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Output reachability analysis and output regulation control design of Boolean control networks

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Abstract This paper investigates the output reachability and output regulation control design of Boolean control networks (BCNs) by using the semi-tensor product method, and presents a number of new results. First, the concept of output reachability is proposed for BCNs, and some necessary and sufficient conditions are presented for the verification of output reachability. Second, based on the output reachability of BCNs and the attractor set of the reference Boolean network, an effective method is proposed for the control design of the output regulation problem. The study of an illustrative example shows the effectiveness of the obtained new results.

Keywords Boolean control network, output reachability, output regulation, control design, semi-tensor product of matrices

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1 Introduction

Output regulation is a fundamental issue in the systems control theory, which aims to design a feedback control such that the output of the closed-loop system tracks reference signals produced by an exosystem. In the last few decades, the output regulation problem has been well studied by lots of scientists [1–3]. In [1], the output regulation problem of linear systems was converted into an eigenvalue placement problem for an augmented linear system via the internal model principle. Huang and Chen [2] established a general framework which converts the robust output regulation problem of nonlinear system into a robust stabilization problem for an appropriately augmented system. It is noted that the output regulation problem is also important for the study of genetic regulatory networks [4]. For example, Julius et al. [4] proposed a novel feedback control design procedure to make the fraction of induced cells in the Escherichia coli bacteria attain a desired level. This is a typical example of the output regulation for genetic regulatory networks.

As a suitable model of genetic regulatory networks, Boolean networks have attracted a great attention from many scholars in the last few decades. The control of Boolean networks is an important issue in both systems theory and medical science [5]. Recently, a semi-tensor product method has been proposed for the

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analysis and control of Boolean networks [6, 7]. The main feature of this method is that one can convert the dynamics of a Boolean control network (BCN) into a bilinear discrete-time system [8], and then one can tackle BCNs by using the classical control theory. Using this novel method, many interesting results have been established for the analysis and control of Boolean networks [9–28]. The semi-tensor product method has also been applied to the modeling, analysis and optimization of networked evolutionary games [29, 30].

It should be pointed out that for BCNs, the output regulation problem was also studied in [31–33]. The output regulation of BCNs to a constant reference signal was studied in [31, 32], and some effective control design methods were presented. The output regulation of BCNs to reference signals produced by some external Boolean network was investigated in [33], and a necessary and sufficient condition was presented for the solvability of the problem. However, the result obtained in [33] cannot be applied to the output regulation control design. Besides, since the reference signals produced by the external Boolean network are time-varying, the methods proposed in [31, 32] can hardly be used for the control design of the above output regulation problem.

In this paper, using the semi-tensor product method, we firstly analyze the output reachability of BCNs, based on which, we then propose an effective method for the control design of the output regulation problem. The main contributions of this paper are as follows. (i) Some necessary and sufficient conditions are presented for the output reachability of BCNs, which are crucial for the control design of the output regulation problem. (ii) A novel method is proposed for the control design of the output regulation problem based on the output reachability of BCNs and the attractor set of the reference Boolean network, which is very effective in dealing with time-varying reference signals.

The rest of this paper is organized as follows. Section 2 gives some preliminary results. In Section 3, we study the output reachability and the control design of the output regulation problem for BCNs, and present the main results of this paper. An example is worked out to illustrate our new results in Section 4, which is followed by some concluding remarks in Section 5.

Notation:

$$\mathcal{D} := \{1, 0\}, \mathcal{D}^n := \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n}, \Delta_n := \{\delta_n^k : 1 \leqslant k \leqslant n\},$$

where δ_n^k is the kth column vector of the identity matrix I_n. For compactness, $\Delta := \Delta_2$. An $n \times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$, which is briefly expressed as $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$. All the $n \times t$ logical matrices form a set $\mathcal{L}_{n\times t}$. Denote $Blk_i(A)$ by the ith $n \times n$ block of an $n \times mn$ matrix A. Given a real matrix $A \in \mathbb{R}^{n\times m}$, $(A)_{i,j}$, $Col_i(A)$ and $Row_i(A)$ denote the (i, j)th element of A, the ith column of A, and the ith row of A, respectively. $A > 0$, if $(A)_{i,j} > 0$ is satisfied for any i and j.

2 Preliminaries

Given two real matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, let $\alpha = lcm(n, p)$ be the least common multiple of n and p. Then, the semi-tensor product of A and B is

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{1}$$

where $\otimes$ is the Kronecker product.

It is well known that the semi-tensor product of matrices is a generalization of the conventional matrix product, which keeps all the properties of the conventional matrix product. In the following, we will omit the symbol "$\ltimes$" if no confusion arises.

Unlike the conventional matrix product, the semi-tensor product of matrices has the following pseudo-commutative law.

Lemma 1 ([6]). Let $X \in \mathbb{R}^{t\times 1}$ be a column vector and $A \in \mathbb{R}^{m\times n}$. Then

$$X \ltimes A = (I_t \otimes A) \ltimes X. \tag{2}$$

Using the semi-tensor product of matrices, one can convert a Boolean function into a matrix expression. To do this, we identify $\Delta \sim \mathcal{D}$, where "$\sim$" denotes two different forms of the same object. We have the following result.

Lemma 2 ([6]). Let $f(x_1, x_2, \ldots, x_s) : \mathcal{D}^s \mapsto \mathcal{D}$ be a Boolean function. Then

$$f(x_1, x_2, \ldots, x_s) = M_f \ltimes_{i=1}^{s} x_i, \quad x_i \in \Delta, \tag{3}$$

where $M_f \in \mathcal{L}_{2\times 2^s}$ is called the structural matrix of f.

