider the global practical tracking for the following uncertain nonlinear system with generalized control coefficients[1)]:

$$\begin{cases} \dot{\eta}_i = g_i(y)\eta_{i+1} + \psi_i(t,\eta), \quad i = 1,\ldots,n-1, \\ \dot{\eta}_n = g_n(y)u + \psi_n(t,\eta), \\ y = \eta_1 - y_r, \end{cases} \tag{1}$$

* Corresponding author (email: lygfr@sdu.edu.cn)

1) Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ denotes the set of all non-negative numbers, $\mathbb{R}^n$ denotes the real $n$-dimensional space. For any given vector or matrix $X$, $X^{\mathrm{T}}$ denotes its transpose, $\|X\|$ denotes the Euclidean norm for vectors, and the corresponding induced norm for matrices, respectively. For any symmetric matrix $P$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote its maximum and minimum eigenvalues, respectively. We use diag$[a_1,\ldots,a_n]$ denote the $n\times n$ diagonal matrix with $a_i$'s on its diagonal.

 

where $\eta = [\eta_1, \ldots, \eta_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state vector with the initial condition $\eta_0 = \eta(0)$; $u \in \mathbb{R}$, $y \in \mathbb{R}$ and $y_r(t)$ are the control input, system output (tracking error) and reference signal, respectively; $\psi_i(t, \eta)$'s are unknown functions but continuous in the first argument and locally Lipschitz in the second one, while $g_i(y)$'s are known and locally Lipschitz functions. System (1) is said to have generalized control coefficients since $g_i(y)$'s are functions of output, rather than constants.

In what follows, we suppose that only system output $y$ is available for feedback, which means that system state $\eta_1$ and $y_r$ are not necessarily known. Furthermore, we make three following assumptions on system (1) and the reference signal $y_r$:

**Assumption 1.** There exist known positive constants $\underline{a}$, $\overline{a}$, $\underline{g}$, $\overline{g}$ and $p$, such that for $\forall\, y \in \mathbb{R}$,

$$0 < \underline{a} \leqslant |g_i(y)| \leqslant \overline{a}(1 + |y|^p), \quad i = 1, \ldots, n,$$
$$\underline{g} \leqslant |g_i(y)/g_{i-1}(y)| \leqslant \overline{g}, \quad i = 2, \ldots, n-1.$$

**Assumption 2.** There exists an unknown positive constant $\theta_s$, such that for $\forall\, t \in \mathbb{R}^+$, $\forall\, \eta \in \mathbb{R}^n$,

$$|\psi_i(t, \eta)| \leqslant \theta_s + \theta_s \sum_{j=1}^{i} |\eta_i|, \quad i = 1, \ldots, n.$$

**Assumption 3.** The reference signal $y_r : \mathbb{R}^+ \to \mathbb{R}$ to be tracked is continuously differentiable, and moreover, there is an unknown positive constant $\theta_r$ such that

$$\sup_{t \geqslant 0} (|y_r(t)| + |\dot{y}_r(t)|) \leqslant \theta_r.$$

Assumption 1 shows that the control coefficients in system (1) are nonlinear functions of output, essentially different from those in the closely related work [1–5] where all the control coefficients are known constants (equal to 1), and hence system (1) allows additional system nonlinearities, compared with the above works. Assumption 2 indicates that the growth of system (1) heavily relies on the unmeasurable states, and has unknown growth rate. Therefore, system (1) possesses serious signal and parameter unknowns, for which some estimation and compensation strategies are usually needed to counteract the caused negative effects. Assumption 3 shows that to practical tracking, merely rather coarse information is required on $y_r(t)$: it is enough that both $y_r(t)$ and its derivative belong to an unknown constant interval, as those in [1–3, 6–12] and unlike those in [4, 5, 13–18].

The objective of global practical tracking in this paper is to construct the following adaptive output-feedback controller for system (1) under Assumptions 1–3:

$$\dot{\chi} = \phi_\lambda(\chi, y), \quad u = \varphi_\lambda(\chi, y), \tag{2}$$

where $\chi$ is the state vector with an appropriate dimension and the initial value $\chi_0 = \chi(0)$; $\lambda$ is a pre-given positive constant used to represent the tracking accuracy/level; and $\phi_\lambda(\cdot)$ and $\varphi_\lambda(\cdot)$ are vector-valued and scalar locally Lipschitz functions, respectively, such that

(i) the solutions of the resulting closed-loop system are well-defined and globally bounded on $[0, +\infty)$;

(ii) for any initial condition and any prescribed constant $\lambda > 0$, there is a finite time $T_\lambda > 0$ such that $\sup_{t \geqslant T_\lambda} |y(t)| = \sup_{t \geqslant T_\lambda} |\eta_1(t) - y_r(t)| \leqslant \lambda$. In this sense, $\lambda$ is called tracking accuracy/level.

Asymptotic tracking requires the tracking error to ultimately converge to zero, rather than enter a interval, which obviously is better than practical tracking. Therefore, the accomplishment of asymptotic tracking needs more rigorous restrictions and more information on the system and the reference signal [15–18]. Although practical tracking has a relatively conservative control objective, it needs weaker restrictions and less related information, and particularly it is enough for many practical problems. Mainly because of this, practical tracking has received much attention over the past decades [1–14], and is still an active field of research. Specifically, work [6–9, 13] considered the practical tracking via state-feedback. When only partial states or output is available for feedback, representative results were obtained in [1–5, 10–12, 14]. In particular, work [1–3] considered the cases with the serious unknowns in the system nonlinearities and the reference signal, but with the rather strong assumption that the control coefficients should be precisely known constants (equal to 1). To the best knowledge of the authors, when the control coefficients are generalized to be functions of output, the global stabilization has been investigated (see, e.g., [19, 20]), but the practical tracking via output-feedback has not been studied and solved for the nonlinear systems with such generalized control coefficients, for example, system (1).

The paper extends the work [1], and considers the global practical tracking via output-feedback for a class of uncertain nonlinear systems with generalized control coefficients. The remarkable feature of the system in question is the function-of-output control coefficients, rather than the constant ones as in [1–5, 11, 12]. Considering the dead zone in [1, 2, 7, 10, 11], a novel updating law of high-gain is introduced to overcome the additional nonlinearities caused by the function-of-output control coefficients and the serious unknowns in system (1) and the reference signal $y_r$, and moreover to ultimately establish the described practical tracking. Based on [20], a high-gain full-order observer is designed to reconstruct the unmeasurable system states, where the gains involved are functions of output and time, and hence are substantially different from the closely related work [1–3, 5]. Then, in terms of the universal controls in [1, 2, 21] and [20], an adaptive output-feedback controller is constructed to establish the global practical tracking of the closed-loop system.

