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Iterative learning control for one-dimensional fourth order distributed parameter systems

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Abstract This paper addresses the problem of iterative learning control algorithm for high order distributed parameter systems in the presence of initial errors. And the considered distributed parameter systems are composed of the one-dimensional fourth order parabolic equations or the one-dimensional fourth order wave equations. According to the characteristics of the systems, iterative learning control laws are proposed for such fourth order distributed parameter systems based on the P-type learning scheme. When the learning scheme is applied to the systems, the output tracking errors on L^2 space are bounded, and furthermore, the tracking errors on L^2 space can tend to zero along the iteration axis in the absence of initial errors. Simulation examples illustrate the effectiveness of the proposed method.

Keywords P-type learning scheme, iterative learning control, distributed parameter systems, fourth order parabolic equation, fourth order wave equation

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1 Introduction

Since the complete algorithm of iterative learning control (ILC) was first proposed by Arimoto et al. [1], it has become the hot issues of cybernetics and has attracted broad attention in recent years [2–5]. The basic idea of ILC is to improve the control signal for the present operation cycle by feeding back the control error in the previous cycle. And the classical formulation of ILC design problem is as follows: find an update mechanism for the output trajectory of a new cycle based on the information from previous cycles so that the output trajectory converges asymptotically to the desired reference trajectory. Owing to its simplicity and effectiveness, ILC has been found to be a good alternative in many areas and applications, see [6] for detailed results. Nowadays, ILC is playing a more and more important role in controlling repeatable processes.

Due to many practical problems can be described by the DPSs (distributed parameter systems) governed by the PDEs (partial differential equations), the applications of DPSs have been involved in many fields in the last few years, and a series of the research achievements have been obtained [7–9]. Since the variables of DPSs are related to infinite dimensional space, studies of ILC for infinite dimensional processes are limited and there have been only a few works reported on ILC for DPSs, while ILC has

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been widely investigated for finite dimensional systems. Furthermore, most of them focused on the first order or the second order DPSs [10–16]. Refs. [10–12] designed the ILC algorithms for parabolic DPSs by using the P-type learning scheme. Ref. [13] discussed both the P-type and the D-type ILC schemes for a parabolic DPS, which was transformed into a linear system on Hilbert space. In [14], ILC was applied to a temporal-spatial discretized first order hyperbolic PDE, guaranteeing stability of the closed loop system and satisfying the requirements of performance. Recently, ref. [15] proposed an ILC algorithm for a DPS which is governed by a second order hyperbolic PDE. In [16], a D-type ILC algorithm for irregular DPSs was introduced with the aid of the weak convergences of functional analysis. In the field of the distributed control for DPSs, hitherto, almost all of the systems involved in ILC have been low order (first order or second order). How to apply ILC algorithm to high order DPSs and conduct the corresponding control design, to the best of our knowledge, there is no relevant literature about this.

Fourth-order PDEs problems arise commonly from the studies of phase separation in cooling binary solutions [17], vibration of beams and thin plates [18–20], and have attracted broad attention in recent years [21–27]. Hitherto, the related research work about fourth-order PDEs have mainly focused on their well-posedness, numerical solutions, etc. [21–27]. In this paper, ILC technique is applied for the first time to a class of fourth-order DPSs. And the considered DPSs are composed of the fourth-order parabolic PDEs in [21] or the fourth-order wave equations in [22]. A P-type learning law and a convergent condition, which can be applied for both the parabolic equation and the wave equation, are proposed. And when the learning law is applied to the systems, the output tracking errors on L^2 space are bounded, and furthermore, the tracking errors on L^2 space can tend to zero along the iteration axis in the absence of initial errors.

In this paper, the following notational conventions are adopted: for function $Q(x,t)$: $[0,1]\times[0,T]\to R$, take the norm: $\|Q(\cdot,t)\|_{L^2}=\sqrt{\int_0^1 Q^2(x,t)\mathrm{d}x}$, and define $\|Q\|_{L^2,s}=\sup_{t\in[0,T]}\|Q(\cdot,t)\|^2_{L^2}$. Denote $H^l(0,1)=\{\theta(x,t)\,|\,\theta(x,t)\in L^2(0,1),\frac{\partial\theta(x,t)}{\partial x}\in L^2(0,1),\ldots,\frac{\partial^l\theta(x,t)}{\partial x^l}\in L^2(0,1)\}$, $H^l_0(0,1)=\{\theta(x,t)|\theta(x,t)\in H^l(0,1)$, and there exists a sequence $\theta_m(x,t)\in C_0^\infty(0,1)$ such that $\theta_m(x,t)\to\theta(x,t)$ in space $H^l(0,1)\}$, $L^n((0,T);H^l(0,1))=\{\theta(x,t)|(\int_0^T\|\theta(\cdot,t)\|^n_{H^l}\mathrm{d}t)^{\frac{1}{n}}<\infty\}$, and $L^\infty(\Omega)=\{\theta(x,t)\,|\,|\theta(x,t)|<\infty$ almost everywhere in $\Omega\}$, where $\Omega=(0,1)\times(0,T)$.

2 Problem description

Consider the following one-dimensional fourth order PDE:

$$\frac{\partial^\alpha Q}{\partial t^\alpha}+Q_{xxxx}+(2-\alpha)gQ=(\alpha-1)Q_{xx}+f(x,t),\quad (x,t)\in\Omega,\quad \alpha=1\text{ or }2, \tag{1}$$

with initial-boundary conditions: $Q(x,0)=\varphi(x)(\alpha=1,2)$, $\frac{\partial Q(x,t)}{\partial t}|_{t=0}=\psi(x)$ $(\alpha=2)$, $x\in[0,1]$; $Q(0,t)=Q(1,t)=\frac{\partial Q(x,t)}{\partial x}|_{x=0}=\frac{\partial Q(x,t)}{\partial x}|_{x=1}=0$, $t\in(0,T]$. And $f\in L^2(\Omega)$, $g\in L^\infty(\Omega)$.

Remark 1. When $\alpha=1$, Eq. (1) is the fourth order parabolic equation in [21]; When $\alpha=2$, Eq. (1) is the fourth order wave equation in [22].

