SCIENCE CHINA
Information Sciences

• **RESEARCH PAPER** •

September 2015, Vol. 58 092101:1–092101:12
doi: 10.1007/s11432-014-5188-3

The sound and complete R-calculus for revising propositional theories

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Received April 2, 2014; accepted September 2, 2014; published online July 21, 2015

Abstract The AGM postulates are for the belief revision (revision by a single belief), and the DP postulates for the iterated revision (revision by a finite sequence of beliefs). Li gave an **R**-calculus for **R**-configurations $\Delta|\Gamma$, where Δ is a set of atomic formulas or the negations of atomic formulas, and Γ is a finite set of formulas. In this paper, two deduction systems for the revision of a theory by another theory are given such that the systems are sound and complete, that is, if $\Delta|\Gamma \Rightarrow \Gamma'$ is provable then $\Gamma' \supseteq \Delta$ is consistent and $\Gamma' - \Delta$ is a maximal subset of Γ such that $(\Gamma' - \Delta) \cup \Delta$ is consistent; and for any finite theories Δ and Γ, there is a finite theory Γ' such that $\Gamma' - \Delta$ is a maximal subset of Γ such that $(\Gamma' - \Delta) \cup \Delta$ is consistent, and $\Delta|\Gamma \Rightarrow \Gamma'$ is provable. Moreover, if $\Delta|\Gamma \Rightarrow \Gamma'$ is provable then Γ' satisfies the AGM and the DP postulates.

Keywords belief revision, **R**-calculus, minimal change, the AGM postulates, the DP postulates

Citation Li W, Sui Y F, Sun M Y. The sound and complete **R**-calculus for revising propositional theories. Sci China Inf Sci, 2015, 58: 092101(12), doi: 10.1007/s11432-014-5188-3

1 Introduction

The AGM postulates [1–4] are for the revision $K \circ \varphi$ of a theory K by a formula φ; while the DP postulates [5] are for the iterated revision $(\cdots(K \circ \varphi_1) \circ \cdots) \circ \varphi_n$ [6–13].

The **R**-calculus [14] brought up a Gentzen-type deduction system to deduce a consistent one $\Gamma' \cup \Delta$ from an inconsistent theory $\Gamma \cup \Delta$, where $\Gamma' \cup \Delta$ should be a maximal consistent subtheory of $\Gamma \cup \Delta$ which includes Δ as a subset (notice that here is the maximal consistent theory, not the maximally consistent theory), where $\Delta|\Gamma$ is an **R**-configuration, Γ is a consistent set of formulas, and Δ is a consistent sets of atomic formulas or the negation of atomic formulas. It was proved that if $\Delta|\Gamma \Rightarrow \Delta|\Gamma'$ is deducible and $\Delta|\Gamma'$ is an **R**-termination, i.e., there is no **R**-rule to reduce $\Delta|\Gamma'$ to another **R**-configuration $\Delta|\Gamma''$, then $\Delta \cup \Gamma'$ is a contraction of Γ by Δ.

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Given two theories Δ, Γ, assume that $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is finite. To find a maximal subset Γ' of Γ such that $\Gamma' \cup \Delta$ is consistent, a natural way is to define

$$\Theta_0 = \Delta;$$
$$\Theta_i = \begin{cases} \Theta_{i-1} \cup \{\varphi_i\}, & \text{if } \varphi_i \cup \Theta_{i-1} \text{ is consistent,} \\ \Theta_{i-1}, & \text{otherwise.} \end{cases}$$

Then, $\Gamma' = \Theta_n - \Delta \subseteq \Gamma$ is a maximal subset such that $\Gamma' \cup \Delta$ is consistent.

Therefore, a revision should be a formula that is revised by a theory, formally, $\Delta|\varphi$ (or $\varphi \circ \Delta$). Correspondingly, a theory Γ revised by a theory Δ is an iterated revision of form

$$\Delta|\Gamma = (\cdots((\Delta|\varphi_1)|\varphi_2)\cdots)|\varphi_n,$$

where $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. Here, we consider only the belief bases and not the belief sets.

In this paper we firstly consider a simple case of revision: $\Delta|\varphi$, where Δ is a set of formulas to revise, and φ is a formula to be revised; and then consider a general case of revision: $\Delta|\Gamma$, which is reduced to the successive revisions $(\cdots((\Delta|\varphi_1)|\varphi_2)\cdots)|\varphi_n$, where $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. Here, formulas are in the propositional logic [15–17].

A set of deductions rules **S** for $\Delta|\varphi$ will be given such that **S** are sound and complete, that is, for any theories Δ, Θ and formula φ, if $\Delta|\varphi \Rightarrow \Theta$ is provable in **S** then Θ is consistent, and if $\Delta \cup \{\varphi\}$ is consistent then $\Theta = \Delta \cup \{\varphi\}$; and if $\Delta \cup \{\varphi\}$ is inconsistent then $\Theta = \Delta$; and conversely, if $\Delta \cup \{\varphi\}$ is consistent then $\Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}$ is provable in **S**; and if $\Delta \cup \{\varphi\}$ is inconsistent then $\Delta|\varphi \Rightarrow \Delta$ is provable in **S**.

Generally, the soundness and completeness theorems hold good for $\Delta|\Gamma$ too. That is, for any consistent sets Δ, Γ, Θ of formulas with Γ being finite, if $\Delta|\Gamma \Rightarrow \Theta$ is provable in **S** then Θ is a maximal consistent set of Γ by Δ; and for any maximal consistent set Θ of Γ by Δ then $\Delta|\Gamma \Rightarrow \Theta$ is provable in **S**. Here, Θ is a maximal consistent set of Γ by Δ if (i) $\Theta \supseteq \Delta$, (ii) Θ is consistent, and (iii) for any Θ' with $\Theta \subset \Theta' \subseteq \Gamma \cup \Delta$, Θ' is inconsistent.