We would like to compare and contrast between this paper and the closely related work [1–5, 10–12, 14], to show the main contributions of this paper. (i) The control coefficients of system (1) are functions of output, rather than constants [1–5, 11, 12], and hence give rise to additional nonlinearities to the system. Moreover, the growth of system (1) heavily relies on the unmeasurable states, and has unknown growth rate. This is unlike the case of known growth rate in [4, 5], and unlike the work [10–12, 14] where the system growth only relies on the measurable output. (ii) The information on the reference signal to be tracked is rather coarse: it is unknown, but itself and its derivative are required to belong to an unknown constant interval. This is unlike the case of known constant interval in [4, 5], and unlike the work [14] where the higher order derivatives of the reference signal are required. (iii) The gains in the observer and the controller to be designed should be chosen as functions of output and time, in order to simultaneously overcome the additional nonlinearities (originated from the function-of-output control coefficients), and the serious unknowns in system (1) and the reference signal. This is essentially different from the works [1–3] where the gains are functions of time. (iv) The establishment, analysis and proof of the main result are motivated by [3] and different from [1, 2], and for the need, a key design parameter (i.e., "$b$" in (8) later) is introduced in the updating law (of high-gain) to be designed. This, we believe, would inspire us study and solve the practical tracking via output-feedback for other uncertain nonlinear systems.

The remainder of the paper is organized as follows. Section 2 provides the control design scheme for the global practical tracking and summarizes the main result obtained in this paper. Section 3 collects the detailed proofs of a technical proposition and an important lemma. Section 4 gives a simulation example, and Section 5 addresses some concluding remarks.

## 2 Global practical tracking via adaptive output-feedback

In this section, an adaptive output-feedback controller is designed to achieve the global practical tracking for system (1) under Assumptions 1–3. Since no system states are measurable and the serious unknowns exist, an observer and a dynamical compensator should be delicately sought, and of course are necessary parts of the desirable controller.

We introduce the new coordinates

$$x_1 = y = \eta_1 - y_r, \quad x_i = \eta_i, \quad i = 2, \ldots, n, \tag{3}$$

and let $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ for later use.

Then by (1), we have

$$\begin{cases} \dot{x}_1 = g_1(y)x_2 + f_1(t, x, y_r, \dot{y}_r), \\ \dot{x}_i = g_i(y)x_{i+1} + f_i(t, x, y_r), \quad i = 2, \dots, n-1, \\ \dot{x}_n = g_n(y)u + f_n(t, x, y_r), \\ y = x_1, \end{cases} \tag{4}$$

where $f_1 = \psi_1(t, x_1 + y_r, x_2, \dots, x_n) - \dot{y}_r$, $f_i = \psi_i(t, x_1 + y_r, x_2, \dots, x_n)$, $i = 2, \dots, n$.

Notably, under (3), the practical tracking for the original system (1) is transformed into the stabilization for the uncertain time-varying nonlinear system (4): search for one controller $u$ in the form (2) such that all the states $x$ of system (4) are globally bounded on $[0, +\infty)$ and furthermore, its output $y$ keeps in $[-\lambda, \lambda]$ after a finite time. Therefore, it establishes the above described stabilization of system (4), as the subsequent sections done.

In view of Assumption 1 and Remark 3 of [20], we can choose known locally Lipschitz functions $h_i(y)$, $i = 1, \dots, n$ be linear constant-coefficient combinations of $g_i(y)$, $i = 1, \dots, n-1$, known locally Lipschitz functions $k_i(y)$, $i = 1, \dots, n$, known symmetric positive-definite matrices $P$ and $Q$, and known positive constants $\nu_o$ and $\nu_c$, such that for $\forall\, y \in \mathbb{R}$,

$$\begin{cases} PA(y) + A^{\mathrm{T}}(y)P \leqslant -\nu_o |g_1(y)| I_n, \\ QB(y) + B^{\mathrm{T}}(y)Q \leqslant -\nu_c |g_1(y)| I_n, \end{cases} \tag{5}$$

where

$$A(y) = \begin{bmatrix} -h_1(y) & & & \\ \vdots & & \mathrm{diag}[g_1, \dots, g_{n-1}] & \\ -h_n(y) & 0 & \cdots & 0 \end{bmatrix}, \quad B(y) = \begin{bmatrix} 0 & & & \\ \vdots & & \mathrm{diag}[g_1, \dots, g_{n-1}] & \\ 0 & & & \\ -k_1(y) & -k_2(y) & \cdots & -k_n(y) \end{bmatrix},$$

The above choice of $h_i$'s and Assumption 1 make a known positive constant $\bar{h}$ exist such that

$$\|H(y)\| = \sqrt{\sum_{i=1}^{n} h_i^2(y)} \leqslant \bar{h}|g_1(y)|, \tag{6}$$

where $H(y) = [h_1(y), \dots, h_n(y)]^{\mathrm{T}}$.

Since $P$ and $Q$ are symmetric positive-definite matrices, it is not hard to show that there are positive constants $\overline{\nu}_o$, $\underline{\nu}_o$, $\overline{\nu}_c$ and $\underline{\nu}_c$, such that

$$\begin{cases} \underline{\nu}_o I \leqslant PD + DP \leqslant \overline{\nu}_o I, \\ \underline{\nu}_c I \leqslant QD + DQ \leqslant \overline{\nu}_c I, \end{cases} \tag{7}$$

where $D = \mathrm{diag}[b, b+1, \dots, b+n-1]$ with $b$ a small design parameter satisfying $0 < b < \frac{1}{2}$.

For any pre-given $\lambda > 0$ which denotes the tracking level, we construct the following adaptive output-feedback controller for system (4):

$$\begin{cases} u = -\dfrac{1}{g_n(y)} \displaystyle\sum_{i=1}^{n} L^{n-i+1}(t) k_i(y) \hat{x}_i, \\ \begin{cases} \dot{\hat{x}}_i = g_i(y)\hat{x}_{i+1} + L^i(t) h_i(y)(y - \hat{x}_1), \quad i = 1, \dots, n-1, \\ \dot{\hat{x}}_n = g_n(y)u + L^n(t) h_n(y)(y - \hat{x}_1), \\ \dot{L} = \max\left\{ \dfrac{1}{L^{2b}} \left( \rho(\varepsilon_1, z_1)\big((y - \hat{x}_1)^2 + \hat{x}_1^2\big) - \dfrac{\lambda^2}{2} \right), 0 \right\}, \quad L(0) = 1, \\ \varepsilon = [\varepsilon_1, \dots, \varepsilon_n]^{\mathrm{T}} =: \left[ \dfrac{x_1 - \hat{x}_1}{L^b}, \dots, \dfrac{x_n - \hat{x}_n}{L^{b+n-1}} \right]^{\mathrm{T}}, \\ z = [z_1, \dots, z_n]^{\mathrm{T}} =: \left[ \dfrac{\hat{x}_1}{L^b}, \dots, \dfrac{\hat{x}_n}{L^{b+n-1}} \right]^{\mathrm{T}}, \end{cases} \end{cases} \tag{8}$$

where $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^{\mathrm{T}}$ with the initial value $\hat{x}(0) = \hat{x}_0$, $\rho = 1 + (1 + (\varepsilon_1 + z_1)^2)^{\frac{p}{2}} = 1 + \frac{(L^{2b}+y^2)^{\frac{p}{2}}}{L^{bp}}$, $p$ is the same as in Assumption 1, and as stated above, $b$ is small such that $0 < b < \frac{1}{2}$.