In this paper, we will expand the ILC framework to the one-dimensional fourth order distributed parameter systems governed by (1). For the requirement of ILC design, we replace $f(x,t)$ given in (1) with the control variable $u(x,t)$. And by adding an output variable $y(x,t)$ with general form, the following fourth order distributed parameter system governed by (1) is given:

$$\begin{cases}\dfrac{\partial^\alpha Q(x,t)}{\partial t^\alpha}+(Q(x,t))_{xxxx}+(2-\alpha)g(x,t)Q(x,t)\\ \quad=(\alpha-1)\left(Q(x,t)\right)_{xx}+u(x,t)\,,\quad (x,t)\in\Omega\,,\ \alpha=1\text{ or }2,\\ y(x,t)=C(t)Q(x,t)+D(t)u(x,t)\,,\quad 0\leqslant t\leqslant T,\ x\in[0,1],\end{cases} \tag{2}$$

with initial-boundary conditions: $Q(x,0)=\varphi(x)(\alpha=1,2),\frac{\partial Q(x,t)}{\partial t}|_{t=0}=\psi(x)(\alpha=2)$, $x\in[0,1]$; $Q(0,t)=Q(1,t)=\frac{\partial Q(x,t)}{\partial x}|_{x=0}=\frac{\partial Q(x,t)}{\partial x}|_{x=1}=0$, $t\in(0,T]$. Where $Q(x,t),u(x,t),y(x,t)\in R$ represent the state, control input and output of the system, respectively.

Remark 2. When $\alpha = 1$, then $\varphi(x) \in L^2(0,1)$; When $\alpha = 2$, then $\varphi(x) \in H_0^2(0,1) \cap H^8(0,1)$, and $\psi(x) \in H^6(0,1)$.

The system (2) is assumed to satisfy the following assumptions.

Assumption 1. $0 < D_1 \leqslant D(t) \leqslant D_2$, where D_1, D_2 are known constants. That is, the system (1) has direct transmission from inputs to outputs. $|C(t)| \leqslant C$, where C is an unknown constant.

Assumption 2. For a given trajectory $y_r(x,t)$, there exists a $u_r(x,t) \in L^2(\Omega)$ such that

$$\begin{cases} \dfrac{\partial^\alpha Q_r(x,t)}{\partial t^\alpha} + (Q_r(x,t))_{xxxx} + (2-\alpha)g(x,t)Q_r(x,t) \\ \quad = (\alpha - 1)\,(Q_r(x,t))_{xx} + u_r(x,t)\,, \quad (x,t) \in \Omega\,,\ \alpha = 1 \text{ or } 2, \\ y_r(x,t) = C(t)Q_r(x,t) + D(t)u_r(x,t)\,, \quad 0 \leqslant t \leqslant T,\ x \in [0,1], \end{cases}$$

with initial-boundary conditions: $Q_r(x,0) = \varphi_r(x)(\alpha = 1,2)$, $\frac{\partial Q_r(x,t)}{\partial t}|_{t=0} = \psi_r(x)$ $(\alpha = 2)$, $x \in [0,1]$; $Q_r(0,t) = Q_r(1,t) = \frac{\partial Q_r(x,t)}{\partial x}|_{x=0} = \frac{\partial Q_r(x,t)}{\partial x}|_{x=1} = 0$, $t \in (0,T]$. And when $\alpha = 2$, then $u_r(0,t) = u_r(1,t) = \frac{\partial u_r(x,t)}{\partial x}|_{x=0} = \frac{\partial u_r(x,t)}{\partial x}|_{x=1} = 0$.

It is assumed that the system (2) is repeatable over $t \in [0,T]$. Rewrite the system (2) at each iteration as

$$\begin{cases} \dfrac{\partial^\alpha Q_k(x,t)}{\partial t^\alpha} + (Q_k(x,t))_{xxxx} + (2-\alpha)g(x,t)Q_k(x,t) \\ \quad = (\alpha - 1)\,(Q_k(x,t))_{xx} + u_k(x,t)\,, \quad (x,t) \in \Omega\,,\ \alpha = 1 \text{ or } 2, \\ y_k(x,t) = C(t)Q_k(x,t) + D(t)u_k(x,t)\,, \quad 0 \leqslant t \leqslant T, x \in [0,1], \end{cases} \tag{3}$$

with initial-boundary conditions: $Q_k(x,0) = \varphi_k(x)(\alpha = 1,2)$, $\frac{\partial Q_k(x,t)}{\partial t}|_{t=0} = \psi_k(x)$ $(\alpha = 2)$, $x \in [0,1]$; $Q_k(0,t) = Q_k(1,t) = \frac{\partial Q_k(x,t)}{\partial x}|_{x=0} = \frac{\partial Q_k(x,t)}{\partial x}|_{x=1} = 0$, $t \in (0,T]$. $k = 0,1,2,\ldots$.

Assumption 3. For all k, the repeatability of the initial setting is satisfied within an admissible deviation level, i.e., $\|\varphi_k(\cdot) - \varphi_r(\cdot)\|_{L^2} \leqslant \varepsilon_1(\alpha = 1)$, $\|\varphi_k(\cdot) - \varphi_r(\cdot)\|_{H^2} = \|\varphi_k(\cdot) - \varphi_r(\cdot)\|_{L^2} + \|\frac{\mathrm{d}}{\mathrm{d}x}(\varphi_k(\cdot) - \varphi_r(\cdot))\|_{L^2} + \|\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\varphi_k(\cdot) - \varphi_r(\cdot))\|_{L^2} \leqslant \varepsilon_2$ $(\alpha = 2)$, $\|\psi_k(\cdot) - \psi_r(\cdot)\|_{L^2} \leqslant \varepsilon_3$, $k = 0,1,2,\ldots$, where ε_1, ε_2, ε_3 are positive constants.

The learning control target is to find an appropriate learning scheme, such that the output tacking errors on L^2 space are bounded, and furthermore, the iterative learning sequence $y_k(x,t)$ uniformly converges to the desired trajectory $y_r(x,t)$ on L^2 space in the absence of initial errors, that is $\lim_{k\to\infty} \|e_k\|_{L^2,s} = 0$, where $e_k(x,t) = y_r(x,t) - y_k(x,t)$.