It will be proved that the AGM postulates and the DP postulates are satisfied by both $\Delta|\varphi$ and $\Delta|\Gamma$.

The deduction rules in **S** are used to decompose formulas in Γ (we call the right decomposition). Symmetrically, there is a set **T** of deduction rules to decompose formulas in Δ, so that the rule $(S^{\neg})$ that if $\Delta \vdash \neg p$ then $\Delta|p \Rightarrow \Delta$ is reduced to the revision of revising l by $\neg l$ by a set of deduction rules for the left decomposition, where l is a literal (atomic formula or the negation of an atomic formula). We shall prove that **T** is sound and complete, that is, for any consistent sets Δ, Γ, Θ of formulas with Γ being finite, if $\Delta|\Gamma \Rightarrow \Theta$ is provable in **T** then Θ is a maximal consistent set of Γ by Δ; and for any maximal consistent set Θ of Γ by Δ then $\Delta|\Gamma \Rightarrow \Theta$ is provable in **T**.

The paper is organized as follows: Section 2 lists the AGM postulates and the **R**-calculus; Section 3 gives an **R**-calculus **S** (a deduction rules) for the revision operator $\Delta|\varphi$, and proves that the deduction rules for $\Delta|\varphi$ are sound and complete; Section 4 discusses the basic properties of $\Delta|\varphi$, and shows that $\Delta|\varphi$ satisfies the AGM postulates for the **R**-calculus (the **R**-calculus is set-based, not closed theory-based, and hence, $\Delta|\Gamma$ is a set of formulas, not a belief base, a deductively closed set of formulas); Section 5 gives a set **T** of deduction rules for the left-decomposition of formulas such that the set of the deduction rules is sound and complete with respect to the maximal consistent sets; and the last section draws conclusion for the whole paper and discusses future work.

2 The AGM postulates for the R-calculus

The **R**-calculus [14] is defined on a first-order logical language [16,17]. Let L' be a logical language of the first-order logic; φ, ψ formulas and Γ, Δ sets of formulas (theories), where Δ is a set of atomic formulas or the negations of atomic formulas, and $\Delta|\Gamma$ is called an **R**-configuration.

The **R**-calculus consists of the following axiom and inference rules:

$$(\mathbf{A}^{\neg})\quad \Delta,\varphi_1|\neg\varphi_1,\Gamma \Rightarrow \varphi_1,\Delta|\Gamma,$$

$$(\mathbf{R}^{\mathrm{cut}})\quad \frac{\begin{array}{l}\Gamma_1,\varphi_1\vdash\varphi_2\quad \varphi_1\mapsto_T\varphi_2\\ \Gamma_2,\varphi_2\vdash\varphi_3\quad \Delta|\varphi_3,\Gamma_2\Rightarrow\Delta|\Gamma_2\end{array}}{\Delta|\varphi_1,\Gamma_1,\Gamma_2\Rightarrow\Delta|\Gamma_1,\Gamma_2},$$

$$(\mathbf{R}^{\wedge})\quad \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\wedge\varphi_2,\Gamma\Rightarrow\Delta|\Gamma},$$

$$(\mathbf{R}^{\vee})\quad \frac{\Delta|\varphi_1,\Gamma\Rightarrow\Delta|\Gamma\quad \Delta|\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\vee\varphi_2,\Gamma\Rightarrow\Delta|\Gamma},$$

$$(\mathbf{R}^{\rightarrow})\quad \frac{\Delta|\neg\varphi_1,\Gamma\Rightarrow\Delta|\Gamma\quad \Delta|\varphi_2,\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\varphi_1\rightarrow\varphi_2,\Gamma\Rightarrow\Delta|\Gamma},$$

$$(\mathbf{R}^{\forall})\quad \frac{\Delta|\varphi[t/x],\Gamma\Rightarrow\Delta|\Gamma}{\Delta|\forall x\varphi,\Gamma\Rightarrow\Delta|\Gamma},$$

where in $\mathbf{R}^{\mathrm{cut}}, \varphi_1\mapsto_T\varphi_2$ means that φ_1 occurs in the proof tree T of φ_2 from Γ_1 and φ_1; and in $\mathbf{R}^{\forall}, t$ is a term, and is free in φ for x.

Definition 1. $\Delta|\Gamma \Rightarrow \Theta|\Gamma'$ is an **R**-theorem, denoted by $\vdash^{\mathbf{R}} \Delta|\Gamma \Rightarrow \Theta|\Gamma'$, if there is a sequence $\{(\Delta_i,\Gamma_i,\Delta_i',\Gamma_i'):i\leqslant n\}$ such that

(i) $\Delta|\Gamma\Rightarrow\Theta|\Gamma'=\Delta_n|\Gamma_n\Rightarrow\Delta_n'|\Gamma_n'$,

(ii) for each $1\leqslant i\leqslant n$, either $\Delta_i|\Gamma_i\Rightarrow\Theta_i|\Gamma_i'$ is an axiom, or $\Delta_i|\Gamma_i\Rightarrow\Theta_i|\Gamma_i'$ is deduced by some **R**-rule of form $\dfrac{\Delta_{i-1}|\Gamma_{i-1}\Rightarrow\Theta_{i-1}|\Gamma_{i-1}'}{\Delta_i|\Gamma_i\Rightarrow\Theta_i|\Gamma_i'}$.

Definition 2. $\Delta|\Gamma\Rightarrow\Delta|\Gamma'$ is valid, denoted by $\models\Delta|\Gamma\Rightarrow\Delta|\Gamma'$, if for any contraction Θ of Γ' by Δ, Θ is a contraction of Γ by Δ.