Thus, from (4) and (8), we derive the dynamics of $\varepsilon$ and $z$ as follows.

$$\begin{cases} \dot{\varepsilon} = L(t)A(y)\varepsilon + F(t, x, y_r, \dot{y}_r, L) - \dfrac{\dot{L}(t)}{L(t)}D\varepsilon, \\ \dot{z} = L(t)B(y)z + L(t)H(y)\varepsilon_1 - \dfrac{\dot{L}(t)}{L(t)}Dz, \end{cases} \tag{9}$$

where $F = [\frac{f_1}{L^b}, \frac{f_2}{L^{b+1}}, \ldots, \frac{f_n}{L^{b+n-1}}]^{\mathrm{T}}$.

For the closed-loop system resulting from (1) and (8), we have the following two propositions.

**Proposition 1.** The gain $L$ determined by (8) is monotone nondecreasing and $L(t) \geqslant 1$, and its dynamics are locally Lipschitz in $(y, \hat{x}_1, L)$.

*Proof.* From (8), we see that $\dot{L}(t) \geqslant 0$, and therefore, $L$ is monotone nondecreasing and $L(t) \geqslant L(0) = 1$. Then, noting that "$\max\{\omega, 0\}$" is locally Lipschitz in $\omega$ (see the proof of Proposition 1 in [1]), and $\frac{1}{L^{2b}}(\rho(\varepsilon_1, z_1)((y-\hat{x}_1)^2 + \hat{x}_1^2) - \frac{\lambda^2}{2})$ is smooth in $(y, \hat{x}_1, L)$, we know that $\dot{L}$ is locally Lipschitz in $(y, \hat{x}_1, L)$.

By (4) (or (1)), (8) and Proposition 1, we can see that the right-hand sides of the ordinary differential equations describing the resulting closed-loop system are locally Lipschitz with respect to $(x, \hat{x}, L)$ (or to $(\eta, \hat{x}, L)$) in an open neighborhood of the initial condition, and hence the closed-loop system has a unique solution on a small interval $[0, t_f)$ (see Theorem 3.1, page 18 of [22]). Let $[0, T_f)$ be the maximal interval on which a unique solution exists, where $0 < T_f \leqslant +\infty$ (see Theorem 2.1, page 17 of [22]). If one can prove $T_f = +\infty$, then all the closed-loop system states would be well-defined on $[0, +\infty)$.

**Proposition 2.** For (9), define $V(\varepsilon, z) = V_1(\varepsilon) + \gamma V_2(z) := \varepsilon^{\mathrm{T}}P\varepsilon + \gamma z^{\mathrm{T}}Qz$ with $\gamma = \frac{\nu_c\nu_o}{4\bar{h}^2\|Q\|^2}$. Then there exists an unknown positive constant $\Theta$, such that on $[0, T_f)$,

$$\dot{V} \leqslant -\frac{\underline{a}\min\{\nu_o, \nu_c\}}{2}(L - \Theta)(\|\varepsilon\|^2 + \gamma\|z\|^2) + \Theta.$$

*Proof.* See Subsection 3.1 later.

Noting that the above defined $V(\varepsilon, z)$ also satisfies

$$\alpha_1(\|\varepsilon\|^2 + \gamma\|z\|^2) \leqslant V(\varepsilon, z) \leqslant \alpha_2(\|\varepsilon\|^2 + \gamma\|z\|^2), \tag{10}$$

with $\alpha_1 = \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$ and $\alpha_2 = \max\{\lambda_{\max}(P), \lambda_{\max}(Q)\}$, by Proposition 2 we conclude that if $L(t) \geqslant \Theta$,

$$\dot{V} \leqslant -\beta(L - \Theta)V + \Theta, \tag{11}$$

where $\beta = \frac{\underline{a}\min\{\nu_o,\nu_c\}}{2\alpha_2}$.

The following lemma shows that the closed-loop system signals $L(t)$ and $(z(t), \varepsilon(t))$ do not blow-up (escape in finite time) and is key to establish the desired practical tracking.

**Lemma 1.** For the closed-loop system consisting of (4) and (8), the gain $L(t)$ and system states $(z(t), \varepsilon(t))$ are globally bounded on $[0, T_f)$.

*Proof.* See Subsection 3.2 later.

We are now in a position to summarize the main result into the following theorem.

**Theorem 1.** For system (1) under Assumptions 1–3, the dynamical output-feedback controller (8) guarantees that, for any initial condition, all the signals (system states and control input) of the closed-loop system consisting of (1) and (8), are well-defined and bounded on $[0, +\infty)$, and furthermore, the global practical tracking can be achieved; that is, for any pre-given tracking level $\lambda > 0$, there exists a finite time $T_\lambda$ such that $|y(t)| \leqslant \lambda$, $\forall\, t \geqslant T_\lambda$.

*Proof.* By (8) and Lemma 1, we know that all the signals of the closed-loop system are bounded on $[0, T_f)$, and therefore $T_f = +\infty$.

By (4), (8) and (9), and the boundedness of $\varepsilon(t)$, $z(t)$, $x(t)$, $\hat{x}(t)$ and $L(t)$ on $[0,+\infty)$, we can see that $\dot{\varepsilon}_1(t)$, $\dot{z}_1(t)$, $\dot{y}(t)$, $\dot{\hat{x}}_1(t)$ and $\dot{L}(t)$ are all bounded on $[0,+\infty)$, and hence $\dot{L}$ can be proved to be uniformly continuous on $[0,+\infty)$ (see the proof of Theorem 1 in [1]). Then, by Barbălat Lemma [1, 23], we have $\lim_{t\to\infty}\dot{L}(t)=0$. This and the boundedness of $L(t)$ on $[0,+\infty)$ means that, for any initial condition $(\eta(0),\hat{x}(0))$, a finite time $T_\lambda>0$ exists such that $\frac{1}{L^{2b}(t)}(\rho(\varepsilon_1(t),z_1(t))\big((y(t)-\hat{x}_1(t))^2+\hat{x}_1^2(t)\big)-\frac{\lambda^2}{2})\leqslant\dot{L}(t)\leqslant\frac{\lambda^2}{2L^{2b}(t)}$ for $\forall\, t>T_\lambda$, which together with $L(t)\geqslant 1$ and $\rho(\varepsilon_1,z_1)\geqslant 2$ implies $(y(t)-\hat{x}_1(t))^2+\hat{x}_1^2(t)\leqslant\frac{1}{\rho(\varepsilon_1(t),z_1(t))}(\frac{\lambda^2}{2}+\frac{\lambda^2}{2})\leqslant\frac{\lambda^2}{2}$, $\forall\, t>T_\lambda$, which implies $|y(t)|=|\eta_1(t)-y_r(t)|\leqslant\lambda$, $\forall\, t\geqslant T_\lambda$.

We subsequently give two remarks to address the peculiar features of controller (8) and to further highlight the contributions in this paper.