For example, the structural matrices of Negation ($\neg$), Conjunction ($\wedge$) and Disjunction ($\vee$) are $M_n = \delta_2[2\ 1]$, $M_c = \delta_2[1\ 2\ 2\ 2]$ and $M_d = \delta_2[1\ 1\ 1\ 2]$, respectively.

3 Main results

In this section, we firstly analyze the output reachability of BCNs, and present some necessary and sufficient conditions, based on which, we then propose an effective method for the control design of the output regulation problem.

3.1 Output reachability analysis of BCNs

Consider the following Boolean control network:

$$\begin{cases} x_1(t+1) = f_1(X(t), U(t)), \\ x_2(t+1) = f_2(X(t), U(t)), \\ \quad \cdots \\ x_n(t+1) = f_n(X(t), U(t)); \\ y_j(t) = h_j(X(t)), \quad j = 1, \ldots, p, \end{cases} \tag{4}$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathcal{D}^n$, $U(t) = (u_1(t), \ldots, u_m(t)) \in \mathcal{D}^m$ and $Y(t) = (y_1(t), \ldots, y_p(t)) \in \mathcal{D}^p$ are the state, the control input and the output of the system (4), respectively, and $f_i : \mathcal{D}^{m+n} \mapsto \mathcal{D}$, $i = 1, \ldots, n$ and $h_j : \mathcal{D}^n \mapsto \mathcal{D}$, $j = 1, \ldots, p$ are Boolean functions. Given a control sequence $\{U(t) : t \in \mathbb{N}\}$, denote the state trajectory of the system (4) starting from an initial state $X(0) \in \mathcal{D}^n$ by $X(t; X(0), U)$, and the output trajectory of the system (4) starting from $X(0) \in \mathcal{D}^n$ by $Y(t; X(0), U)$.

Using the vector form of Boolean values and setting $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^{m} u_i(t) \in \Delta_{2^m}$ and $y(t) = \ltimes_{i=1}^{p} y_i(t) \in \Delta_{2^p}$, by Lemma 2, one can convert (4) into the following algebraic form:

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases} \tag{5}$$

where $L \in \mathcal{L}_{2^n \times 2^{m+n}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

Now, we define the concept of output-reachability for BCNs as follows.

Definition 1. Consider the system (5). $y_f \in \Delta_{2^p}$ is said to be s-output-reachable from the initial state $x(0) \in \Delta_{2^n}$, if one can find a control sequence $u(0), \ldots, u(s-1) \in \Delta_{2^m}$, under which $y(s; x(0), u(0), \ldots, u(s-1)) = y_f$. y_f is said to be s-output-reachable from the initial state set Δ_{2^n}, if y_f is s-output-reachable from any $x(0) \in \Delta_{2^n}$. y_f is said to be output-reachable from the initial state $x(0) \in \Delta_{2^n}$, if one can find an integer $s > 0$ such that y_f is s-output-reachable from $x(0)$. $y_f \in \Delta_{2^p}$ is said to be output-reachable from the initial state set Δ_{2^n}, if y_f is output-reachable from any $x(0) \in \Delta_{2^n}$.

In order to give a necessary and sufficient condition for the output-reachability of BCNs, we recall a useful result on the reachability of BCNs. For details, please refer to [34].

Lemma 3. Consider the system (5). Let $x_f = \delta_{2^n}^{q}$ and $x(0) = \delta_{2^n}^{r}$ be given. Then,

1) x_f is reachable from $x(0)$ at time s, if and only if

$$(M^s)_{q,r} > 0, \tag{6}$$

where

$$M = \sum_{i=1}^{2^m} Blk_i(L); \tag{7}$$

2) x_f is reachable from $x(0)$, if and only if

$$\sum_{s=1}^{2^{m+n}} (M^s)_{q,r} > 0. \tag{8}$$

Based on Definition 1 and Lemma 3, we have the following result.

Theorem 1. Consider the system (5). Let $y_f = \delta_{2^p}^k$ and $x(0) = \delta_{2^n}^l$ be given. Then,

1) y_f is s-output-reachable from $x(0)$, if and only if

$$(HM^s)_{k,l} > 0; \tag{9}$$

2) y_f is s-output-reachable from Δ_{2^n}, if and only if

$$Row_k(HM^s) > 0; \tag{10}$$

3) y_f is output-reachable from $x(0)$, if and only if

$$\sum_{s=1}^{2^{m+n}} (HM^s)_{k,l} > 0; \tag{11}$$

4) y_f is output-reachable from Δ_{2^n}, if and only if

$$\sum_{s=1}^{2^{m+n}} Row_k(HM^s) > 0. \tag{12}$$

Proof. We just need to prove Conclusions 1) and 3). Conclusion 2) follows from Definition 1 and Conclusion 1), while Conclusion 4) follows from Definition 1 and Conclusion 3).

Firstly, we prove Conclusion 1).

(Sufficiency) Suppose that (9) holds. Since

$$(HM^s)_{k,l} = \sum_{i=1}^{2^n} (H)_{k,i}(M^s)_{i,l}, \tag{13}$$

there must exist an integer $1 \leqslant i_0 \leqslant 2^n$ such that $(H)_{k,i_0} = 1$ and $(M^s)_{i_0,l} > 0$.

By Lemma 3, $\delta_{2^n}^{i_0}$ is reachable from $\delta_{2^n}^l$ at time s, that is, one can find a control sequence $u(0), \dots, u(s-1) \in \Delta_{2^m}$ such that

$$x(s; \delta_{2^n}^l, u(0), \dots, u(s-1)) = \delta_{2^n}^{i_0}.$$

Thus,

$$y(s; \delta_{2^n}^l, u(0), \dots, u(s-1)) = H\delta_{2^n}^{i_0} = Col_{i_0}(H) = \delta_{2^p}^k,$$

which implies that $\delta_{2^p}^k$ is s-output-reachable from $x(0)$.