For D_1, D_2 given in Assumption 1, take a positive number ε satisfying

$$\frac{D_2}{D_1} < \frac{\sqrt{1+\varepsilon}+1}{\sqrt{1+\varepsilon}-1}. \tag{4}$$

Constructing the learning scheme for the system (3) as follows:

$$u_{k+1}(x,t) = u_k(x,t) + q e_k(x,t), \tag{5}$$

where $q > 0$ is the learning gain. Take q so that

$$\rho = \sup_{t\in[0,T]} |1 - qD(t)| < \frac{1}{\sqrt{1+\varepsilon}} \tag{6}$$

holds. Denote $\delta Q_k(x,t) = Q_{k+1}(x,t) - Q_k(x,t)$, $\delta u_k(x,t) = u_{k+1}(x,t) - u_k(x,t)$. It follows from (3) and (5) that

$$\begin{aligned} e_{k+1}(x,t) &= e_k(x,t) + y_k(x,t) - y_{k+1}(x,t) \\ &= e_k(x,t) - C(t)\delta Q_k(x,t) - D(t)\delta u_k(x,t) \\ &= e_k(x,t) - C(t)\delta Q_k(x,t) - qD(t)e_k(x,t) \\ &= (1 - qD(t))e_k(x,t) - C(t)\delta Q_k(x,t). \end{aligned}$$

From (6) and Assumption 1, we have

$$|e_{k+1}(x,t)| \leqslant \rho\,|e_k(x,t)| + C\,|\delta Q_k(x,t)|\,.$$

Using the basic inequality, it yields

$$(e_{k+1}(x,t))^2 \leqslant (1+\varepsilon)\rho^2(e_k(x,t))^2 + \left(1+\frac{1}{\varepsilon}\right)C^2(\delta Q_k(x,t))^2.$$

Integrating both sides with respect to x from 0 to 1, we get

$$\|e_{k+1}(\cdot,t)\|_{L^2}^2 \leqslant (1+\varepsilon)\rho^2\,\|e_k(\cdot,t)\|_{L^2}^2 + \left(1+\frac{1}{\varepsilon}\right)C^2\,\|\delta Q_k(\cdot,t)\|_{L^2}^2\,.$$

For $\lambda > 0$, we have

$$\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\,\|e_{k+1}(\cdot,t)\|_{L^2}^2\right\} \leqslant (1+\varepsilon)\rho^2\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\,\|e_k(\cdot,t)\|_{L^2}^2\right\} + \left(1+\frac{1}{\varepsilon}\right)C^2\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\,\|\delta Q_k(\cdot,t)\|_{L^2}^2\right\}. \tag{7}$$

It follows from Assumption 3 that

$$\|\delta Q_k(\cdot,0)\|_{L^2} = \|\varphi_{k+1}(\cdot)-\varphi_r(\cdot)+\varphi_r(\cdot)-\varphi_k(\cdot)\|_{L^2} \leqslant 2\varepsilon_1, \quad \alpha=1, \tag{8}$$

$$\|\delta Q_k(\cdot,0)\|_{H^2} = \|\varphi_{k+1}(\cdot)-\varphi_r(\cdot)+\varphi_r(\cdot)-\varphi_k(\cdot)\|_{H^2} \leqslant 2\varepsilon_2, \quad \alpha=2, \tag{9}$$

$$\left\|\left.\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right|_{t=0}\right\|_{L^2} = \|\psi_{k+1}(\cdot)-\psi_r(\cdot)+\psi_r(\cdot)-\psi_k(\cdot)\|_{L^2} \leqslant 2\varepsilon_3. \tag{10}$$

Lemma 1 ([28]). If $\{a_k\}$, $k\in\{0,1,\ldots,\infty\}$ is a sequence of real numbers such that

$$|a_{k+1}| \leqslant \hat{\rho}\,|a_k| + \beta, \quad 0\leqslant\hat{\rho}<1, \quad \beta>0,$$

then

$$\limsup_{k\to\infty}|a_k| \leqslant \frac{\beta}{1-\hat{\rho}}.$$

3 ILC for the fourth order parabolic system

When $\alpha = 1$, we have the following theorem.

Theorem 1. Let Assumptions 1–3 and (6) be satisfied, then the output tracking error on L^2 space is bounded under the effect of the learning control law (5). Furthermore, when $\varphi_k(x) = \varphi_r(x)$, $x\in[0,1]$, $k = 0,1,2,\ldots$, the output tracking error on L^2 space can tend to zero along the iteration axis, under the effect of the learning control law (5), i.e., $\lim_{k\to\infty}\|e_k\|_{L^2,s} = 0$.

Proof. It follows from (3) ($\alpha = 1$) and (5) that

$$\frac{\partial(\delta Q_k(x,t))}{\partial t} + (\delta Q_k(x,t))_{xxxx} + g(x,t)\delta Q_k(x,t) = qe_k(x,t)\,,\ (x,t)\in\Omega. \tag{11}$$

Multiplying both sides of (11) by $\delta Q_k(x,t)$ and integrating with respect to x from 0 to 1, we can get

$$\int_0^1\left\{\delta Q_k(x,t)\frac{\partial(\delta Q_k(x,t))}{\partial t}\right\}\mathrm{d}x + \int_0^1\left\{\delta Q_k(x,t)\,(\delta Q_k(x,t))_{xxxx}\right\}\mathrm{d}x = q\int_0^1\delta Q_k(x,t)e_k(x,t)\mathrm{d}x - \int_0^1 g(x,t)(\delta Q_k(x,t))^2\mathrm{d}x, \quad 0<t\leqslant T, \tag{12}$$

while

$$\int_0^1 \left\{\delta Q_k(x,t)\frac{\partial(\delta Q_k(x,t))}{\partial t}\right\}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\delta Q_k(x,t))^2\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2, \tag{13}$$

$$\begin{aligned}\int_0^1 \delta Q_k(x,t)e_k(x,t)\mathrm{d}x &\leqslant \frac{1}{2}\int_0^1 (\delta Q_k(x,t))^2\mathrm{d}x + \frac{1}{2}\int_0^1 (e_k(x,t))^2\mathrm{d}x \\ &= \frac{1}{2}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2 + \frac{1}{2}\left\|e_k(\cdot,t)\right\|_{L^2}^2,\end{aligned} \tag{14}$$

$$-\int_0^1 g(x,t)(\delta Q_k(x,t))^2\mathrm{d}x \leqslant \left|\int_0^1 g(x,t)(\delta Q_k(x,t))^2\mathrm{d}x\right| \leqslant \sigma\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2, \tag{15}$$

where $\sigma = \|g\|_{L^\infty(\Omega)}$. Integrating by parts and combining with the boundary conditions of the system (3), we can derive