**Remark 1.** We can see three-fold essential differences between the controller described by (8) and that in [1], which are caused by the functions-of-output control coefficients, rather than the constant ones in [1]. (i) The novel updating law of $L(t)$, motivated by [1–3], is quite essential to effectively overcome the system nonlinearities caused by the functions-of-output control coefficients, and the serious unknowns in the system and the reference signal. (ii) The gains $k_i(y)$'s and $h_i(y)$'s, respectively in the controller $u$ and the observer $\dot{\hat{x}}_i$'s, are functions of output, cannot be constants as in [1]. (iii) The design parameter $b$ should satisfy $0<b<\frac{1}{2}$ and cannot be picked to be 1 as in [1], and otherwise, the later proof for the global practical tracking would be invalid.

**Remark 2.** In (8), function $\rho(\cdot)$ has two different expressions. The former expression is preferable for the tracking performance analysis, for example, the proof of boundedness of $(\varepsilon(t),z(t))$ on $[0,T_f)$ in Lemma 1 below. The latter one is preferable for the simulation examples since it is more direct.

## 3 Proofs of Proposition 2 and Lemma 1

This section collects the proofs of a technical proposition and an important lemma, which are required to establish the desired global practical tracking.

### 3.1 Proof of Proposition 2

By (5), (7) and the fact $\dot{L}(t)\geqslant 0$, $L(t)\geqslant 1$ for $\forall\, t\in[0,T_f)$, along the trajectories of (9), we have

$$\begin{aligned}\dot{V} &= L(\varepsilon^{\mathrm T}A^{\mathrm T}P\varepsilon+\varepsilon^{\mathrm T}PA\varepsilon)+2\varepsilon^{\mathrm T}PF-\frac{\dot{L}}{L}(\varepsilon^{\mathrm T}DP\varepsilon+\varepsilon^{\mathrm T}PD\varepsilon)\\ &\quad+\gamma\left(L\big(z^{\mathrm T}B^{\mathrm T}Qz+z^{\mathrm T}QBz\big)+2L\varepsilon_1z^{\mathrm T}QH-\frac{\dot{L}}{L}\big(z^{\mathrm T}DQz+z^{\mathrm T}QDz\big)\right)\\ &\leqslant -\nu_oL|g_1(y)|\|\varepsilon\|^2-\gamma\nu_cL|g_1(y)|\|z\|^2+2\varepsilon^{\mathrm T}PF+2\gamma L\varepsilon_1z^{\mathrm T}QH. \end{aligned}\tag{12}$$

We next deal with the last two terms on the right-hand side of (12). By Assumptions 2 and 3, and noting $L(t)\geqslant 1$, we have that for $i=1,\ldots,n$,

$$\begin{aligned}\frac{|f_i|}{L^{b+i-1}} &\leqslant \frac{1}{L^{b+i-1}}\big(\theta_s(|x_1+y_r|+|x_2|+\cdots+|x_i|)+\theta_s+\theta_r\big)\\ &\leqslant \frac{1}{L^{b+i-1}}\left(\theta_s\sum_{j=1}^{i}(|x_j-\hat{x}_j|+|\hat{x}_j|)+\theta_r\theta_s+\theta_r+\theta_s\right)\\ &\leqslant \frac{\theta}{\sqrt{n}}(\|\varepsilon\|+\|z\|+1),\end{aligned}$$

where $\theta=\sqrt{n}\max\{\theta_s\sqrt{n},\theta_r(\theta_s+1)+\theta_s\}$ is an unknown positive constant.

Then, the last second term on the right-hand side of (12) satisfies

$$2\varepsilon^{\mathrm T}PF\leqslant 2\theta\|P\|\cdot\|\varepsilon\|\cdot(\|\varepsilon\|+\|z\|+1)\leqslant 4\theta\|P\|(\|\varepsilon\|^2+\|z\|^2)+\theta\|P\|. \tag{13}$$

By (6), the last term on the right-hand side of (12) satisfies

$$\begin{aligned} 2\gamma L\varepsilon_1 z^{\mathrm{T}}QH &\leqslant 2\gamma L|\varepsilon_1|\cdot\|z\|\cdot\|Q\|\cdot\overline{h}|g_1(y)| \\ &\leqslant \frac{\gamma\nu_c}{2}L|g_1(y)|\cdot\|z\|^2+\frac{2\gamma\overline{h}^2\|Q\|^2}{\nu_c}L|g_1(y)|\cdot\|\varepsilon\|^2. \end{aligned} \tag{14}$$

Substituting (13) and (14) into (12) first, and then noting $\gamma=\frac{\nu_c\nu_o}{4\overline{h}^2\|Q\|^2}$ and by Assumption 1, we arrive at

$$\begin{aligned} \dot{V} &\leqslant -L\left(\nu_o-\frac{2\gamma\overline{h}^2\|Q\|^2}{\nu_c}\right)|g_1(y)|\cdot\|\varepsilon\|^2-\frac{\gamma\nu_c}{2}L|g_1(y)|\cdot\|z\|^2+4\theta\|P\|(\|\varepsilon\|^2+\|z\|^2)+\theta\|P\| \\ &\leqslant -\frac{\underline{a}\nu_o}{2}L\|\varepsilon\|^2-\frac{\underline{a}\gamma\nu_c}{2}L\|z\|^2+4\theta\|P\|(\|\varepsilon\|^2+\|z\|^2)+\theta\|P\| \\ &\leqslant -\frac{\underline{a}\min\{\nu_o,\nu_c\}}{2}L(\|\varepsilon\|^2+\gamma\|z\|^2)+4\theta\|P\|\max\left\{1,\frac{1}{\gamma}\right\}(\|\varepsilon\|^2+\gamma\|z\|^2)+\theta\|P\| \\ &\leqslant -\frac{\underline{a}\min\{\nu_o,\nu_c\}}{2}\left(L-\Theta\right)(\|\varepsilon\|^2+\gamma\|z\|^2)+\Theta, \end{aligned}$$

where $\Theta=8\theta\|P\|\max\{\frac{1}{8},\frac{1}{\underline{a}\nu_o},\frac{1}{\underline{a}\nu_c},\frac{1}{\underline{a}\gamma\nu_o},\frac{1}{\underline{a}\gamma\nu_c}\}$ is an unknown positive constant.

The proof of Proposition 2 is completed.

### 3.2 Proof of Lemma 1

The whole proof of Lemma 1 is divided into two parts, to respectively prove the boundedness of $L(t)$ and $\big(\varepsilon(t),z(t)\big)$, both on $[0,T_f)$.