(Necessity) Assume that $\delta_{2^p}^k$ is s-output-reachable from $\delta_{2^n}^l$. Then, there exists a control sequence $u(0), \dots, u(s-1) \in \Delta_{2^m}$ such that $y(s; \delta_{2^n}^l, u(0), \dots, u(s-1)) = \delta_{2^p}^k$.

Since

$$y(s; \delta_{2^n}^l, u(0), \dots, u(s-1)) = Hx(s; \delta_{2^n}^l, u(0), \dots, u(s-1)),$$

setting $x(s; \delta_{2^n}^l, u(0), \dots, u(s-1)) = \delta_{2^n}^{i_0}$, one can obtain that $Col_{i_0}(H) = \delta_{2^p}^k$, and $\delta_{2^n}^{i_0}$ is reachable from $\delta_{2^n}^l$ at time s, that is, $(H)_{k,i_0} = 1$ and $(M^s)_{i_0,l} > 0$.

Thus,

$$(HM^s)_{k,l} = \sum_{i=1}^{2^n} (H)_{k,i}(M^s)_{i,l} \geqslant (H)_{k,i_0}(M^s)_{i_0,l} > 0,$$

which implies that (9) holds.

Next, we prove Conclusion 3).

(Sufficiency) Suppose that (11) holds. Then, there exists an integer $1 \leqslant s \leqslant 2^{m+n}$ such that $(HM^s)_{k,l} > 0$. By Conclusion 1), y_f is s-output-reachable from $x(0)$. Thus, y_f is output-reachable from $x(0)$.

(Necessity) Assume that y_f is output-reachable from $x(0)$. Then, there exists an integer $s > 0$ such that $(HM^s)_{k,l} > 0$. From (13), there must exist an integer $1 \leqslant i_0 \leqslant 2^n$ such that $(H)_{k,i_0} = 1$ and $(M^s)_{i_0,l} > 0$. By Lemma 3, we only need to consider $1 \leqslant s \leqslant 2^{m+n}$.

Therefore,

$$\sum_{j=1}^{2^{m+n}} (HM^j)_{k,l} \geqslant (HM^s)_{k,l} > 0,$$

which implies that (11) holds.

Finally, suppose that $\delta_{2^p}^k$ is s-output-reachable from $\delta_{2^n}^l$. From the proof of Theorem 1, one can design a control sequence to realize the output-reachability by Algorithm 1.

Algorithm 1

1) Find an integer $1 \leqslant i_0 \leqslant 2^n$ such that $(H)_{k,i_0} = 1$ and $(M^s)_{i_0,l} > 0$;
2) Find an integer $1 \leqslant \alpha \leqslant 2^m$ such that $(Blk_\alpha(M^{s-1}L))_{i_0,l} > 0$. Set $u(0) = \delta_{2^m}^\alpha$. If $s = 1$, stop. Otherwise, go to the next step;
3) Find two integers $1 \leqslant j \leqslant 2^n$ and $1 \leqslant \beta \leqslant 2^m$ such that $(Blk_\beta(L))_{i_0,j} > 0$ and $(Blk_\alpha(M^{s-2}L))_{j,l} > 0$. Set $u(s-1) = \delta_{2^m}^\beta$ and $x(s-1) = \delta_{2^n}^j$. If $s - 1 = 1$, stop. Otherwise, replace s and i_0 by $s - 1$ and j, respectively, and go to 2).

Remark 1. The main differences between the state reachability [34] and output reachability of BCNs are shown below.

- The state reachability reflects the possibility of reachability between two states in the state space, while the output reachability reflects the possibility of reachability between a state in the state space and an output in the output space.
- The state reachability can be applied to the state feedback control design of BCNs (see [15]), while the output reachability can be applied to the output related control problem of BCNs (see [12] and Theorem 4 below).

3.2 Output regulation control design of BCNs

In this subsection, based on the output-reachability of BCNs, we study how to design the state feedback gain for the output regulation of BCNs. To this end, we give the dynamics of the reference Boolean network as follows.

$$\begin{cases} \widehat{x}_1(t+1) = \widehat{f}_1(\widehat{X}(t)), \\ \widehat{x}_2(t+1) = \widehat{f}_2(\widehat{X}(t)), \\ \quad \cdots \\ \widehat{x}_{n_1}(t+1) = \widehat{f}_{n_1}(\widehat{X}(t)); \\ \widehat{y}_j(t) = \widehat{h}_j(\widehat{X}(t)), \quad j = 1, \ldots, p, \end{cases} \tag{14}$$

where $\widehat{X}(t) = (\widehat{x}_1(t), \widehat{x}_2(t), \ldots, \widehat{x}_{n_1}(t)) \in \mathcal{D}^{n_1}$ and $\widehat{Y}(t) = (\widehat{y}_1(t), \ldots, \widehat{y}_p(t)) \in \mathcal{D}^p$ are the state and the output of the system (14), respectively, and $\widehat{f}_i : \mathcal{D}^{n_1} \mapsto \mathcal{D}$, $i = 1, \ldots, n_1$ and $\widehat{h}_j : \mathcal{D}^{n_1} \mapsto \mathcal{D}$, $j = 1, \ldots, p$ are Boolean functions. Given an initial state $\widehat{X}(0) \in \mathcal{D}^{n_1}$, the state trajectory of the system (14) is denoted by $\widehat{X}(t; \widehat{X}(0))$, and the output trajectory of the system (14) is denoted by $\widehat{Y}(t; \widehat{X}(0))$.