Theorem 1 (The soundness and completeness theorem of the **R**-calculus)**.** For any theories Γ,Γ' and Δ,

$$\vdash\Delta|\Gamma\Rightarrow\Delta|\Gamma'$$

if and only if

$$\models\Delta|\Gamma\Rightarrow\Delta|\Gamma'.$$

In the following we discuss about the propositional logic. Let L be a logical language of the propositional logic which contains the following symbols:

- propositional variables $p_0,p_1,\ldots$;
- logical connectives $\neg,\wedge,\vee,\rightarrow$.

Formulas are defined as follows: $\varphi=p|\neg p|\varphi_1\wedge\varphi_2|\varphi_1\vee\varphi_2|\varphi_1\rightarrow\varphi_2$.

The AGM postulates are for the logically-closed theory revision; and the **R**-calculus is for the set-theoretic theory revision. Therefore, the AGM postulates should be rewritten as follows to fit in the set-theoretic theory revision.

The AGM postulates for the **R**-calculus:

- Success: $\Delta\subseteq\Delta|\Gamma$;
- Inclusion: $\Delta|\Gamma\subseteq\Delta\cup\Gamma$;
- Vacuity: if $\Gamma\nvdash\neg\varphi_1,\Gamma_1=\Gamma\cup\{\varphi_1\}\nvdash\neg\varphi_2,\ldots,\Gamma_{n-1}=\Gamma\cup\{\varphi_1,\ldots,\varphi_{n-1}\}\nvdash\neg\varphi_n$, where $\Delta=\{\varphi_1,\ldots,\varphi_n\}$, then $\Delta|\Gamma=\Delta\cup\Gamma$;
- Extensionality: $\Delta|\Gamma$ is consistent if Δ is consistent;
- Extensionality: if $\Delta\equiv\Delta'$, that is, $\Delta\vdash\Delta'$ and $\Delta'\vdash\Delta$ then $\Delta|\Gamma\equiv\Delta'|\Gamma$;
- Superexpansion: $(\Delta_1\cup\Delta_2)|\Gamma\subseteq(\Delta_1|\Gamma)\cup\Delta_2$;
- Subexpansion: if $\Delta_1|\Gamma\nvdash\neg\Delta_2$ then $(\Delta_1|\Gamma)\cup\Delta_2\subseteq(\Delta_1\cup\Delta_2)|\Gamma$.

Definition 3. Given any sets Δ,Γ,Θ of formulas, Θ is called a maximal consistent set of Γ by Δ if

(i) $\Theta\subseteq\Delta\cup\Gamma$;

(ii) $\Delta \subseteq \Theta$, and

(iii) for any set Θ' of formulas with $\Theta \subset \Theta' \subseteq \Delta \cup \Gamma, \Theta'$ is inconsistent.

Proposition 1. Θ is a maximal consistent set of Γ by Δ if and only if $\Theta - \Delta$ is a maximal subset of Γ such that $(\Theta - \Delta) \cup \Delta$ is consistent.

3 The R-calculus S

The rules for the calculus **S** are the composing rules, which compose subformulas (e.g. φ_1, φ_2) in the precondition of a rule into a complex formula (e.g. $\varphi_1 \wedge \varphi_2$) in the postcondition of the rule.

S consists of the following rules:

$$(S^{\mathrm{con}})\ \frac{\varphi \cup \Delta \text{ is consistent}}{\Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}},$$

$$(S^{\neg})\ \frac{\Delta \vdash \neg p}{\Delta|p \Rightarrow \Delta},$$

$$(S_1^{\wedge})\ \frac{\Delta|\varphi_1 \Rightarrow \Delta}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \text{if } \Delta \cup \{\varphi_1\} \text{ is inconsistent},$$

$$(S_2^{\wedge})\ \frac{\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \text{if } \Delta \cup \{\varphi_1\} \text{ is consistent},$$

$$(S^{\vee})\ \frac{\Delta|\varphi_1 \Rightarrow \Delta \quad \Delta|\varphi_2 \Rightarrow \Delta}{\Delta|\varphi_1 \vee \varphi_2 \Rightarrow \Delta},$$

$$(S^{\rightarrow})\ \frac{\Delta|\neg\varphi_1 \Rightarrow \Delta \quad \Delta|\varphi_2 \Rightarrow \Delta}{\Delta|\varphi_1 \rightarrow \varphi_2 \Rightarrow \Delta}.$$

Remark 1. The rules

$$\frac{\Delta|\varphi_1 \Rightarrow \Delta}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \text{and} \quad \frac{\Delta|\varphi_2 \Rightarrow \Delta}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta}$$

are too weak to revise $\Delta|\varphi_1 \wedge \varphi_2$ when both $\Delta \cup \{\varphi_1\}$ and $\Delta \cup \{\varphi_2\}$ are consistent and $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$ is inconsistent.

We need $(S_1^{\wedge})$ and $(S_2^{\wedge})$ instead of the weak ones.

For example, let $\Delta = \{p \rightarrow q, q \rightarrow r\}$ and $\varphi_1 \wedge \varphi_2 = p \wedge \neg r$. Then, both $\Delta \cup \{p\}$ and $\Delta \cup \{\neg r\}$ are consistent and $\Delta \cup \{p \wedge \neg r\}$ is inconsistent. Therefore, we have the following deduction:

$$\begin{array}{ll} p \rightarrow q, q \rightarrow r, [p] \ \vdash \ r & \\ p \rightarrow q, q \rightarrow r, [p]|\neg r \ \Rightarrow \ p \rightarrow q, q \rightarrow r, [p]| & (S^{\neg}) \\ p \rightarrow q, q \rightarrow r|p \wedge \neg r \ \Rightarrow \ p \rightarrow q, q \rightarrow r| & (S_2^{\wedge}) \end{array}$$