**Claim 1: Boundedness of $L(t)$ on $[0,T_f)$**

Suppose for contradiction that $L(t)$ is unbounded on $[0,T_f)$. This and noting $L(t)\geqslant 1$ imply $\lim_{t\to T_f}L(t)=+\infty$, and hence a finite time $t_1\in[0,T_f)$ exists such that

$$L(t)\geqslant\Theta+1,\quad\forall\,t\in[t_1,T_f),$$

from which and (11), it follows that

$$\dot{V}\leqslant-\beta V(t)+\Theta,\quad\forall\,t\in[t_1,T_f),$$

and hence

$$V(t)\leqslant\frac{\Theta}{\beta}+V(t_1)\exp\big(-\beta(t-t_1)\big),\quad\forall\,t\in[t_1,T_f),$$

where and in what follows, $V(t)$ denotes $V(\varepsilon(t),z(t))$ if no confusion occurs. This and (10) imply the following:

$$\lim_{t\to T_f}L(t)=+\infty\Longrightarrow\sup_{t\in[0,T_f)}V(t)<+\infty\Longleftrightarrow\sup_{t\in[0,T_f)}\big(\|\varepsilon(t)\|+\|z(t)\|\big)<+\infty. \tag{15}$$

We next show that this is impossible, whether $T_f<+\infty$ or $T_f=+\infty$.

When $T_f<+\infty$, by (15), the expression of $\dot{L}(t)$ in (8), the fact $L(t)\geqslant 1$, and the smoothness of $\rho(\cdot)$, we have

$$+\infty=L(T_f)-L(0)=\int_0^{T_f}\dot{L}(t)\mathrm{d}t\leqslant\int_0^{T_f}\rho\big(\varepsilon_1(t),z_1(t)\big)(\varepsilon_1^2(t)+z_1^2(t))\mathrm{d}t<+\infty,$$

a contradiction.

When $T_f=+\infty$, (15) still leads to a contradiction. We first prove the following implication:

$$\lim_{t\to+\infty}L(t)=+\infty\Longrightarrow\lim_{t\to+\infty}V(t)=0\Longleftrightarrow\lim_{t\to+\infty}\big(\|\varepsilon(t)\|+\|z(t)\|\big)=0. \tag{16}$$

For this, let $\delta$ be any positive constant. Since $\lim_{t\to T_f} L(t) = +\infty$ implies that a finite time $t_2$ exists such that $L(t) \geqslant \Theta + \frac{2\Theta}{\beta\delta}$, $\forall\, t \geqslant t_2$, by (11), we have

$$\dot{V} \leqslant -\frac{2\Theta}{\delta} V(t) + \Theta, \quad \forall\, t \geqslant t_2,$$

and in turn

$$V(t) \leqslant \frac{\delta}{2} + V(t_2)\exp\left(-\frac{2\Theta}{\delta}(t-t_2)\right), \quad \forall\, t \geqslant t_2. \tag{17}$$

On the other hand, another finite time $t_3 \geqslant t_2$ exists such that $V(t_2)\exp\big(-\frac{2\Theta}{\delta}(t-t_2)\big) \leqslant \frac{\delta}{2}$, $\forall\, t \geqslant t_3$, which and (17) imply $V(t) \leqslant \delta$, $\forall\, t \geqslant t_3$. This and the arbitrariness of $\delta$ conclude $\lim_{t\to+\infty} V(t) = 0$.

We introduce the function $\Gamma(L(t), \varepsilon, z) = L(t)V(\varepsilon, z)$. Since $\lim_{t\to T_f} L(t) = +\infty$ implies that a finite time $t_4$ exists such that $L(t) \geqslant 2\Theta$, $\forall\, t \geqslant t_4$, by (11) and noting $L(t) \geqslant 1$, we obtain

$$\dot{\Gamma} \leqslant \dot{L}(t)V(t) + L(t)\left(-\frac{\beta}{2}L(t)V(t) + \Theta\right) = L(t)\left(-\left(\frac{\beta}{2} - \frac{\dot{L}(t)}{L^2(t)}\right)\Gamma(t) + \Theta\right), \quad \forall\, t \geqslant t_4. \tag{18}$$

Considering the expression of $\dot{L}(t)$ in (8), the implication (16) illustrates $\lim_{t\to+\infty} \dot{L}(t) = 0$, and hence by $L(t) \geqslant 1$, a finite time $t_5 \geqslant t_4$ exists such that $\frac{\dot{L}(t)}{L^2(t)} \leqslant \frac{\beta}{4}$, $\forall\, t \geqslant t_5$. Thus, from (18), it follows that

$$\dot{\Gamma} \leqslant -L(t)\left(\frac{\beta}{4}\Gamma(t) - \Theta\right), \quad \forall\, t \geqslant t_5. \tag{19}$$

This implies that $\Gamma(t)$ is bounded on $[0, +\infty)$ [2)], namely $\sup_{t\in[0,+\infty)} \Gamma(t) \leqslant M_1 \min\{\lambda_{\min}(P), \gamma\lambda_{\min}(Q)\}$ for a constant $M_1 > 0$.

Then, we have

$$L^{1-2b}(t)\big((x_1(t) - \hat{x}_1(t))^2 + \hat{x}_1^2(t)\big) = L(t)\left(\varepsilon_1^2(t) + z_1^2(t)\right) \leqslant \frac{\Gamma(t)}{\min\{\lambda_{\min}(P), \gamma\lambda_{\min}(Q)\}} \leqslant M_1, \quad \forall\, t \geqslant 0,$$

and in turn

$$(x_1(t) - \hat{x}_1(t))^2 + \hat{x}_1^2(t) \leqslant \frac{M_1}{L^{1-2b}(t)}, \quad \forall\, t \geqslant 0. \tag{20}$$

Since (16) implies that a constant $M_2 > 0$ exists such that $0 < \sup_{t\in[0,+\infty)} \rho(\varepsilon_1(t), z_1(t)) < M_2$, from (20), it follows that

$$\rho\big(\varepsilon_1(t), z_1(t)\big)\big((x_1(t) - \hat{x}_1(t))^2 + \hat{x}_1^2(t)\big) - \frac{\lambda^2}{2} \leqslant \frac{M_1M_2}{L^{1-2b}(t)} - \frac{\lambda^2}{2}, \quad \forall\, t \geqslant 0.$$

Noting $1 - 2b > 0$, $\lim_{t\to T_f} L(t) = +\infty$ implies that a finite time $t_6 \geqslant t_5$ exists such that

$$\frac{M_1M_2}{L^{1-2b}(t)} - \frac{\lambda^2}{2} \leqslant 0, \quad \forall\, t \geqslant t_6.$$

By this and the expression of $\dot{L}(t)$ in (8), we know that $\dot{L}(t) \equiv 0$ when $t \geqslant t_6$, which contradicts

$$\lim_{t\to+\infty} L(t) = +\infty.$$

The above two contradictions mean that $L(t)$ is bounded on $[0, T_f)$, regardless of whether $T_f$ is finite or not.

2) Otherwise, there is $t_5' \in (t_5, +\infty)$ such that $\Gamma(t_5') > \max\{\Gamma(t_5), \frac{4\Theta}{\beta}\}$. Then, by the continuity of $\Gamma(t)$, there is $t_5'' \in [t_5, t_5')$ such that $\Gamma(t) > \Gamma(t_5'') = \max\{\Gamma(t_5), \frac{4\Theta}{\beta}\}$, $\forall\, t \in (t_5'', t_5']$. From this and (19), it follows that $\dot{\Gamma}(t) \leqslant 0$, $\forall\, t \in [t_5'', t_5']$ which implies $\Gamma(t) \leqslant \Gamma(t_5'')$, $\forall\, t \in (t_5'', t_5']$, and a contradiction occurs.