The output regulation problem is to design a state feedback control in the form of

$$\begin{cases} u_1(t) = g_1(X(t), \widehat{X}(t)), \\ \quad \cdots \\ u_m(t) = g_m(X(t), \widehat{X}(t)), \end{cases} \tag{15}$$

where $g_i : \mathcal{D}^{n+n_1} \mapsto \mathcal{D}$, $i = 1, \dots, m$ are Boolean functions, under which there exists an integer $\tau > 0$ such that

$$Y(t; X_0, U) = \widehat{Y}(t; \widehat{X}_0)$$

holds for $\forall\, t \geqslant \tau$, $\forall\, X_0 \in \mathcal{D}^n$ and $\forall\, \widehat{X}_0 \in \mathcal{D}^{n_1}$.

Using the vector form of Boolean values and setting $\widehat{x}(t) = \ltimes_{i=1}^{n_1} \widehat{x}_i(t) \in \Delta_{2^{n_1}}$ and $\widehat{y}(t) = \ltimes_{i=1}^{p} \widehat{y}_i(t) \in \Delta_{2^p}$, the system (14) and the control (15) can be converted to

$$\begin{cases} \widehat{x}(t+1) = \widehat{L}\widehat{x}(t), \\ \widehat{y}(t) = \widehat{H}\widehat{x}(t), \end{cases} \tag{16}$$

and

$$u(t) = Gx(t)\widehat{x}(t), \tag{17}$$

respectively, where $\widehat{L} \in \mathcal{L}_{2^{n_1} \times 2^{n_1}}$, $\widehat{H} \in \mathcal{L}_{2^p \times 2^{n_1}}$ and $G \in \mathcal{L}_{2^m \times 2^{n+n_1}}$. Hence, the output regulation problem becomes the design of the state feedback gain matrix $G \in \mathcal{L}_{2^m \times 2^{n+n_1}}$.

In the following, based on the output-reachability of BCNs, we study how to design the state feedback gain for the output regulation problem.

For the system (16), starting from any initial state $\widehat{x}(0) \in \Delta_{2^{n_1}}$, the state trajectory will converge to a fixed point or a cycle. Correspondingly, the output trajectory will converge to a constant output or a set of periodic outputs. Thus, to design the state feedback gain for the output regulation problem, we just need to consider the states of the system (16) belonging to the attractor set. We firstly consider the following two special cases:

• Case I: The attractor set of the system (16) only has fixed points. Denote the set of fixed points by $\Pi = \{\delta_{2^{n_1}}^{\pi_1}, \dots, \delta_{2^{n_1}}^{\pi_q}\}$.

• Case II: The attractor set of the system (16) only has cycles with length greater than 1. Denote the set of cycles by $\Gamma = \{C_1, \dots, C_r\}$, where $C_i = \{\delta_{2^{n_1}}^{\gamma_1^i}, \dots, \delta_{2^{n_1}}^{\gamma_{d_i}^i}\}$ is a cycle with length d_i, $i = 1, \dots, r$. We assume that $\delta_{2^{n_1}}^{\gamma_1^i} \to \cdots \to \delta_{2^{n_1}}^{\gamma_{d_i}^i} \to \delta_{2^{n_1}}^{\gamma_1^i}$.

Consider the system (5) with $L = \delta_{2^n}[l_1\ l_2\ \cdots\ l_{2^{m+n}}]$. For each integer $1 \leqslant j \leqslant 2^p$, define

$$O(\delta_{2^p}^j) = \{\delta_{2^n}^i : Col_i(H) = \delta_{2^p}^j\}. \tag{18}$$

For a nonempty set $S \subseteq \Delta_{2^n}$ and an integer $k \in \mathbb{Z}_+$, denote by $R_k(S)$ the set of all the initial states which reach S at the kth step, that is,

$$R_k(S) = \{x_0 \in \Delta_{2^n} : \text{there exist } u(0), \dots, u(k-1) \in \Delta_{2^m} \text{ such that } x(k; x_0, u(0), \dots, u(k-1)) \in S\}. \tag{19}$$

Then, for Case I, we have the following result.

Theorem 2. The output regulation problem is solvable for Case I, if and only if the following two conditions hold:

1) $O(\widehat{H}\delta_{2^{n_1}}^{\pi_i}) \neq \emptyset$, $\forall\, i = 1, \dots, q$;

2) for each $i \in \{1, \dots, q\}$, there exist a nonempty set $S_i \subseteq O(\widehat{H}\delta_{2^{n_1}}^{\pi_i})$ and an integer $1 \leqslant \tau_i \leqslant 2^n$ such that

$$S_i \subseteq R_1(S_i) \text{ and } R_{\tau_i}(S_i) = \Delta_{2^n}. \tag{20}$$

Proof. (Sufficiency) Assuming that Conditions 1) and 2) hold, we prove that the output regulation problem is solvable for Case I by a constructed state feedback control.

For each $i \in \{1, \ldots, q\}$, set

$$R_k^{\circ}(S_i) = R_k(S_i) \setminus R_{k-1}(S_i), \quad k = 1, \ldots, \tau_i, \tag{21}$$

where $R_0(S_i) := \emptyset$. Then, it is easy to see that $R_{k_1}^{\circ}(S_i) \bigcap R_{k_2}^{\circ}(S_i) = \emptyset$, $\forall\, k_1, k_2 \in \{1, \ldots, \tau_i\}, k_1 \neq k_2$, and $\bigcup_{k=1}^{\tau_i} R_k^{\circ}(S_i) = \Delta_{2^n}$. Thus, for any integer $1 \leqslant j \leqslant 2^n$, there exists a unique integer $1 \leqslant k_j^i \leqslant \tau_i$ such that $\delta_{2^n}^j \in R_{k_j^i}^{\circ}(S_i)$.