Example 1. Revision $p, \neg r, p \rightarrow q|q \rightarrow r$ has the following sub-revisions:

$$\begin{array}{ll} (1)\ p, \neg r, p \rightarrow q \vdash q & \\ (2)\ p, \neg r, p \rightarrow q|\neg q \Rightarrow p, \neg r, p \rightarrow q| & (S^{\neg}) \\ (3)\ p, \neg r, p \rightarrow q \vdash \neg r & \\ (4)\ p, \neg r, p \rightarrow q|r \Rightarrow p, \neg r, p \rightarrow q| & (S^{\neg}) \\ (5)\ p, \neg r, p \rightarrow q|q \rightarrow r \Rightarrow p, \neg r, p \rightarrow q| & (S^{\rightarrow}) \end{array}$$

Definition 4. $\Delta|\varphi \Rightarrow \Delta, \varphi^i$ is provable if there is a sequence $\{\theta_1, \ldots, \theta_m\}$ of statements such that

$$\theta_1 = \Delta|\varphi_1 \Rightarrow \Delta|\varphi_2,$$
$$\cdots$$
$$\theta_m = \Delta|\varphi_m \Rightarrow \Delta, \varphi^i;$$

and for each $j < m, \Delta|\varphi_j \Rightarrow \Delta|\varphi_{j+1}$ is deduced from the previous statements by a deduction rule, where $i \in \{0, 1\}, \varphi^1 = \varphi$ and $\varphi^0 = \lambda$, the empty string.

Intuitively, we decompose φ into literals according to the structure of φ, and delete/add literals by rule $(S^\neg)/(S^{\rm con})$.

Theorem 2. For any consistent theory Δ and formula φ, if $\Delta|\varphi \Rightarrow \Delta, \varphi^i$ is provable then if $i = 0$ then $\Delta \cup \{\varphi\}$ is inconsistent; otherwise, $\Delta \cup \{\varphi\}$ is consistent.

Proof. Assume that $\Delta|\varphi \Rightarrow \Delta, \varphi^i$ is provable.

If $i = 1$, i.e., $\Delta|\varphi \Rightarrow \Delta, \varphi$ is provable then by $(S^{\rm con}), \Delta \cup \{\varphi\}$ is consistent.

If $i = 0$ then we prove that $\Delta \cup \{\varphi\}$ is inconsistent by induction on the structure of φ.

If $\varphi = p$ or $\neg p$ is a literal then $\Delta|\varphi \Rightarrow \Delta$ only if $\Delta \vdash \neg\varphi$, and by $(S^\neg), \Delta \cup \{\varphi\}$ is inconsistent;

If $\varphi = \varphi_1 \wedge \varphi_2$ then there are two cases: either $\Delta|\varphi_1 \Rightarrow \Delta$ or $\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]$ is provable. By the induction assumption, if $\Delta|\varphi_1 \Rightarrow \Delta$ then $\Delta \cup \{\varphi_1\}$ is inconsistent, and so is $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$; and if $\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]$ then $\Delta \cup \{\varphi_1, \varphi_2\}$ is inconsistent, and so is $\Delta \cup \{\varphi_1 \wedge \varphi_2\}$;

If $\varphi = \varphi_1 \vee \varphi_2$ then $\Delta|\varphi_1 \Rightarrow \Delta$ and $\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]$ are provable. By the induction assumption, both $\Delta \cup \{\varphi_1\}$ and $\Delta \cup \{\varphi_2\}$ are inconsistent, and so is $\Delta \cup \{\varphi_1 \vee \varphi_2\}$;

Similarly for $\varphi = \varphi_1 \rightarrow \varphi_2$.

Theorem 2 is the soundness theorem for the deduction, that is, if $\Delta|\varphi \Rightarrow \Delta|\varphi^i$ is provable then

$$\Delta||\varphi = \begin{cases} \Delta \cup \{\varphi\} & \text{if } \Delta \cup \{\varphi\} \text{ is consistent,} \\ \Delta & \text{if } \Delta \text{ is consistent,} \end{cases}$$

where $||$ is a consistent operator. And the following Theorem 3 is the completeness theorem for the deduction, that is, if $\Delta||\varphi \Rightarrow \Delta, \varphi^i$ then $\Delta|\varphi \Rightarrow \Delta, \varphi^i$ is provable.

Theorem 3. For any consistent theory Δ and formula φ, if $\Delta \cup \{\varphi\}$ is consistent then $\Delta|\varphi \Rightarrow \Delta, \varphi$ is provable; and if $\Delta \cup \{\varphi\}$ is inconsistent then $\Delta|\varphi \Rightarrow \Delta$ is provable.

Proof. If $\Delta \cup \{\varphi\}$ is consistent then, by $(S^{\rm con})$,

$$\Delta|\varphi \Rightarrow \Delta, \varphi;$$

If $\Delta \cup \{\varphi\}$ is inconsistent then we prove that $\Delta|\varphi \Rightarrow \Delta$ is provable by induction on the structure of φ.

If $\varphi = p$ or $\neg p$ is a literal then $\Delta \vdash \neg\varphi$, and $\Delta|\varphi \Rightarrow \Delta$ by $(S^{\rm con})$.

If $\varphi = \varphi_1 \wedge \varphi_2$ then either $\Delta \cup \{\varphi_1\}$ is inconsistent or $\Delta \cup \{\varphi_1\} \cup \{\varphi_2\}$ is inconsistent. By the induction assumption, either $\Delta|\varphi_1 \Rightarrow \Delta$, or $\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]$; and by $(S_1^\wedge)$ and $(S_2^\wedge)$, $\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta$.

If $\varphi = \varphi_1 \vee \varphi_2$ then $\Delta \cup \{\varphi_1\}$ and $\Delta \cup \{\varphi_2\}$ are inconsistent. By the induction assumption, $\Delta|\varphi_1 \Rightarrow \Delta$, and $\Delta|\varphi_2 \Rightarrow \Delta$; and by $(S_1^\vee)$, $\Delta|\varphi_1 \vee \varphi_2 \Rightarrow \Delta$.