**Claim 2: Boundedness of $\varepsilon(t)$ and $z(t)$ on $[0, T_f)$**

We first prove the boundedness of $z(t)$ on $[0, T_f)$. For this, we define the function $V_2(z) = z^{\mathrm{T}}Qz$. Then by (5), (7) and noting $\dot{L}(t) \geqslant 0$ and $L(t) \geqslant 1$, we obtain

$$\begin{aligned}\dot{V}_2 &= L\big(z^{\mathrm{T}}B^{\mathrm{T}}Qz + z^{\mathrm{T}}QBz\big) + 2L\varepsilon_1 z^{\mathrm{T}}QH - \frac{\dot{L}}{L}\big(z^{\mathrm{T}}(DQ+QD)z\big)\\ &\leqslant -\nu_c L|g_1(y)|\cdot\|z\|^2 + 2L\varepsilon_1 z^{\mathrm{T}}QH.\end{aligned} \tag{21}$$

Considering (6), the last term on the right-hand side of (21) satisfies

$$2L\varepsilon_1 z^{\mathrm{T}}QH \leqslant 2L|\varepsilon_1|\cdot\|z\|\cdot\|Q\|\cdot\overline{h}|g_1(y)| \leqslant \frac{\nu_c}{2}L|g_1(y)|\cdot\|z\|^2 + \frac{2\overline{h}^2\|Q\|^2}{\nu_c}L|g_1(y)|\varepsilon_1^2. \tag{22}$$

Moreover, by Assumption 1, the definitions of $\varepsilon_1$, $z_1$ and $\rho(\cdot)$ in (8), and considering $L(t) \geqslant 1$, we have

$$\begin{aligned}g_1(y) &\leqslant \overline{a}(1+|y|^p) = \overline{a}\big(1+|L^b(\varepsilon_1+z_1)|^p\big)\\ &\leqslant \overline{a}L^{pb}(1+|\varepsilon_1+z_1|^p)\\ &\leqslant \overline{a}L^{pb}(1+(1+(\varepsilon_1+z_1)^2)^{\frac{p}{2}}) = \overline{a}L^{pb}\rho(\varepsilon_1,z_1).\end{aligned} \tag{23}$$

Substituting (22) and (23) into (21), and considering Assumption 1, $L(t) \geqslant 1$ and the expression of $\dot{L}(t)$ yield

$$\begin{aligned}\dot{V}_2 &\leqslant -\frac{\nu_c}{2}L|g_1(y)|\cdot\|z\|^2 + \frac{2\overline{h}^2\|Q\|^2}{\nu_c}L|g_1(y)|\varepsilon_1^2\\ &\leqslant -\frac{\underline{a}\nu_c}{2}\|z\|^2 + \frac{2\overline{a}\overline{h}^2\|Q\|^2}{\nu_c}L^{1+pb}\rho(\varepsilon_1,z_1)\varepsilon_1^2\\ &\leqslant -\frac{\underline{a}\nu_c}{2}\|z\|^2 + \frac{2\overline{a}\overline{h}^2\|Q\|^2}{\nu_c}L^{1+pb}\dot{L}(t) + \frac{\overline{a}\lambda^2\overline{h}^2\|Q\|^2}{\nu_c}L^{1+(p-2)b}.\end{aligned}$$

Then by (8) and noting $1+(p-2)b > 0$ and letting $L_f = \lim_{t\to T_f} L(t)$ (its boundedness has been proved), we have

$$\dot{V}_2 \leqslant -\frac{\underline{a}\nu_c}{2}\|z\|^2 + \frac{2\overline{a}\overline{h}^2\|Q\|^2}{\nu_c}L^{1+pb}\dot{L}(t) + \frac{\overline{a}\lambda^2\overline{h}^2\|Q\|^2L_f^{1+(p-2)b}}{\nu_c} \leqslant -q_1V_2(t) + q_2L^{1+pb}\dot{L}(t) + q_3, \tag{24}$$

where $q_1 = \frac{\underline{a}\nu_c}{2\lambda_{\max}(Q)}$, $q_2 = \frac{2\overline{a}\overline{h}^2\|Q\|^2}{\nu_c}$ and $q_3 = \frac{\overline{a}\lambda^2\overline{h}^2\|Q\|^2L_f^{1+(p-2)b}}{\nu_c}$ are positive constants.

From (24), it follows that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{q_1t}V_2(t)) \leqslant q_2\mathrm{e}^{q_1t}L^{1+pb}\dot{L} + q_3\mathrm{e}^{q_1t}$, $\forall t \in [0, T_f)$. Then integrating both sides yields

$$\begin{aligned}\mathrm{e}^{q_1t}V_2(t) &\leqslant V_2(0) + q_2\int_0^t \mathrm{e}^{q_1s}L^{1+pb}(s)\mathrm{d}L(s) + q_3\int_0^t \mathrm{e}^{q_1s}\mathrm{d}s\\ &\leqslant V_2(0) + \frac{q_2}{2+pb}L_f^{2+pb}\mathrm{e}^{q_1t} + \frac{q_3}{q_1}\mathrm{e}^{q_1t}, \quad \forall t \in [0, T_f),\end{aligned}$$

which implies $V(t) \leqslant V_2(0)\mathrm{e}^{-q_1t} + \frac{q_2}{2+pb}L_f^{2+pb} + \frac{q_3}{q_1}$, $\forall t \in [0, T_f)$, and hence $z(t)$ is bounded on $[0, T_f)$.

We next prove the boundedness of $\varepsilon(t)$ on $[0, T_f)$. For this, we introduce the following change of coordinates:

$$\xi_i = \frac{x_i - \hat{x}_i}{(L^*)^{b+i-1}}, \quad i = 1, \ldots, n, \tag{25}$$

where $L^*$ is a constant large enough such that

$$L^* \geqslant \max\left\{L_f, \frac{8\theta^*\|P\|}{\nu_o\underline{a}} + 1\right\}, \qquad \theta^* = \sqrt{n}\max\left\{\sqrt{n}\theta_s, \frac{\theta_r(\theta_s+1)+\theta_s}{L_f^b}\right\}. \tag{26}$$

Then, by (4) and (8), we have

$$\begin{cases}\dot{\xi}_i = L^* g_i(y)\xi_{i+1} - L^* h_i(y)\xi_1 + L^* h_i(y)\xi_1 - L\left(\dfrac{L}{L^*}\right)^{i-1} h_i(y)\xi_1 + \dfrac{f_i}{(L^*)^{b+i-1}}, \quad i=1,\ldots,n-1,\\ \dot{\xi}_n = -L^* h_n(y)\xi_1 + L^* h_n(y)\xi_1 - L\left(\dfrac{L}{L^*}\right)^{n-1} h_n(y)\xi_1 + \dfrac{f_n}{(L^*)^{b+n-1}},\end{cases}$$

which can also be written in the following compact form:

$$\dot{\xi} = L^* A(y)\xi + L^* H(y)\xi_1 - L(t)\Lambda(t)H(y)\xi_1 + F^*(t,x,y_r,\dot{y}_r), \tag{27}$$

where $\xi = [\xi_1,\ldots,\xi_n]^{\mathrm{T}}$, $\Lambda = \mathrm{diag}[1, \frac{L}{L^*},\ldots,(\frac{L}{L^*})^{n-1}]$, $F^* = [\frac{f_1}{(L^*)^b},\frac{f_i}{(L^*)^{b+i-1}},\ldots,\frac{f_n}{(L^*)^{b+n-1}}]^{\mathrm{T}}$, and $H$ has been defined above.