For $k_j^i = 1$, there exists an integer $1 \leqslant \theta_j^i \leqslant 2^m$ such that

$$L \ltimes \delta_{2^m}^{\theta_j^i} \ltimes \delta_{2^n}^j = \delta_{2^n}^{l_{(\theta_j^i - 1)2^n + j}} \in S_i.$$

For $2 \leqslant k_j^i \leqslant \tau_i$, there exists an integer $1 \leqslant \theta_j^i \leqslant 2^m$ such that $\delta_{2^n}^{l_{(\theta_j^i - 1)2^n + j}} \in R_{k_j^i - 1}(S_i)$.

Now, we set $G = \delta_{2^m}[v_1\ v_2\ \cdots\ v_{2^{n+n_1}}] \in \mathcal{L}_{2^m \times 2^{n+n_1}}$, where

$$\begin{cases} v_k = \theta_j^i, & \text{for } k = (j-1)2^{n_1} + \pi_i,\ i = 1, \ldots, q,\ j = 1, \ldots, 2^n, \\ v_k \in \{1, 2, \ldots, 2^m\}, & \text{otherwise.} \end{cases} \tag{22}$$

Then, starting from any $x(0) \in \Delta_{2^n}$ and any $\widehat{x}(0) \in \Delta_{2^{n_1}}$, under the control $u(t) = Gx(t)\widehat{x}(t)$, there exist three integers $1 \leqslant \sigma \leqslant 2^{n_1}$, $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant 2^n$ such that

$$\begin{aligned} &\widehat{x}(t; \widehat{x}(0)) = \delta_{2^{n_1}}^{\pi_i},\ \forall\, t \geqslant \sigma, \\ &x(\sigma; x(0), u) = \delta_{2^n}^j, \\ &x(\sigma + k_j^i; x(0), u) \in S_i. \end{aligned}$$

Since $S_i \subseteq R_1(S_i)$, one can see that

$$x(t; x(0), u) \in S_i,\ \forall\, t \geqslant \sigma + k_j^i,$$

which implies that

$$y(t; x(0), u) = Hx(t; x(0), u) = \widehat{H}\delta_{2^{n_1}}^{\pi_i} = \widehat{y}(t; \widehat{x}(0)), \quad \forall\, t \geqslant \sigma + k_j^i.$$

From the arbitrariness of $x(0)$ and $\widehat{x}(0)$, one can see that the output regulation problem is solvable for Case I by $u(t) = Gx(t)\widehat{x}(t)$.

(Necessity) Suppose that the output regulation problem is solvable for Case I by a state feedback control, say, $u(t) = Gx(t)\widehat{x}(t)$, $G \in \mathcal{L}_{2^m \times 2^{n+n_1}}$. Then, the closed-loop system consisting of the system (5) and the control $u(t) = Gx(t)\widehat{x}(t)$ becomes

$$\begin{cases} x(t+1) = \widetilde{L}(\widehat{x}(t))x(t), \\ y(t) = Hx(t), \end{cases} \tag{23}$$

where $\widetilde{L}(\widehat{x}(t)) = LG(I_{2^n} \otimes \widehat{x}(t))M_{r,2^n}$.

Firstly, we prove that Condition 1) holds. In fact, if Condition 1) is not true, then there exists an integer $1 \leqslant i \leqslant q$ such that $O(\widehat{H}\delta_{2^{n_1}}^{\pi_i}) = \emptyset$, that is, $y(t; x(0), u) \neq \widehat{H}\delta_{2^{n_1}}^{\pi_i}$ holds for any $t \in \mathbb{N}$, any $x(0) \in \Delta_{2^n}$ and any control sequence $\{u(t) : t \in \mathbb{N}\}$. On the other hand, $\widehat{y}(t; \delta_{2^{n_1}}^{\pi_i}) = \widehat{H}\delta_{2^{n_1}}^{\pi_i}$, $\forall\, t \in \mathbb{N}$. Thus, starting from $\widehat{x}(0) = \delta_{2^{n_1}}^{\pi_i}$ and under the control $u(t) = Gx(t)\widehat{x}(t)$, we have $y(t; x(0), u) \neq \widehat{y}(t; \widehat{x}(0))$, $\forall\, x(0) \in \Delta_{2^n}$, $\forall\, t \in \mathbb{N}$, which is a contradiction to the solvability of the output regulation problem by $u(t) = Gx(t)\widehat{x}(t)$. Hence, Condition 1) holds.

Next, we prove that Condition 2) holds.

For the system (16), starting from any $\widehat{x}(0) = \delta_{2^{n_1}}^{\pi_i}$, $i = 1, \ldots, q$, it is easy to see that $\widehat{x}(t; \widehat{x}(0)) = \delta_{2^{n_1}}^{\pi_i}$, $\forall\, t \in \mathbb{N}$. In this case, $\widetilde{L}(\widehat{x}(t)) = LG(I_{2^n} \otimes \delta_{2^{n_1}}^{\pi_i})M_{r,2^n}$, $\forall\, t \in \mathbb{N}$.

For each $i \in \{1, \ldots, q\}$, denote all the states belonging to the attractor set of the system (23) with $\widetilde{L}(\widehat{x}(t)) \equiv LG(I_{2^n} \otimes \delta_{2^{n_1}}^{\pi_i})M_{r,2^n}$ by S_i. In addition, let $1 \leqslant \tau_i \leqslant 2^n$ be the transient period [6] of the system (23) with $\widetilde{L}(\widehat{x}(t)) \equiv LG(I_{2^n} \otimes \delta_{2^{n_1}}^{\pi_i})M_{r,2^n}$. Obviously, $S_i \subseteq R_1(S_i)$ and $R_{\tau_i}(S_i) = \Delta_{2^n}$. We just need to prove that $S_i \subseteq O(\widehat{H}\delta_{2^{n_1}}^{\pi_i})$.