Similarly for $\varphi = \varphi_1 \rightarrow \varphi_2$.

Let Δ, Γ be two finite consistent sets of formulas such that $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$. Then we define

$$\Delta|\Gamma = (\cdots((\Delta|\varphi_1)|\varphi_2)|\cdots)|\varphi_n.$$

Thus, Theorems 2 and 3 can be generalized to any finite set Γ in the following Theorems 4 and 5, respectively.

Theorem 4. For any consistent theories Δ, Γ and Θ, if $\Delta|\Gamma \Rightarrow \Theta$ is provable then Θ is a maximal consistent set of Γ by Δ.

Theorem 5. For any consistent theories Δ and Γ and any maximal consistent set Θ of Γ by Δ, $\Delta|\Gamma \Rightarrow \Theta$ is provable.

4 The basic logical properties of $\Delta|\Gamma$

The AGM postulates for revision:

Success: $\Delta \subset \Delta|\varphi$;
Inclusion: $\Delta|\varphi \subseteq \{\varphi\} \cup \Delta$;
Vacuity: if $\Delta \nvdash \neg\varphi$ then $\Delta|\varphi = \Delta \cup \{\varphi\}$;
Extensionality: $\Delta|\varphi$ is consistent if Δ is consistent;
Extensionality: if $\models \Delta \leftrightarrow \Delta'$ then $\Delta|\varphi \equiv \Delta'|\varphi$;
Superexpansion: $(\Delta_1, \Delta_2)|\varphi \subseteq (\Delta_2|\varphi) \cup \Delta_1$;
Subexpansion: if $\Delta_2|\varphi \nvdash \neg\Delta_1$ then $(\Delta_2|\varphi) \cup \Delta_1 \equiv \Delta_1, \Delta_2|\varphi$.

Theorem 6. $\Delta|\varphi$ satisfies the AGM postulates.

Proof. It is obvious that $\Delta \subseteq \Delta|\varphi$ and $\Delta|\varphi \subseteq \{\varphi\} \cup \Delta$.

(Vacuity) If $\Delta \nvdash \neg\varphi$ then $\Delta \cup \varphi$ is consistent, and by $(S^{\rm con}), \Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}$.

(Extensionality) By the proof of the completeness theorem, if Δ is consistent with φ then $\Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}$; and if Δ is inconsistent with φ then $\Delta|\varphi \Rightarrow \Delta$, and Δ is consistent.

(Extensionality) If $\Delta \equiv \Delta'$ then Δ is consistent with φ if and only if Δ' is consistent with φ, and $\Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}$ iff $\Delta'|\varphi \Rightarrow \Delta' \cup \{\varphi\}$, and hence,

$$\Delta|\varphi \equiv \Delta'|\varphi.$$

(Superexpansion) By the definition

$$\Delta_1, \Delta_2|\varphi = \Delta_1|(\Delta_2|\varphi)$$

and (Inclusion), $\Delta_1, \Delta_2|\varphi = \Delta_1|(\Delta_2|\varphi) \subseteq (\Delta_2|\varphi) \cup \Delta_1$.

(Subexpansion) Assume that $\Delta_2|\varphi \nvdash \neg\Delta_1$. Then, $(\Delta_2|\varphi)$ is consistent with Δ_1, and $\Delta_1|(\Delta_2|\varphi) \equiv \Delta_1 \cup (\Delta_2|\varphi)$.

The DP postulates:

(C1) If $\Delta_2 \models \Delta_1$ then $\Delta_2|(\Delta_1|\varphi) \equiv \Delta_2|\varphi$;
(C2) If $\Delta_2 \models \neg\Delta_1$ then $\Delta_2|(\Delta_1|\varphi) \equiv \Delta_2|\varphi$;
(C3) If $(\Delta_2|\varphi) \models \Delta_1$ then $\Delta_2|(\Delta_1|\varphi) \models \Delta_1$;
(C4) If $(\Delta_2|\varphi) \not\models \neg\Delta_1$ then $\Delta_2|(\Delta_1|\varphi) \not\models \neg\Delta_1$.

Lemma 1. If $\Delta_1 \equiv \Delta_2$ then $\Delta_1|\varphi \equiv \Delta_2|\varphi$.

Proof. Assume that $\Delta_1 \equiv \Delta_2$.

If φ is consistent with Δ_1 then φ is consistent with Δ_2, and $\Delta_1|\varphi = \Delta_1 \cup \{\varphi\} \equiv \Delta_2 \cup \{\varphi\} = \Delta_2|\varphi$.

If φ is inconsistent with Δ_1 then φ is inconsistent with Δ_2, and $\Delta_1|\varphi \Rightarrow \Delta_1, \Delta_2|\varphi \Rightarrow \Delta_2$, and hence, $\Delta_1|\varphi \equiv \Delta_2|\varphi$.

Lemma 2. Assume that $\Delta_1 \cup \Delta_2$ is consistent. Then, $\Delta_1, \Delta_2|\varphi \equiv \Delta_1|(\Delta_2|\varphi) \equiv \Delta_2|(\Delta_1|\varphi)$.