$$\begin{aligned}&\int_0^1 \left\{\delta Q_k(x,t)\left(\delta Q_k(x,t)\right)_{xxxx}\right\}\mathrm{d}x \\ &= \left\{\delta Q_k(x,t)\left(\delta Q_k(x,t)\right)_{xxx}\right\}\big|_0^1 - \int_0^1 \left\{\left(\delta Q_k(x,t)\right)_x\left(\delta Q_k(x,t)\right)_{xxx}\right\}\mathrm{d}x \\ &= -\int_0^1 \left\{\left(\delta Q_k(x,t)\right)_x\left(\delta Q_k(x,t)\right)_{xxx}\right\}\mathrm{d}x \\ &= -\left\{\left(\delta Q_k(x,t)\right)_x\left(\delta Q_k(x,t)\right)_{xx}\right\}\big|_0^1 + \int_0^1 \left\{\left(\delta Q_k(x,t)\right)_{xx}\left(\delta Q_k(x,t)\right)_{xx}\right\}\mathrm{d}x \\ &= \int_0^1 \left(\delta Q_k(x,t)\right)_{xx}^2\mathrm{d}x \geqslant 0.\end{aligned} \tag{16}$$

Substituting (13)–(16) into (12), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2 \leqslant (q+2\sigma)\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2 + q\left\|e_k(\cdot,t)\right\|_{L^2}^2, \quad 0 < t \leqslant T.$$

Applying Gronwall lemma and combining with (8), we have

$$\begin{aligned}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2 &\leqslant q\int_0^t \mathrm{e}^{(q+2\sigma)(t-\eta)}\left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta + \mathrm{e}^{(q+2\sigma)t}\left\|\delta Q_k(\cdot,0)\right\|_{L^2}^2 \\ &\leqslant q\mathrm{e}^{(q+2\sigma)T}\int_0^t \left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta + 4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T} \\ &= q\mathrm{e}^{(q+2\sigma)T}\int_0^t \mathrm{e}^{\lambda\eta}\mathrm{e}^{-\lambda\eta}\left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta + 4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T} \\ &\leqslant q\mathrm{e}^{(q+2\sigma)T}\int_0^t \mathrm{e}^{\lambda\eta}\mathrm{d}\eta\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} + 4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T} \\ &= q\mathrm{e}^{(q+2\sigma)T}\frac{\mathrm{e}^{\lambda t}-1}{\lambda}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} + 4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T}.\end{aligned}$$

Therefore

$$\begin{aligned}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2\right\} &\leqslant q\mathrm{e}^{(q+2\sigma)T}\sup_{t\in[0,T]}\left\{\frac{1-\mathrm{e}^{-\lambda t}}{\lambda}\right\}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} \\ &\quad + \sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\right\}4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T} \\ &= q\mathrm{e}^{(q+2\sigma)T}\frac{1-\mathrm{e}^{-\lambda T}}{\lambda}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} + 4\varepsilon_1^2\mathrm{e}^{(q+2\sigma)T}.\end{aligned}$$

Substituting the above expression into (7), it yields

$$\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_{k+1}(\cdot,t)\right\|_{L^2}^2\right\} \leqslant \left\{(1+\varepsilon)\rho^2 + \left(1+\frac{1}{\varepsilon}\right)C^2q\mathrm{e}^{(q+2\sigma)T}\frac{1-\mathrm{e}^{-\lambda T}}{\lambda}\right\}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}$$

$$+ 4\varepsilon_1^2\left(1+\frac{1}{\varepsilon}\right)C^2\mathrm{e}^{(q+2\sigma)T}$$
$$= \hat{\rho}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} + 4\varepsilon_1^2\left(1+\frac{1}{\varepsilon}\right)C^2\mathrm{e}^{(q+2\sigma)T},$$

where

$$\hat{\rho} = (1+\varepsilon)\rho^2 + \left(1+\frac{1}{\varepsilon}\right)C^2 q\mathrm{e}^{(q+2\sigma)T}\frac{1-\mathrm{e}^{-\lambda T}}{\lambda}.$$

Since $0 \leqslant (1+\varepsilon)\rho^2 < 1$ by (6), it is possible to choose λ sufficiently large so that $\hat{\rho} < 1$. So it can be derived by Lemma 1 that

$$\limsup_{k\to\infty}\left\{\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}\right\} \leqslant \frac{1}{1-\hat{\rho}}4\varepsilon_1^2\left(1+\frac{1}{\varepsilon}\right)C^2\mathrm{e}^{(q+2\sigma)T}.$$

Further we have

$$\limsup_{k\to\infty}\|e_k\|_{L^2,s} = \limsup_{k\to\infty}\left\{\sup_{t\in[0,T]}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\} \leqslant \limsup_{k\to\infty}\left\{\sup_{t\in[0,T]}\left\{\mathrm{e}^{\lambda T}\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}\right\}$$
$$= \mathrm{e}^{\lambda T}\limsup_{k\to\infty}\left\{\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}\right\} \leqslant \frac{1}{1-\hat{\rho}}4\varepsilon_1^2\mathrm{e}^{\lambda T}\left(1+\frac{1}{\varepsilon}\right)C^2\mathrm{e}^{(q+2\sigma)T}.$$

That is, the output tracking error on L^2 space is bounded. Furthermore, if $\varphi_k(x) = \varphi_r(x)$, $x \in [0,1]$, $k = 0,1,2,\ldots$, then $\varepsilon_1 = 0$, and we can obtain

$$\lim_{k\to\infty}\|e_k\|_{L^2,s} = 0.$$

This completes the proof.

Remark 3. Ref. [21] discussed the problem of observability estimate for a one-dimensional fourth order parabolic equation, which is sometimes known as Cahn-Hilliard type equation and appears in the study of phase separation in cooling binary solutions and in other contexts generating spatial pattern formation. And the observability inequality for the system was given. The following one-dimensional fourth order parabolic PDE was given in [21], with the initial-boundary conditions:

$$\begin{cases} Q_t + Q_{xxxx} + gQ = f\,, & (x,t)\in\Omega, \\ Q(0,t) = Q(1,t) = 0\,, & t\in(0,T), \\ Q_x(0,t) = Q_x(1,t) = 0\,, & t\in(0,T), \\ Q(x,0) = Q_0(x)\,, & x\in(0,1). \end{cases} \tag{17}$$

It is easy to see that Eq. (17) is the same as Eq. (1) in this paper (when $\alpha = 1$). And the uniqueness and existence of the solution of (17) have been given in [21] as follows: for each $g \in L^\infty(\Omega)$, each $f \in L^2(\Omega)$ and each $Q_0 \in L^2(0,1)$, Eq. (17) admits a unique function $Q \in C([0,T];L^2(0,1)) \cap L^2((0,T);H_0^2(0,1))$. Moreover, $Q \in L^2((\delta,T);H^4(0,1))$ and $Q_t \in L^2((\delta,T)\times(0,1))$ for all $\delta \in (0,T)$. From above, we know that if the initial control $u_0(x,t)$ in (3) is taken from $L^2(\Omega)$, then the solution of (3) at $k=0$ is existing and unique. Furthermore, by (5) and Assumption 2, $u_1(x,t) \in L^2(\Omega)$ is obtained to guarantee the uniqueness and existence of the solution of (3) at $k=1$. So, the solution of the system (3) at kth iteration is always existing and unique ($k = 0,1,2,\ldots$).