In fact, if $S_i \nsubseteq O(\widehat{H}\delta_{2^{n_1}}^{\pi_i})$, then there exists $\delta_{2^n}^j \in S_i$ such that $H\delta_{2^n}^j \neq \widehat{H}\delta_{2^{n_1}}^{\pi_i}$. Since $\delta_{2^n}^j$ is a fixed point or a state belonging to some cycle of the system (23) with $\widetilde{L}(\widehat{x}(t)) \equiv LG(I_{2^n} \otimes \delta_{2^{n_1}}^{\pi_i})M_{r,2^n}$, there exists a positive integer T such that $\delta_{2^n}^j = x(kT; \delta_{2^n}^j)$ holds for all $k \in \mathbb{N}$. Thus, $y(kT; \delta_{2^n}^j) = Hx(kT; \delta_{2^n}^j) \neq \widehat{H}\delta_{2^{n_1}}^{\pi_i}$, $\forall\, k \in \mathbb{N}$, which is a contradiction to the solvability of the output regulation problem by $u(t) = Gx(t)\widehat{x}(t)$.

Therefore, Condition 2) holds for S_i and τ_i. This completes the proof.

In what follows, we study Case II.

For the system (16), starting from any initial state $\widehat{x}(0) \in \Delta_{2^{n_1}}$, the state trajectory will converge to a cycle, say $C_i = \{\delta_{2^{n_1}}^{\gamma_1^i}, \ldots, \delta_{2^{n_1}}^{\gamma_{d_i}^i}\}$. Then, the output trajectory also converges to a set of periodic outputs $\widehat{O}(C_i) := \{\widehat{H}\delta_{2^{n_1}}^{\gamma_1^i}, \ldots, \widehat{H}\delta_{2^{n_1}}^{\gamma_{d_i}^i}\}$. Thus, if the output regulation problem is solvable for Case II, there exists a state feedback control such that the output trajectory of the system (5) starting from any $x(0) \in \Delta_{2^n}$ converges to $\widehat{O}(C_i)$, $i = 1, \ldots, r$. Hence, a necessary condition for the solvability of the output regulation problem for Case II is as follows.

Proposition 1. If the output regulation problem is solvable for Case II, then $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$, $i = 1, \ldots, r$, $j = 1, \ldots, d_i$ are output-reachable from Δ_{2^n}.

However, the condition presented in Proposition 1 is not sufficient. For example, consider the systems (5) and (16) with $L = \delta_4[1\ 1\ 2\ 1\ 3\ 4\ 4\ 4]$, $H = \delta_2[1\ 1\ 1\ 2]$, $\widehat{L} = \delta_4[2\ 3\ 2\ 1]$ and $\widehat{H} = \delta_2[1\ 1\ 2\ 1]$. Obviously, the attractor set of the system (16) only has a cycle with length 2, that is, $C_1 = \{\delta_4^2, \delta_4^3\}$, and $\widehat{O}(C_1) = \{\delta_2^1, \delta_2^2\}$. A simple calculation shows that $\sum_{s=1}^{8} Row_i(HM^s) > 0$, $i = 1, 2$. By Theorem 1, both δ_2^1 and δ_2^2 are output-reachable from Δ_4. However, one can easily check all $G \in \mathcal{L}_{2\times 16}$ and find that the output regulation problem is not solvable for this example.

Now, we give a sufficient condition for the solvability of the output regulation problem for Case II as follows.

Theorem 3. The output regulation problem is solvable for Case II, if $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$, $i = 1, \ldots, r$, $j = 1, \ldots, d_i$ are 1-output-reachable from Δ_{2^n}.

Proof. Suppose that $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$, $i = 1, \ldots, r$, $j = 1, \ldots, d_i$ are 1-output-reachable from Δ_{2^n}. Then, for any $i \in \{1, \ldots, r\}$, any $j \in \{1, \ldots, d_i\}$ and any $k \in \{1, 2, \ldots, 2^n\}$, from Algorithm 1, one can design a control $\delta_{2^m}^{\eta_{i,j,k}}$ such that $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$ is 1-output-reachable from $\delta_{2^n}^k$.

We set $G = \delta_{2^m}[v_1\ v_2\ \cdots\ v_{2^{n+n_1}}] \in \mathcal{L}_{2^m \times 2^{n+n_1}}$, where

$$\begin{cases} v_\mu = \eta_{i,j,k}, & \text{for } \mu = (k-1)2^{n_1} + \gamma_{j-1}^i,\ i = 1, \ldots, r,\ j = 1, \ldots, d_i,\ k = 1, \ldots, 2^n, \\ v_\mu \in \{1, 2, \ldots, 2^m\}, & \text{otherwise}, \end{cases} \tag{24}$$

and $\gamma_0^i := \gamma_{d_i}^i$. Then, starting from any $x(0) \in \Delta_{2^n}$ and any $\widehat{x}(0) \in \Delta_{2^{n_1}}$, for the system (16), there exist three integers $1 \leqslant \sigma \leqslant 2^{n_1}$, $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant d_i$ such that $\widehat{x}(\sigma; \widehat{x}(0)) = \delta_{2^{n_1}}^{\gamma_j^i}$; for the system (5), under the control $u(t) = Gx(t)\widehat{x}(t)$, there exists an integer $1 \leqslant k \leqslant 2^n$ such that $x(\sigma - 1; x(0), u) = \delta_{2^n}^k$. Since $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$ is 1-output-reachable from $\delta_{2^n}^k$ under the control $\delta_{2^m}^{\eta_{i,j,k}}$, we have

$$y(\sigma; x(0), u) = H \ltimes L \ltimes G \ltimes \delta_{2^n}^k \ltimes \delta_{2^{n_1}}^{\gamma_{j-1}^i} \ltimes \delta_{2^n}^k = HL\delta_{2^m}^{\eta_{i,j,k}}\delta_{2^n}^k = \widehat{H}\delta_{2^{n_1}}^{\gamma_j^i} = \widehat{y}(\sigma; \widehat{x}(0)).$$