Proof. If φ is consistent with $\Delta_1 \cup \Delta_2$ then φ is consistent with Δ_1 and Δ_2, respectively, and $\Delta_1, \Delta_2|\varphi; \Delta_2, \Delta_1|\varphi$ are consistent, and

$$\begin{aligned} \Delta_1, \Delta_2|\varphi &= \Delta_1, \Delta_2, \varphi \\ &= \Delta_1|(\Delta_2|\varphi) \\ &\equiv \Delta_2|(\Delta_1|\varphi). \end{aligned}$$

If φ is inconsistent with Δ_1 and Δ_2 then

$$\begin{aligned} \Delta_1|(\Delta_2|\varphi) &= \Delta_1|\Delta_2 \\ &= \Delta_1, \Delta_2 \end{aligned}$$

$$\begin{aligned} &\equiv \Delta_2|\Delta_1 \\ &\equiv \Delta_2|(\Delta_1|\varphi). \end{aligned}$$

If φ is consistent with Δ_1 and inconsistent with Δ_2 then

$$\begin{aligned} \Delta_1|(\Delta_2|\varphi) &= \Delta_1|\Delta_2 \\ &= \Delta_1, \Delta_2 \\ &\equiv \Delta_2, \Delta_1|\varphi) \\ &\equiv \Delta_2|(\Delta_1, \varphi) \\ &\equiv \Delta_2|(\Delta_1|\varphi). \end{aligned}$$

If φ is consistent with Δ_1 and Δ_2, and inconsistent with $\Delta_1 \cup \Delta_2$ then

$$\begin{aligned} \Delta_1|(\Delta_2|\varphi) &= \Delta_1|(\Delta_2, \varphi) \\ &= \Delta_1, \Delta_2|\varphi \\ &= \Delta_1, \Delta_2 \\ &\equiv \Delta_2, \Delta_1|\varphi) \\ &\equiv \Delta_2|(\Delta_1, \varphi) \\ &\equiv \Delta_2|(\Delta_1|\varphi). \end{aligned}$$

Theorem 7. By introducing a postulate

$$\Delta_2, \Delta_1, \neg\Delta_1|\varphi \equiv \Delta_2|\varphi,$$

$\Delta|\varphi$ satisfies the DP postulates.

Proof. (C1) Assume that $\Delta_2 \models \Delta_1$. Then,

$$\begin{aligned} \Delta_2|\varphi &\equiv (\Delta_2, \Delta_1)|\varphi \\ &\equiv \Delta_2|(\Delta_1)|\varphi. \end{aligned}$$

(C2) Assume that $\Delta_2 \models \neg\Delta_1$. Then,

$$\begin{aligned} \Delta_2, \Delta_1|\varphi &\equiv \Delta_2, \neg\Delta_1, \Delta_1|\varphi \\ &\equiv \Delta_2|\varphi. \end{aligned}$$

(C3) Assume that $\Delta_2|\varphi \models \Delta_1$. Then

$$\begin{aligned} \Delta_2, \Delta_1|\varphi &\equiv \Delta_2|(\Delta_1|\varphi) \\ &\supseteq \Delta_2 \\ &\models \Delta_1. \end{aligned}$$

(C4) Assume that $\Delta_2|\varphi \not\models \neg\Delta_1$. Then,

$$\begin{aligned} \Delta_2, \Delta_1|\varphi &\equiv \Delta_1|(\Delta_2|\varphi) \\ &\equiv \Delta_1 \cup (\Delta_2|\varphi) \\ &\not\models \neg\Delta_1, \end{aligned}$$

because $\Delta_1 \cup (\Delta_2|\varphi)$ is consistent.

It is rather clear that $\Delta|\varphi$ satisfies the principle of the minimal change [18–20] with respect to set. Considering the fact that if Δ is consistent with φ then $\Delta|\varphi \Rightarrow \Theta = \Delta\cup\{\varphi\}$, and for any set $A \subseteq \Delta\cup\{\varphi\}$, if $A \supseteq \Delta$ then $A \subseteq \Theta$; and if Δ is inconsistent with φ then $\Delta|\varphi \Rightarrow \Theta = \Delta$, and for any set $A \subseteq \Delta$, if $A \supseteq \Delta$ then $A \subseteq \Theta$.

$\Delta|\varphi$ also satisfies the principle of the minimal change with respect to inference.

Theorem 8. Assume that $\Delta|\varphi \Rightarrow \Theta$ is provable. Then,

(i) $\Theta \vdash \Delta$; and

(ii) for any theory Δ_0, if $\Delta_0 \vdash \Delta$ and $\Delta_0 \vdash \varphi$ then $\Delta_0 \vdash \Theta$.

Proof. If φ is consistent with Δ then $\Theta = \Delta \cup \{\varphi\}$, and (i),(ii) are satisfied.

If φ is inconsistent with Δ then $\Theta = \Delta$, and for any theory Δ_0, if $\Delta_0 \vdash \Delta$ and $\Delta_0 \vdash \varphi$ then Δ_0 is inconsistent, and so $\Delta_0 \vdash \Theta$.

Similarly, the following theorem is also true.

Theorem 9. (i) $\Delta|\Gamma$ satisfies the AGM postulates; and

(ii) $\Delta|\Gamma$ satisfies the AGM postulates.

Theorem 10. Assume that $\Delta|\Gamma \Rightarrow \Theta$ is provable. Then,

(i) $\Theta \subseteq \Delta \cup \Gamma$;

(ii) $\Theta \vdash \Delta$; and

(iii) for any consistent theory Δ_0, if $\Delta_0 \vdash \Delta$ and $\Delta_0 \vdash \Gamma$ then $\Delta_0 \vdash \Theta$.

5 The left decomposition rules

The **S**-rules are used to decompose the formula φ to be revised by Δ. The rule (S^{con}) can be replaced by a set of the T-rules to decompose formulas in Δ.

The rules for the calculus **T** are decomposed into two classes: the right-side rules (denoted by T) and the left-side rules (denoted by S).