4 ILC for the fourth order wave system

When $\alpha = 2$, we have the following theorem:

Theorem 2. Let Assumptions 1–3 and (6) are satisfied, then the output tracking error on L^2 space is bounded under the effect of the learning control law (5). Furthermore, when $\varphi_k(x) = \varphi_r(x)$, $\psi_k(x) =$

$\psi_r(x)$, $x \in [0,1]$, $k = 0,1,2,\ldots$, the output tracking error on L^2 space can tend to zero along the iteration axis, under the effect of the control law (5), i.e., $\lim_{k\to\infty} \|e_k\|_{L^2,s} = 0$.

Proof. It follows from (3) ($\alpha = 2$) and (5) that

$$\frac{\partial^2(\delta Q_k(x,t))}{\partial t^2} + (\delta Q_k(x,t))_{xxxx} = (\delta Q_k(x,t))_{xx} + q e_k(x,t)\,,\ (x,t) \in \Omega. \tag{18}$$

Multiplying both sides of (18) by $\frac{\partial(\delta Q_k(x,t))}{\partial t}$ and integrating with respect to x from 0 to 1, we can get

$$\begin{aligned} &\int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}\frac{\partial^2(\delta Q_k(x,t))}{\partial t^2}\right\}\mathrm{d}x + \int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{xxxx}\right\}\mathrm{d}x \\ &= \int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{xx}\right\}\mathrm{d}x + q\int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}e_k(x,t)\right\}\mathrm{d}x\,, \quad 0 < t \leqslant T, \end{aligned} \tag{19}$$

while

$$\begin{aligned} \int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}\frac{\partial^2(\delta Q_k(x,t))}{\partial t^2}\right\}\mathrm{d}x &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{\partial(\delta Q_k(x,t))}{\partial t}\right)^2 \mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2, \end{aligned} \tag{20}$$

$$\begin{aligned} \int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}e_k(x,t)\right\}\mathrm{d}x &\leqslant \frac{1}{2}\int_0^1 \left(\frac{\partial(\delta Q_k(x,t))}{\partial t}\right)^2\mathrm{d}x + \frac{1}{2}\int_0^1 (e_k(x,t))^2\mathrm{d}x \\ &= \frac{1}{2}\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2 + \frac{1}{2}\|e_k(\cdot,t)\|_{L^2}^2. \end{aligned} \tag{21}$$

Integrating by parts and combining with the boundary conditions of the system (3), we can derive

$$\begin{aligned} &\int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{xxxx}\right\}\mathrm{d}x \\ &= \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{xxx}\right\}\Big|_0^1 - \int_0^1 \{(\delta Q_k(x,t))_{tx}(\delta Q_k(x,t))_{xxx}\}\mathrm{d}x \\ &= -\int_0^1 \{(\delta Q_k(x,t))_{tx}(\delta Q_k(x,t))_{xxx}\}\mathrm{d}x \\ &= -\{(\delta Q_k(x,t))_{tx}(\delta Q_k(x,t))_{xx}\}|_0^1 + \int_0^1 \{(\delta Q_k(x,t))_{txx}(\delta Q_k(x,t))_{xx}\}\mathrm{d}x \\ &= \int_0^1 \{(\delta Q_k(x,t))_{txx}(\delta Q_k(x,t))_{xx}\}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial^2(\delta Q_k(\cdot,t))}{\partial x^2}\right\|_{L^2}^2. \end{aligned} \tag{22}$$

$$\begin{aligned} &\int_0^1 \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{xx}\right\}\mathrm{d}x \\ &= \left\{\frac{\partial(\delta Q_k(x,t))}{\partial t}(\delta Q_k(x,t))_{x}\right\}\Big|_0^1 - \int_0^1 \{(\delta Q_k(x,t))_{tx}(\delta Q_k(x,t))_{x}\}\mathrm{d}x \\ &= -\int_0^1 \{(\delta Q_k(x,t))_{tx}(\delta Q_k(x,t))_{x}\}\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial x}\right\|_{L^2}^2. \end{aligned} \tag{23}$$

Substituting (20)–(23) into (19), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2 + \left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial^2(\delta Q_k(\cdot,t))}{\partial x^2}\right\|_{L^2}^2\right\}$$

$$\leqslant q\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2+q\left\|e_k(\cdot,t)\right\|_{L^2}^2$$
$$\leqslant q\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2+q\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial x}\right\|_{L^2}^2+q\left\|\frac{\partial^2(\delta Q_k(\cdot,t))}{\partial x^2}\right\|_{L^2}^2+q\left\|e_k(\cdot,t)\right\|_{L^2}^2$$
$$=q\left\{\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2+\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial^2(\delta Q_k(\cdot,t))}{\partial x^2}\right\|_{L^2}^2\right\}+q\left\|e_k(\cdot,t)\right\|_{L^2}^2,\quad 0<t\leqslant T.$$