Similarly, since $\widehat{H}\delta_{2^{n_1}}^{\gamma_{j+1}^i}$ is 1-output-reachable from $x(\sigma; x(0), u)$ under the control $Gx(\sigma; x(0), u)\delta_{2^{n_1}}^{\gamma_j^i}$, one can see that

$$y(\sigma + 1; x(0), u) = \widehat{H}\delta_{2^{n_1}}^{\gamma_{j+1}^i} = \widehat{y}(\sigma + 1; \widehat{x}(0)).$$

In general, for any integer $t \geqslant \sigma$, we have

$$y(t; x(0), u) = \widehat{H}\delta_{2^{n_1}}^{\gamma_\alpha^i} = \widehat{y}(t; \widehat{x}(0)),$$

where $\alpha = (j + t - \sigma) \mod (d_i)$.

From the arbitrariness of $x(0)$ and $\widehat{x}(0)$, one can see that the output regulation problem is solvable for Case II by $u(t) = Gx(t)\widehat{x}(t)$.

Finally, based on Cases I and II, we study the general case, that is, the attractor set of the system (16) has fixed points and cycles with length greater than 1. Denote the set of fixed points by $\Pi = \{\delta_{2^{n_1}}^{\pi_1}, \ldots, \delta_{2^{n_1}}^{\pi_q}\}$, and the set of cycles by $\Gamma = \{C_1, \ldots, C_r\}$, where $C_i = \{\delta_{2^{n_1}}^{\gamma_1^i}, \ldots, \delta_{2^{n_1}}^{\gamma_{d_i}^i}\}$ is a cycle with length d_i. We assume that $\delta_{2^{n_1}}^{\gamma_1^i} \to \cdots \to \delta_{2^{n_1}}^{\gamma_{d_i}^i} \to \delta_{2^{n_1}}^{\gamma_1^i}$.

For the general case, combining the proofs of Theorems 2 and 3 together, we have the following result.

Theorem 4. Suppose that the following three conditions hold:

1) $O(\widehat{H}\delta_{2^{n_1}}^{\pi_i}) \neq \emptyset$, $\forall\, i = 1, \ldots, q$;

2) for each $i \in \{1, \ldots, q\}$, there exist a nonempty set $S_i \subseteq O(\widehat{H}\delta_{2^{n_1}}^{\pi_i})$ and an integer $1 \leqslant \tau_i \leqslant 2^n$ such that $S_i \subseteq R_1(S_i)$ and $R_{\tau_i}(S_i) = \Delta_{2^n}$;

3) $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$, $i = 1, \ldots, r$, $j = 1, \ldots, d_i$ are 1-output-reachable from Δ_{2^n}.

Then, the output regulation problem is solvable.

Algorithm 2

1) For each $i \in \{1, \ldots, q\}$, calculate $R_k(S_i)$ and $R_k^\circ(S_i), k = 1, \ldots, \tau_i$ according to (19) and (21), respectively;

2) For any $i \in \{1, \ldots, q\}$ and any $j \in \{1, 2, \ldots, 2^n\}$, find the unique integer $1 \leqslant k_j^i \leqslant \tau_i$ such that $\delta_{2^n}^j \in R_{k_j^i}^\circ(S_i)$. If $k_j^i = 1$, find an integer $1 \leqslant \theta_j^i \leqslant 2^m$ such that $\delta_{2^n}^{l_{(\theta_j^i-1)2^n+j}} \in S_i$. If $2 \leqslant k_j^i \leqslant \tau_i$, find an integer $1 \leqslant \theta_j^i \leqslant 2^m$ such that $\delta_{2^n}^{l_{(\theta_j^i-1)2^n+j}} \in R_{k_j^i-1}(S_i)$;

3) For any $i \in \{1, \ldots, r\}$, any $j \in \{1, \ldots, d_i\}$ and any $k \in \{1, 2, \ldots, 2^n\}$, from Algorithm 1, design a control $\delta_{2^m}^{\eta_{i,j,k}}$ such that $\widehat{H}\delta_{2^{n_1}}^{\gamma_j^i}$ is 1-output-reachable from $\delta_{2^n}^k$;

4) The state feedback gain matrix can be designed as

$$G = \delta_{2^m}[v_1\ v_2\ \cdots\ v_{2^{n+n_1}}],$$

where

$$\begin{cases} v_\mu = \theta_j^i, & \text{for } \mu = (j-1)2^{n_1} + \pi_i,\ i = 1, \ldots, q,\ j = 1, \ldots, 2^n, \\ v_\mu = \eta_{i,j,k}, & \text{for } \mu = (k-1)2^{n_1} + \gamma_{j-1}^i,\ i = 1, \ldots, r,\ j = 1, \ldots, d_i,\ k = 1, \ldots, 2^n, \\ v_\mu \in \{1, 2, \ldots, 2^m\}, & \text{otherwise}, \end{cases} \tag{25}$$

and $\gamma_0^i := \gamma_{d_i}^i$.