The right-side rules:

$$(T^{\text{incon}})\ \frac{\Delta \text{ is inconsistent}}{\Delta|\varphi \Rightarrow \Delta|},$$

$$(T_1^{\neg})\ \frac{\neg l \in \Delta}{\Delta|l \Rightarrow \Delta}, \qquad (T_2^{\neg})\ \frac{}{\Delta, l', \neg l'|l \Rightarrow \Delta, l', \neg l'},$$

$$(T_1^{\wedge})\ \frac{\Delta, \psi_1|l \Rightarrow \Delta, \psi_1}{\Delta, \psi_1 \wedge \psi_2|l \Rightarrow \Delta, \psi_1 \wedge \psi_2},$$

$$(T_2^{\wedge})\ \frac{\Delta, \psi_2|l \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \wedge \psi_2|l \Rightarrow \Delta, \psi_1 \wedge \psi_2},$$

$$(T^{\vee})\ \frac{\Delta, \psi_1|l \Rightarrow \Delta, \psi_1 \quad \Delta, \psi_2|l \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \vee \psi_2|l \Rightarrow \Delta, \psi_1 \vee \psi_2},$$

$$(T^{\rightarrow})\ \frac{\Delta, \neg\psi_1|l \Rightarrow \Delta \quad \Delta, \psi_2|l \Rightarrow \Delta}{\Delta, \psi_1 \rightarrow \psi_2|l \Rightarrow \Delta, \psi_1 \rightarrow \psi_2},$$

where $l = p|\neg p$.

The left-side rules:

$$(S^{\text{con}})\ \frac{\varphi \cup \Delta \text{ is consistent}}{\Delta|\varphi \Rightarrow \Delta \cup \{\varphi\}},$$

$$(S_1^{\wedge})\ \frac{\Delta|\varphi_1 \Rightarrow \Delta}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \text{if } \Delta \cup \{\varphi_1\} \text{ is inconsistent},$$

$$(S_1^{\wedge})\ \frac{\Delta, [\varphi_1]|\varphi_2 \Rightarrow \Delta, [\varphi_1]}{\Delta|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta} \quad \text{if } \Delta \cup \{\varphi_1\} \text{ is consistent},$$

$$(S^{\vee})\ \frac{\Delta|\varphi_1 \Rightarrow \Delta \quad \Delta|\varphi_2 \Rightarrow \Delta}{\Delta|\varphi_1 \vee \varphi_2 \Rightarrow \Delta},$$

$$(S^{\rightarrow})\ \frac{\Delta|\neg\varphi_1 \Rightarrow \Delta \quad \Delta|\varphi_2 \Rightarrow \Delta}{\Delta|\varphi_1 \rightarrow \varphi_2 \Rightarrow \Delta}.$$

The reason for which we need rule $(T_2^{\neg})$ is given in the following example. Let $\Delta = \{p, p \to q\}$ and $l = \neg q$. To use $(T^{\to})$, we need

$$\text{(i) } p, \neg p | \neg q \Rightarrow p, \neg p;$$
$$\text{(ii) } p, q | \neg q \Rightarrow p, q$$

and a rule to get

$$p, p \to q | \neg q \Rightarrow p, p \to q.$$

(ii) is obtained by rule $(T_1^{\neg})$. Without $(T_1^{\neg})$ we cannot prove (i).

Example 2. Revision $p, \neg r, p \to q | q \to r$ has the following sub-revisions:

$$\begin{array}{ll}
(1)\ p, \neg r, \neg p | \neg q \Rightarrow p, \neg r, \neg p| & (T_2^{\neg}) \\
(2)\ p, \neg r, q | \neg q \Rightarrow p, \neg r, q| & (T_1^{\neg}) \\
(3)\ p, \neg r, \neg p | r \Rightarrow p, \neg r, \neg p| & (T_1^{\neg}) \\
(4)\ p, \neg r, q | r \Rightarrow p, \neg r, q| & (T_1^{\neg}) \\
(5)\ p, \neg r, p \to q | \neg q \Rightarrow p, \neg r, p \to q| & (1, 2, T^{\to}) \\
(6)\ p, \neg r, p \to q | r \Rightarrow p, \neg r, p \to q| & (3, 4, T^{\to}) \\
(7)\ p, \neg r, p \to q | q \to r \Rightarrow p, \neg r, p \to q| & (5, 6, S^{\to}).
\end{array}$$

Therefore, we have

$$p, \neg r, p \to q | q \to r \Rightarrow p, \neg r, p \to q| = \{p, \neg r, p \to q\}.$$

Intuitively, φ can be decomposed into literals by the decomposition **S**-rules, and some formulas in Δ can be decomposed into literals by the decomposition T-rules. By (S^{con}) and (T^{incon}) rules, we delete some literals, and then, compose the formulas in Δ by the composition T-rules.

Proposition 2. For any formula φ, we have

$$(T_1') \ \frac{\neg\varphi \in \Delta}{\Delta | \varphi \Rightarrow \Delta}, \qquad (T_1'') \ \frac{}{\Delta, \psi, \neg\psi | \varphi \Rightarrow \Delta, \psi, \neg\psi},$$

$$(T_2) \ \frac{\Delta, \psi_1 | \varphi \Rightarrow \Delta, \psi_1 \text{ or } \Delta, \psi_1 | \varphi \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \wedge \psi_2 | \varphi \Rightarrow \Delta, \psi_1 \wedge \psi_2},$$

$$(T_3) \ \frac{\Delta, \psi_1 | \varphi \Rightarrow \Delta, \psi_1 \quad \Delta, \psi_2 | \varphi \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \vee \psi_2 | \varphi \Rightarrow \Delta, \psi_1 \vee \psi_2},$$

$$(T_4) \ \frac{\Delta, \neg\psi_1 | \varphi \Rightarrow \Delta \quad \Delta, \psi_2 | \varphi \Rightarrow \Delta}{\Delta, \psi_1 \to \psi_2 | \varphi \Rightarrow \Delta, \psi_1 \to \psi_2},$$

Proof. We prove (T_3) by the induction on the structure of φ, and can prove others similarly.