Applying Gronwall lemma and combining with (9) and (10), we have

$$\begin{aligned}
&\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2+\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial^2(\delta Q_k(\cdot,t))}{\partial x^2}\right\|_{L^2}^2\\
&\quad\leqslant q\int_0^t \mathrm{e}^{q(t-\eta)}\left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta\\
&\qquad+\mathrm{e}^{qt}\left\{\left\|\left.\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right|_{t=0}\right\|_{L^2}^2+\left\|\frac{\partial(\delta Q_k(\cdot,0))}{\partial x}\right\|_{L^2}^2+\left\|\frac{\partial^2(\delta Q_k(\cdot,0))}{\partial x^2}\right\|_{L^2}^2\right\}\\
&\quad\leqslant q\mathrm{e}^{Tq}\int_0^t\left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta+4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}\\
&\quad=q\mathrm{e}^{Tq}\int_0^t\mathrm{e}^{\lambda\eta}\mathrm{e}^{-\lambda\eta}\left\|e_k(\cdot,\eta)\right\|_{L^2}^2\mathrm{d}\eta+4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}\\
&\quad\leqslant q\mathrm{e}^{Tq}\int_0^t\mathrm{e}^{\lambda\eta}\mathrm{d}\eta\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}+4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}\\
&\quad=q\mathrm{e}^{Tq}\frac{\mathrm{e}^{\lambda t}-1}{\lambda}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}+4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2\right\}&\leqslant q\mathrm{e}^{Tq}\sup_{t\in[0,T]}\left\{\frac{1-\mathrm{e}^{-\lambda t}}{\lambda}\right\}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}\\
&\quad+\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\right\}4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}\\
&=q\mathrm{e}^{Tq}\frac{1-\mathrm{e}^{-\lambda T}}{\lambda}\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|e_k(\cdot,t)\right\|_{L^2}^2\right\}+4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq}. \qquad (24)
\end{aligned}$$

On the other hand, by using the basic inequality, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2&=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(\delta Q_k(x,t))^2\mathrm{d}x=2\int_0^1\delta Q_k(x,t)\frac{\partial(\delta Q_k(x,t))}{\partial t}\mathrm{d}x\\
&\leqslant\int_0^1(\delta Q_k(x,t))^2\mathrm{d}x+\int_0^1\left(\frac{\partial(\delta Q_k(x,t))}{\partial t}\right)^2\mathrm{d}x\\
&=\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2+\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2,\quad 0<t\leqslant T.
\end{aligned}$$

Applying Gronwall lemma and combining with (9), we can obtain

$$\begin{aligned}
\left\|\delta Q_k(\cdot,t)\right\|_{L^2}^2&\leqslant\int_0^t\left\{\mathrm{e}^{t-\eta}\left\|\frac{\partial(\delta Q_k(\cdot,\eta))}{\partial\eta}\right\|_{L^2}^2\right\}\mathrm{d}\eta+\mathrm{e}^t\left\|\delta Q_k(\cdot,0)\right\|_{L^2}^2\\
&\leqslant\mathrm{e}^t\int_0^t\left\{\mathrm{e}^{(\lambda-1)\eta}\mathrm{e}^{-\lambda\eta}\left\|\frac{\partial(\delta Q_k(\cdot,\eta))}{\partial\eta}\right\|_{L^2}^2\right\}\mathrm{d}\eta+4\varepsilon_2^2\mathrm{e}^T\\
&\leqslant\mathrm{e}^t\int_0^t\mathrm{e}^{(\lambda-1)\eta}\mathrm{d}\eta\sup_{t\in[0,T]}\left\{\mathrm{e}^{-\lambda t}\left\|\frac{\partial(\delta Q_k(\cdot,t))}{\partial t}\right\|_{L^2}^2\right\}+4\varepsilon_2^2\mathrm{e}^T
\end{aligned}$$

$$= \frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{t}}{\lambda - 1} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \left\| \frac{\partial(\delta Q_k(\cdot,t))}{\partial t} \right\|_{L^2}^2 \right\} + 4\varepsilon_2^2 \mathrm{e}^T.$$

Take $\lambda > 1$, then

$$\begin{aligned} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|\delta Q_k(\cdot,t)\|_{L^2}^2 \right\} &\leqslant \sup_{t\in[0,T]} \left\{ \frac{1-\mathrm{e}^{-(\lambda-1)t}}{\lambda-1} \right\} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \left\| \frac{\partial(\delta Q_k(\cdot,t))}{\partial t} \right\|_{L^2}^2 \right\} \\ &\quad + \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \right\} 4\varepsilon_2^2 \mathrm{e}^T \\ &= \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \left\| \frac{\partial(\delta Q_k(\cdot,t))}{\partial t} \right\|_{L^2}^2 \right\} + 4\varepsilon_2^2 \mathrm{e}^T. \end{aligned} \tag{25}$$

Substituting (24) into (25) results

$$\begin{aligned} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|\delta Q_k(\cdot,t)\|_{L^2}^2 \right\} &\leqslant q\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} \frac{1-\mathrm{e}^{-\lambda T}}{\lambda} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|e_k(\cdot,t)\|_{L^2}^2 \right\} \\ &\quad + 4(\varepsilon_3^2 + \varepsilon_2^2)\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} + 4\varepsilon_2^2 \mathrm{e}^T. \end{aligned}$$

Substituting the above expression into (7), it yields

$$\begin{aligned} &\sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|e_{k+1}(\cdot,t)\|_{L^2}^2 \right\} \\ &\leqslant \left\{ (1+\varepsilon)\rho^2 + \left(1+\frac{1}{\varepsilon}\right) C^2 q\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} \frac{1-\mathrm{e}^{-\lambda T}}{\lambda} \right\} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|e_k(\cdot,t)\|_{L^2}^2 \right\} \\ &\quad + \left(4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} + 4\varepsilon_2^2\mathrm{e}^T \right) \left(1+\frac{1}{\varepsilon}\right) C^2 \\ &= \bar{\rho} \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|e_k(\cdot,t)\|_{L^2}^2 \right\} + \left(4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} + 4\varepsilon_2^2\mathrm{e}^T \right) \left(1+\frac{1}{\varepsilon}\right) C^2, \end{aligned}$$

where

$$\bar{\rho} = (1+\varepsilon)\rho^2 + \left(1+\frac{1}{\varepsilon}\right) C^2 q\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} \frac{1-\mathrm{e}^{-\lambda T}}{\lambda}.$$

So, it can be derived by Lemma 1 that

$$\limsup_{k\to\infty} \left\{ \sup_{t\in[0,T]} \left\{ \mathrm{e}^{-\lambda t} \|e_k(\cdot,t)\|_{L^2}^2 \right\} \right\} \leqslant \frac{1}{1-\bar{\rho}} \left(4(\varepsilon_3^2+\varepsilon_2^2)\mathrm{e}^{Tq} \frac{1-\mathrm{e}^{-(\lambda-1)T}}{\lambda-1} + 4\varepsilon_2^2\mathrm{e}^T \right) \left(1+\frac{1}{\varepsilon}\right) C^2.$$

That is, the output tracking error on L^2 space is bounded. Furthermore, if $\varphi_k(x) = \varphi_r(x)$, $\psi_k(x) = \psi_r(x)$, $x \in [0,1]$, $k = 0,1,2,\ldots$, then $\varepsilon_2 = 0$, $\varepsilon_3 = 0$, and we can obtain

$$\lim_{k\to\infty} \|e_k\|_{L^2,s} = 0.$$

This completes the proof.