Remark 2. It should be pointed out that the conditions presented in Theorem 4 are not necessary for the output regulation problem of BCNs. For instance, consider the systems (5) and (16) with $L = \delta_4[4\ 3\ 3\ 2\ 1\ 4\ 2\ 4\ 3\ 2\ 1\ 2\ 4\ 4\ 4\ 3]$, $H = \delta_2[1\ 2\ 1\ 1]$, $\widehat{L} = \delta_8[1\ 4\ 1\ 6\ 6\ 2\ 4\ 7]$ and $\widehat{H} = \delta_2[2\ 1\ 1\ 1\ 2\ 2\ 1\ 1]$. One can easily see that for this example, the system (16) has a fixed point δ_8^1 and a cycle with length 3, that is, $C_1 = \{\delta_8^2, \delta_8^4, \delta_8^6\}$. Moreover, $\widehat{H}\delta_8^2 = \widehat{H}\delta_8^4 = \delta_2^1$, and $\widehat{H}\delta_8^6 = \delta_2^2$. For this example, it is easy to obtain from (7) that

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.$$

Thus,

$$HM = \begin{bmatrix} 4 & 3 & 3 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

which implies that δ_2^2 is not 1-output-reachable from Δ_4. However, it is easy to check that the output regulation problem is solvable for this example under the state feedback gain $G = \delta_4[3\ 2\ 1\ 4\ 1\ 1\ 2\ 3\ 3\ 1\ 1\ 3\ 1\ 1\ 2\ 2\ 2\ 4\ 4\ 2\ 3\ 3\ 4\ 4\ 1\ 2\ 2\ 3\ 2\ 2\ 1\ 1]$. Therefore, the condition presented in Theorem 4 is not necessary. A necessary and sufficient condition needs to be obtained in the future work.

Finally, suppose that all the conditions of Theorem 4 hold. One can design a state feedback gain for the output regulation problem by Algorithm 2.

Remark 3. From (25), one can see that the state feedback gain is completely determined by the attractor set of the system (16). Therefore, Theorems 2–4 establish an attractor-driven method for the control design of the output regulation problem.

Remark 4. The computational complexity of Algorithm 1 is $O(2^{m+n+p})$. Moreover, the computational complexity of Algorithm 2 is $O(2^{m+n+n_1+p})$.

4 An illustrative example

In this section, we present an illustrative example to show the effectiveness of our main results.

Example 1. Consider the following BCN:

$$\begin{cases} x_1(t+1) = [u(t) \wedge (x_1(t) \vee x_2(t))] \vee [\neg u(t) \wedge \neg(x_1(t) \vee x_2(t))], \\ x_2(t+1) = (u(t) \wedge x_1(t)) \vee (\neg u(t) \wedge \neg x_1(t)); \\ y(t) = x_1(t). \end{cases} \tag{26}$$

The dynamics of the reference Boolean network is

$$\begin{cases} \widehat{x}_1(t+1) = (\widehat{x}_1(t) \wedge \widehat{x}_2(t)) \vee \neg\widehat{x}_1(t), \\ \widehat{x}_2(t+1) = \widehat{x}_1(t) \vee (\neg\widehat{x}_1(t) \wedge \neg\widehat{x}_2(t)); \\ \widehat{y}(t) = \widehat{x}_1(t) \vee (\neg\widehat{x}_1(t) \wedge \neg\widehat{x}_2(t)). \end{cases} \tag{27}$$

Our objective is to design a state feedback control (if possible) such that the output regulation problem is solvable for (26) and (27).

Using the vector form of Boolean values and setting $x(t) = \ltimes_{i=1}^2 x_i(t)$ and $\widehat{x}(t) = \ltimes_{i=1}^2 \widehat{x}_i(t)$, by Lemma 2, one can convert (26) and (27) into

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases} \tag{28}$$

and

$$\begin{cases} \widehat{x}(t+1) = \widehat{L}\widehat{x}(t), \\ \widehat{y}(t) = \widehat{H}\widehat{x}(t), \end{cases} \tag{29}$$

respectively, where $L = \delta_4[1\ 1\ 2\ 4\ 4\ 4\ 3\ 1]$, $H = \delta_2[1\ 1\ 2\ 2]$, $\widehat{L} = \delta_4[1\ 3\ 2\ 1]$ and $\widehat{H} = \delta_2[1\ 1\ 2\ 1]$. Moreover, one can easily see that the attractor set of the system (27) has a fixed point δ_4^1 and a cycle with length 2: $\{\delta_4^2, \delta_4^3\}$. A simple calculation gives $O(\widehat{H}\delta_4^1) = \{\delta_4^1, \delta_4^2\}$ and $\widehat{O}(\{\delta_4^2, \delta_4^3\}) = \{\delta_2^1, \delta_2^2\}$.

For the fixed point δ_4^1, setting $S = \{\delta_4^1\} \subseteq O(\widehat{H}\delta_4^1)$, one can easily obtain that $S \subseteq R_1(S)$ and $R_2(S) = \Delta_4$. For the cycle $\{\delta_4^2, \delta_4^3\}$, it follows from

$$HM = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

that both δ_2^1 and δ_2^2 are 1-output-reachable from Δ_4. Thus, all the conditions of Theorem 4 hold. By Theorem 4, the output regulation problem is solvable for this example.

According to Algorithm 2, we can obtain 16 state feedback gain matrices as follows:

$$G = \delta_2[1\ 1\ 2\ v_4\ 1\ 1\ 2\ v_8\ 1\ 1\ 2\ v_{12}\ 2\ 2\ 1\ v_{16}],$$

where $v_i \in \{1, 2\}, i = 4, 8, 12, 16$.

5 Conclusion

In this paper, we have studied the output reachability and output regulation control design of Boolean control networks by using the semi-tensor product method. We have proposed the concept of output reachability for BCNs, and presented some necessary and sufficient conditions for the verification of output reachability. We have proposed an attractor-driven method for the control design of the output regulation problem based on the output reachability of BCNs and the attractor set of the reference Boolean network. The study of an illustrative example has shown that the new results obtained in this paper are very effective.

It should be pointed out that Theorem 4 only presents a sufficient condition for the controller design of the output regulation problem. One possible way of getting a necessary and sufficient condition is to use the classical augmented system method [2].

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