We define $V_3(\xi) = \xi^{\mathrm{T}} P\xi$. Then, by (27) and (5), we have

$$\begin{aligned}\dot{V}_3 &= L^*\xi^{\mathrm{T}}(A^{\mathrm{T}}P + PA)\xi + 2L^*\xi_1 H^{\mathrm{T}}P\xi - 2L\xi_1 H^{\mathrm{T}}\Lambda P\xi + 2{F^*}^{\mathrm{T}}P\xi\\ &\leqslant -L^*\nu_o|g_1(y)|\cdot\|\xi\|^2 + 2L^*\xi_1 H^{\mathrm{T}}P\xi - 2L\xi_1 H^{\mathrm{T}}\Lambda P\xi + 2{F^*}^{\mathrm{T}}P\xi.\end{aligned} \tag{28}$$

Considering $\xi_1^2 = \left(\frac{L}{L^*}\right)^{2b}\varepsilon_1^2$ and by (6), the second term on the right-hand side of (28) satisfies

$$2L^*\xi_1 H^{\mathrm{T}}P\xi \leqslant 2L^*|\xi_1|\cdot\|P\|\cdot\|\xi\|\cdot\overline{h}|g_1(y)| \tag{29}$$

$$\leqslant L^*|g_1(y)|\left(\frac{\nu_o}{4}\|\xi\|^2 + \frac{4\overline{h}^2\|P\|^2}{\nu_o}\xi_1^2\right) \tag{30}$$

$$= \frac{L^*\nu_o}{4}|g_1(y)|\cdot\|\xi\|^2 + \frac{4(L^*)^{1-2b}\overline{h}^2\|P\|^2}{\nu_o}L^{2b}|g_1(y)|\varepsilon_1^2, \tag{31}$$

and quite similarly, the third term satisfies (noting $\|\Lambda\| = 1$)

$$-2L\xi_1 H^{\mathrm{T}}\Lambda P\xi \leqslant \frac{L^*\nu_o}{4}|g_1(y)|\cdot\|\xi\|^2 + \frac{4\overline{h}^2\|P\|^2}{(L^*)^{1+2b}\nu_o}L^{2+2b}|g_1(y)|\varepsilon_1^2. \tag{32}$$

Moreover, similar to the deduction of (13) and by the definitions of $\varepsilon_i$'s, $\xi_i$'s and $L^*$, we have that for $i=1,\ldots,n$,

$$\frac{|f_i|}{(L^*)^{b+i-1}} \leqslant \frac{\theta^*}{\sqrt{n}}(\|z\| + \|\xi\| + 1).$$

Thus, the last term on the right-hand side of (28) satisfies

$$\begin{aligned}2{F^*}^{\mathrm{T}}P\xi &\leqslant 2\|F^*\|\cdot\|P\|\cdot\|\xi\| \;\leqslant\; 2\theta^*\|P\|\cdot\|\xi\|(\|z\| + \|\xi\| + 1)\\ &\leqslant 2\theta^*\|P\|\left(2\|\xi\|^2 + \frac{1}{2}\|z\|^2 + \frac{1}{2}\right) \;=\; \theta^*\|P\|(4\|\xi\|^2 + \|z\|^2 + 1).\end{aligned}$$

From this, (29) and (32), it follows that

$$\begin{aligned}\dot{V}_3 &\leqslant -\frac{L^*\nu_o}{2}|g_1(y)|\cdot\|\xi\|^2 + \theta^*\|P\|(4\|\xi\|^2 + \|z\|^2 + 1)\\ &\quad + \left(\frac{4(L^*)^{1-2b}\overline{h}^2\|P\|^2}{\nu_o}L^{2b} + \frac{4\overline{h}^2\|P\|^2}{(L^*)^{1+2b}\nu_o}L^{2+2b}\right)|g_1(y)|\varepsilon_1^2\\ &= -\frac{L^*\nu_o}{2}|g_1(y)|\cdot\|\xi\|^2 + \theta^*\|P\|(4\|\xi\|^2 + \|z\|^2 + 1) + \tau(L)|g_1(y)|\varepsilon_1^2,\end{aligned}$$

where $\tau(L) = \frac{4(L^*)^{1-2b}\overline{h}^2\|P\|^2}{\nu_o}L^{2b} + \frac{4\overline{h}^2\|P\|^2}{(L^*)^{1+2b}\nu_o}L^{2+2b}$ clearly satisfies $0 < \tau(L(t)) \leqslant \tau(L_f)$, $\forall\, t\in[0,T_f)$.

Thus, by Assumption 1, (23) and (26), we have

$$\begin{aligned}\dot{V}_3 &\leqslant -\frac{\nu_o\underline{a}}{2}\left(L^* - \frac{8\theta^*\|P\|}{\nu_o\underline{a}}\right)\|\xi\|^2 + \theta^*\|P\|(\|z\|^2+1) + \tau(L)|g_1(y)|\varepsilon_1^2 \\ &\leqslant -\frac{\nu_o\underline{a}}{2\lambda_{\max}(P)}V_3 + \overline{a}\tau(L)L^{pb}\dot{L} + \theta^*\|P\|(\|z\|^2+1) + \frac{\overline{a}\lambda^2\tau(L)L^{(p-2)b}}{2} \\ &\leqslant -\frac{\nu_o\underline{a}}{2\lambda_{\max}(P)}V_3 + \overline{a}\tau(L_f)L^{pb}\dot{L} + \theta^*\|P\|\sup_{t\in[0,T_f)}\|z(t)\|^2 + \theta^*\|P\| + \frac{\overline{a}\lambda^2\tau(L_f)\max\{1, L_f^{(p-2)b}\}}{2},\end{aligned}$$

from which, quite similar to the above, we can prove the boundedness of $\xi(t)$ on $[0, T_f)$. Considering the relation between $\varepsilon$ and $\xi$ described by (25) and (8), we directly obtain the boundedness of $\varepsilon$ on $[0, T_f)$.

Thus, the proof of Lemma 1 is completed.

## 4 A simulation example

In this section, an example is given to illustrate the effectiveness of theoretical results for the following system:

$$\begin{cases}\dot{\eta}_1 = (1+y^2)\eta_2, \\ \dot{\eta}_2 = u + \theta(\eta_1+\eta_2), \\ y = \eta_1 - y_r,\end{cases} \tag{33}$$

where $y_r$ is the signal to be tracked. Suppose $\theta = 0.8$, $y_r = \sin(t)$.

We can check that system (33) satisfies Assumption 1 with $p = 2$ and Assumption 2 with $\theta = 0.8$, and the the reference signal $y_r$ satisfies Assumption 3 with $\theta_r = 2$.