Remark 4. Ref. [22] studied the initial boundary value problem for a class of one-dimensional fourth order wave equations, and by using potential well method, the global existence of weak solutions, generalized solutions and classical solutions under some assumptions was proved. The following one-dimensional fourth order wave equation was given in [22], with the initial-boundary conditions:

$$\begin{cases} Q_{tt} + Q_{xxxx} = \sigma(Q_x)_x + f(x,t)\,, \ x \in (0,1),\ t > 0, \\ Q(0,t) = Q(1,t) = 0\,,\ t \geqslant 0, \\ Q_x(0,t) = Q_x(1,t) = 0\,,\ t \geqslant 0, \\ Q(x,0) = \varphi(x)\,,\ Q_t(x,0) = \psi(x),\ x \in [0,1]. \end{cases} \tag{26}$$

In (26), we take $t \in [0,T]$ and let the initial conditions $\varphi(x), \psi(x)$ satisfy the corresponding assumptions in Remark 2. Furthermore, take the function $\sigma(s) = s$, then it is easy to see that the system (2) (when $\alpha = 2$) is composed of the fourth order wave equation (26). For $\sigma(s) = s$, we have (i) $\sigma \in C^5(R)$, $\sigma(0) = 0$, $\sigma'(0) \geqslant 0$, $\sigma''(0) = 0$, and $\sigma(s)$ is not identically zero in the neighborhood of the origin; (ii) $\frac{1}{4}s\sigma(s) - \int_0^s \sigma(\rho)\mathrm{d}\rho \leqslant 0$ holds for all $s \in R$. That is, the assumptions for $\sigma(s)$ in [22] are satisfied, see (2.4) and Theorem 4.1 in [22] for detailed results. It follows from Theorem 4.1 in [22] that, if $f(x,t)$ is taken to satisfy $f(x,t) \in H^1((0,T];\ H^4(0,1))$ and $f(0,t) = f(1,t) = \frac{\partial f(x,t)}{\partial x}|_{x=0} = \frac{\partial f(x,t)}{\partial x}|_{x=1} = 0$, then the global classical solution of (26) is existing and unique.

From (3), (5) and Assumption 2, we know that if the initial control $u_0(x,t)$ in (3) is taken to satisfy $u_0(x,t) \in H^1((0,T];\ H^4(0,1))$ and $u_0(0,t) = u_0(1,t) = \frac{\partial u_0(x,t)}{\partial x}|_{x=0} = \frac{\partial u_0(x,t)}{\partial x}|_{x=1} = 0$, then $u_1(x,t) \in H^1((0,T];\ H^4(0,1))$ and $u_1(0,t) = u_1(1,t) = \frac{\partial u_1(x,t)}{\partial x}|_{x=0} = \frac{\partial u_1(x,t)}{\partial x}|_{x=1} = 0$. So, the global classical solution of the system (3) at kth iteration is always existing and unique ($k = 0,1,2,\ldots$).

Remark 5. For Eq. (26), the global classical solutions are selected to be considered in this paper. In fact, if the corresponding assumptions in [22], which can guarantee the global existence of weak solutions and generalized solutions, are adopted in this paper, then the corresponding results may also be obtained.

Remark 6 ([15,16])**.** From Assumption 1, when we choose the learning gain q such that $\frac{\sqrt{1+\varepsilon}-1}{D_1\sqrt{1+\varepsilon}} < q < \frac{\sqrt{1+\varepsilon}+1}{D_2\sqrt{1+\varepsilon}}$, the convergence condition (6) holds. And from (4), the learning gain q satisfying the convergence condition (6) always exists.

5 Simulation examples

(1) Let $\alpha = 1$ and take $g(x,t) = 1$, $C(t) = D(t) = 1$, $T = 0.5$, then the system (2) is as follows:

$$\begin{cases} \dfrac{\partial Q(x,t)}{\partial t} + (Q(x,t))_{xxxx} + Q(x,t) = u(x,t)\ ,\ (x,t) \in \Omega, \\ y(x,t) = Q(x,t) + u(x,t)\ ,\ 0 \leqslant t \leqslant 0.5,\ \ x \in [0,1]. \end{cases}$$

For the given desired trajectory: $y_r(x,t) = 3\mathrm{e}^t x^2(x-1)^2 + 24\mathrm{e}^t$, we have $Q_r(x,t) = \mathrm{e}^t x^2(x-1)^2$, $u_r(x,t) = 2\mathrm{e}^t x^2(x-1)^2 + 24\mathrm{e}^t$.

Construct the kth iteration

$$\begin{cases} \dfrac{\partial Q_k(x,t)}{\partial t} + (Q_k(x,t))_{xxxx} + Q_k(x,t) = u_k(x,t)\ ,\ (x,t) \in \Omega, \\ y_k(x,t) = Q_k(x,t) + u_k(x,t)\ ,\ 0 \leqslant t \leqslant 0.5,\ \ x \in [0,1]. \end{cases}$$

Combining with Assumption 3, we take the initial-boundary values at kth iteration:

$$Q_k(x,0) = (1+\varepsilon_{1k})x^2(x-1)^2, \quad x \in [0,1];$$

$$Q_k(0,t) = Q_k(1,t) = \left.\frac{\partial Q_k(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial Q_k(x,t)}{\partial x}\right|_{x=1} = 0, \quad t \in (0,T].$$

Take the initial control $u_0(x,t) = 22\mathrm{e}^t$, and construct the following ILC:

$$u_{k+1}(x,t) = u_k(x,t) + qe_k(x,t),$$

$k = 0,1,2,\ldots$. Taking $\varepsilon = \frac{9}{16}$, it follows from Remark 6 that the iteration is convergent for $0.2 < q < 1.8$. Therefore, we take $q = 1.5$. By using the mathematical software Mathematica, we have the following cases.

Case 1. ε_{1k} is selected randomly from the interval $[-30,30]$, $k = 0,1,2,\ldots$. The simulation result of $\|e_k\|_{L^2,s}$ with the change of k is shown in Figure 1.

Case 2. $\varepsilon_{1k} = 0$, $k = 0,1,2,\ldots$. The simulation result of $\|e_k\|_{L^2,s}$ with the change of k is shown in Figure 2.

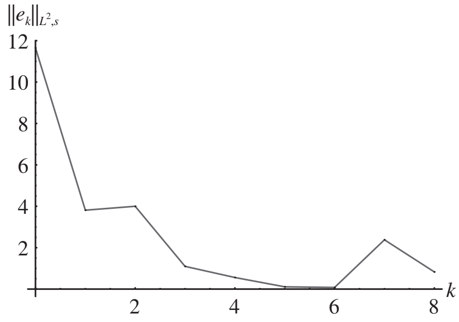

Figure 1 Iterations for the output tracking errors on L^2 space in the presence of initial errors (α= 1).

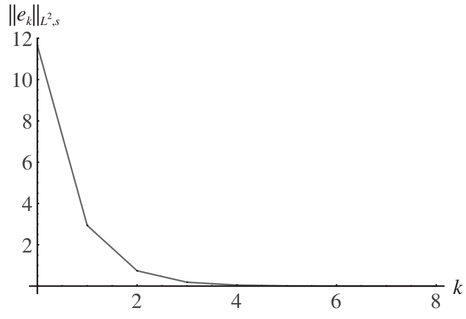

Figure 2 Iterations for the output tracking errors on L^2 space in the absence of initial errors ($\alpha = 1$).

(2) Let $\alpha = 2$ and take $C(t) = D(t) = 1$, $T = 0.5$, then the system (2) is as follows:

$$\begin{cases} \dfrac{\partial^2 Q(x,t)}{\partial t^2} + (Q(x,t))_{xxxx} = (Q(x,t))_{xx} + u(x,t)\,,\ (x,t) \in \Omega, \\ y(x,t) = Q(x,t) + u(x,t)\,,\ 0 \leqslant t \leqslant 0.5,\ x \in [0,1]. \end{cases}$$

For the given desired trajectory: $y_r(x,t) = 0.01\left(2x^4 - 4x^3 - 10x^2 + 12x + 22\right)\mathrm{e}^t$, we have $Q_r(x,t) = 0.01\mathrm{e}^t x^2(x-1)^2$, $u_r(x,t) = 0.01\left(x^4 - 2x^3 - 11x^2 + 12x + 22\right)\mathrm{e}^t$.

Construct the kth iteration

$$\begin{cases} \dfrac{\partial^2 Q_k(x,t)}{\partial t^2} + (Q_k(x,t))_{xxxx} = (Q_k(x,t))_{xx} + u_k(x,t)\,,\ (x,t) \in \Omega, \\ y_k(x,t) = Q_k(x,t) + u_k(x,t)\,,\ 0 \leqslant t \leqslant 0.5,\ x \in [0,1]. \end{cases}$$

Combining with Assumption 3, we take the initial-boundary values at kth iteration:

$$Q_k(x,0) = (0.01 + \varepsilon_{2k})x^2(x-1)^2, \quad \left.\frac{\partial Q_k(x,t)}{\partial t}\right|_{t=0} = (0.01 + \varepsilon_{3k})x^2(x-1)^2, \quad x \in [0,1];$$

$$Q_k(0,t) = Q_k(1,t) = \left.\frac{\partial Q_k(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial Q_k(x,t)}{\partial x}\right|_{x=1} = 0, \quad t \in (0,T].$$

Take the initial control $u_0(x,t) = 0.22\mathrm{e}^t$, and construct the following ILC:

$$u_{k+1}(x,t) = u_k(x,t) + qe_k(x,t),$$

$k = 0, 1, 2, \ldots$. Taking $\varepsilon = \frac{9}{16}$, it follows from Remark 6 that the iteration is convergent for $0.2 < q < 1.8$. Therefore, we take $q = 1.5$. By using the mathematical software Mathematica, we have the following cases.

Case 1. ε_{2k} and ε_{3k} are selected randomly from the interval $[-0.1, 0.1]$, $k = 0, 1, 2, \ldots$. The simulation result of $\|e_k\|_{L^2,s}$ with the change of k is shown in Figure 3.

Case 2. $\varepsilon_{2k} = \varepsilon_{3k} = 0$, $k = 0, 1, 2, \ldots$. The simulation result of $\|e_k\|_{L^2,s}$ with the change of k is shown in Figure 4.

6 Conclusion

This paper considers the ILC problem for a class of high order DPSs in the presence of initial errors. And the considered DPSs are composed of the one-dimensional fourth order parabolic PDEs or the one-dimensional fourth order wave equations. According to the characteristics of the systems, the ILC laws are constructed based on the P-type learning scheme. When the learning scheme is applied to the systems, the output tracking errors on L^2 space are bounded, and furthermore, the tracking errors on L^2

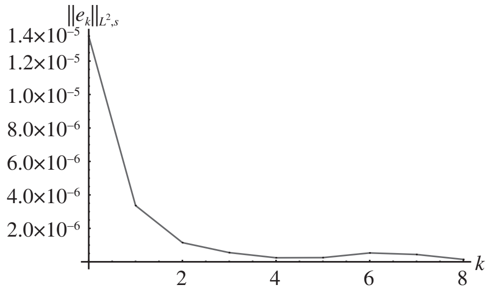

Figure 3 Iterations for the output tracking errors on L^2 space in the presence of initial errors ($\alpha = 2$).

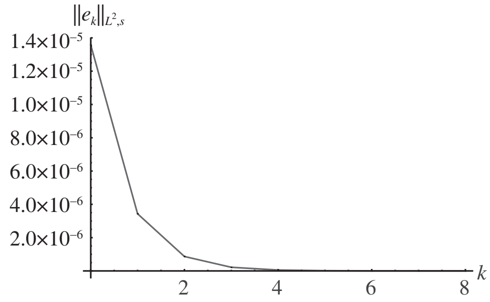

Figure 4 Iterations for the output tracking errors on L^2 space in the absence of initial errors ($\alpha = 2$).

space can tend to zero along the iteration axis in the absence of initial errors. The simulation results are consistent with theoretical analysis. Since the DPSs involved in ILC have been low order (first order or second order) until now, the result of this paper extends the range of the application of ILC design for DPSs to some extent.

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