We choose $h_1(y) = 2.1(1+y^2)$, $h_2(y) = 1.2(1+y^2)$, $k_1(y) = 1.3(1+y^2)$ and $k_2(y) = 1.5(1+y^2)$. These functions are suitable since from which solving (5) and (7) with $\nu_o = 0.1$, $\nu_c = 0.2$ can yield symmetric positive definite matrices

$$P = \begin{bmatrix}1.2 & -0.3 \\ -0.3 & 0.5\end{bmatrix}, \quad Q = \begin{bmatrix}2.8 & 0.2 \\ 0.2 & 2.1\end{bmatrix}.$$

In view of the design procedure developed in Section 2, under the pre-given tracking level $\lambda = 0.1$, we obtain the following dynamical output-feedback controller to system (33):

$$\begin{cases}u = -1.3L^2(t)(1+y^2)\hat{x}_1 - 1.5L(t)(1+y^2)\hat{x}_2, \\ \begin{cases}\dot{\hat{x}}_1 = (1+y^2)\hat{x}_2 + 2.1L(t)(1+y^2)(y-\hat{x}_1), \\ \dot{\hat{x}}_2 = u + 1.2L^2(t)(1+y^2)(y-\hat{x}_1), \\ \dot{L} = \max\left\{\frac{1}{L^{0.02}(t)}\left(\left(1+\frac{L^{0.02}(t)+y^2}{L^{0.02}(t)}\right)\left((y-\hat{x}_1)^2+\hat{x}_1^2\right) - 0.005\right), 0\right\}, \quad L(0) = 1.\end{cases}\end{cases}$$

Setting the initial conditions of the closed-loop system by $\eta_1(0) = -3.5$, $\eta_2(0) = -1.5$, $\hat{x}_1(0) = 1.2$, $\hat{x}_2(0) = -1$ and $L(0) = 1$, the simulation results are shown in Figures 1–4 where the logarithmic X-coordinates have been adopted to show the transient behavior prominently. These figures show that all the closed-loop system states are bounded, and especially, Figure 4 shows that the tracking error satisfies $|\eta_1 - y_r| \leqslant 0.1$ after about one second, which means that the prescribed tracking performance is achieved.

## 5 Conclutions

In this paper, the global practical tracking has been accomplished via adaptive output-feedback for a class of uncertain nonlinear systems with generalized control coefficients. The novelty of work lies in the

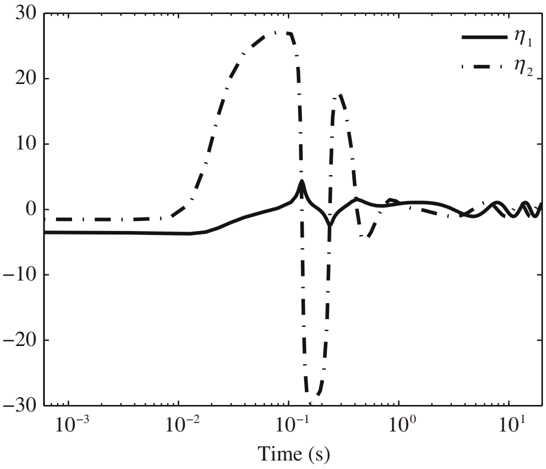


**Figure 1** The trajectories of state $\eta_1$ and $\eta_2$.

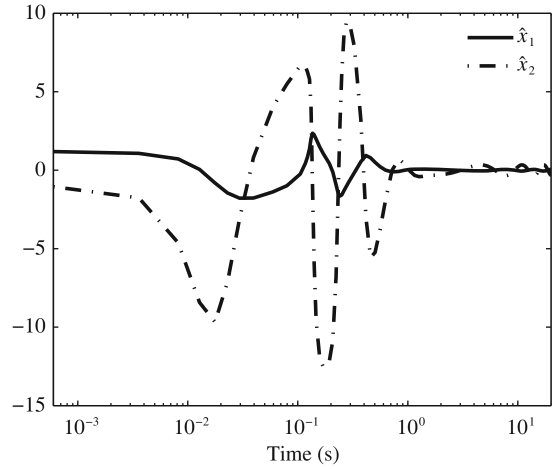


**Figure 2** The trajectories of state $\hat{x}_1$ and $\hat{x}_2$.

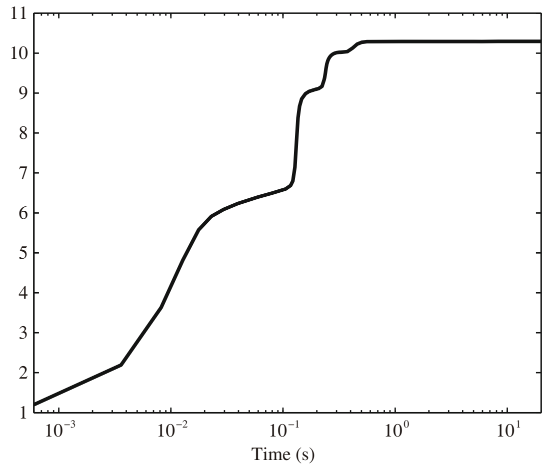


**Figure 3** The trajectory of gain $L$.

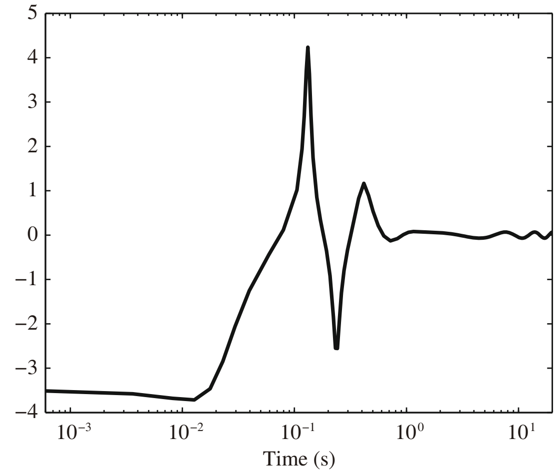


**Figure 4** The trajectory of the tracking error $y$.

introduction of a new updating law of high-gain, based on which, a high-gain observer and an adaptive output-feedback controller have been designed (involved gains are functions of output and time, and essentially different from the existing related literature). This paper rather extends our earlier result in [1], while the analysis and proof of our main result are motivated by [3] and different from those in [1, 2]. It is noteworthy that system (1) is typically representative but far from general: the upper bounds of the control coefficients are merely polynomials of output, and the system growth linearly relies on the unmeasurable system states and its growth rate is only a constant. Apparently, the proposed approach in this paper is inapplicable for the cases where the upper bounds of the control coefficients are non-polynomials of output, the system growth nonlinearly relies on the unmeasurable system states, or where the growth rate is function of output. Nevertheless, the proposed approach in this paper provides insight, for solving the global practical tracking for more general nonlinear systems, as the cases that have been mentioned for example. This will be attempted in our future research.

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**Conflict of interest** The authors declare that they have no conflict of interest.

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