Case 1. $\varphi = l$ is a literal. Then, (T_3) is $(T^{\vee})$;

Case 2. $\varphi = \varphi_1 \wedge \varphi_2$. Assume that

$$\frac{\Delta, \psi_1 | \varphi_1 \Rightarrow \Delta, \psi_1 \quad \Delta, \psi_2 | \varphi_1 \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \vee \psi_2 | \varphi_1 \Rightarrow \Delta, \psi_1 \vee \psi_2}$$

$$\frac{\Delta, \psi_1 | \varphi_2 \Rightarrow \Delta, \psi_1 \quad \Delta, \psi_2 | \varphi_2 \Rightarrow \Delta, \psi_2}{\Delta, \psi_1 \vee \psi_2 | \varphi_2 \Rightarrow \Delta, \psi_1 \vee \psi_2},$$

and

$$\Delta, \psi_1 | \varphi \Rightarrow \Delta, \psi_1$$
$$\Delta, \psi_2 | \varphi \Rightarrow \Delta, \psi_2$$

are provable, i.e.,

$$\Delta, \psi_1 | \varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1$$
$$\Delta, \psi_2 | \varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_2$$

are provable. Then, either $\Delta, \psi_1|\varphi_1 \Rightarrow \Delta, \psi_1$ or $\Delta, \psi_1, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_1$; and either $\Delta, \psi_2|\varphi_2 \Rightarrow \Delta, \psi_2$ or $\Delta, \psi_2, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_2$ are provable.

If $\Delta, \psi_1|\varphi_1 \Rightarrow \Delta, \psi_1$ and $\Delta, \psi_2|\varphi_2 \Rightarrow \Delta, \psi_2$ are provable then

$$\begin{array}{l} \Delta, \psi_1|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \\ \Delta, \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_2 \end{array}$$

are provable and by $(T^\vee)$, we have $\Delta, \psi_1 \vee \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \vee \psi_2$ is provable.

If $\Delta, \psi_1|\varphi_1 \Rightarrow \Delta, \psi_1$ and $\Delta, \psi_2, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_2, \varphi_1$ are provable then

$$\begin{array}{l} \Delta, \psi_1|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \\ \Delta, \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_2 \end{array}$$

are provable and by $(T^\vee)$, we have $\Delta, \psi_1 \vee \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \vee \psi_2$ is provable.

If $\Delta, \psi_1, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_1, \varphi_1$ and $\Delta, \psi_2|\varphi_2 \Rightarrow \Delta, \psi_2$ are provable then

$$\begin{array}{l} \Delta, \psi_1|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \\ \Delta, \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_2 \end{array}$$

are provable and by $(T^\vee)$, we have $\Delta, \psi_1 \vee \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \vee \psi_2$ is provable.

If $\Delta, \psi_1, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_1, \varphi_1$ and $\Delta, \psi_2, \varphi_1|\varphi_2 \Rightarrow \Delta, \psi_2, \varphi_1$ are provable then

$$\begin{array}{l} \Delta, \psi_1|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \\ \Delta, \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_2 \end{array}$$

are provable and by $(T^\vee)$, we have $\Delta, \psi_1 \vee \psi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \Delta, \psi_1 \vee \psi_2$ is provable.

Similarly, there is the following soundness and completeness theorem for the deduction.

Theorem 11. For any consistent theory Δ and formula φ, if $\Delta|\varphi \Rightarrow \Delta, \varphi^i$ is provable then if $i = 0$ then $\Delta \cup \{\varphi\}$ is inconsistent; otherwise, $\Delta \cup \{\varphi\}$ is consistent.

We cannot prove that for any consistent theory Δ and formula φ, if $\Delta \cup \{\varphi\}$ is inconsistent then $\Delta|\varphi \Rightarrow \Delta$ is provable. For example, let $\Delta = \{\neg p \vee q, \neg p \vee \neg q\}$ and $\varphi = p$, we cannot prove that either $\neg p \vee q|p \Rightarrow \neg p \vee q$, or $\neg p \vee q \neg q|p \Rightarrow \neg p \vee q$, and hence it cannot be inferred that

$$\neg p \vee q, \neg p \vee \neg q|p \Rightarrow \neg p \vee q, \neg p \vee \neg q \equiv \neg p.$$

Because $\neg p \vee q, \neg p \vee \neg q \equiv \neg p$ and $\neg p|p \Rightarrow \neg p$, we have the following theorem.

Theorem 12. For any consistent theory Δ and formula φ, if $\Delta \cup \{\varphi\}$ is consistent then $\Delta|\varphi \Rightarrow \Delta, \varphi$ is provable; and if $\Delta \cup \{\varphi\}$ is inconsistent then there is a theory Δ' such that $\Delta' \equiv \Delta$ and $\Delta'|\varphi \Rightarrow \Delta$ is provable.

Proof. Assume that $\Delta \cup \{\varphi\}$ is inconsistent. Then, there is a theory Δ' such that $\neg\varphi \in \Delta'$ and $\Delta' \equiv \Delta$.

Therefore, we prove that $\neg\varphi|\varphi \Rightarrow \varphi$ is provable. We prove it by the induction on the structure of φ.

If $\varphi = p$ is a propositional variable then by $(T^\neg)$, $\neg p|p \Rightarrow \neg p$.

If $\varphi = \varphi_1 \wedge \varphi_2$ then $\neg\varphi|\varphi \equiv \neg\varphi_1 \vee \neg\varphi_2|\varphi_1 \wedge \varphi_2$, and

$$\begin{array}{lr} \neg\varphi_1|\varphi_1 \Rightarrow \neg\varphi_1 & \text{(induction assumption)} \\ \neg\varphi_1|\varphi_1 \wedge \varphi_2 \Rightarrow \neg\varphi_1 & (S^\wedge) \\ \neg\varphi_2|\varphi_2 \Rightarrow \neg\varphi_2 & \text{(induction assumption)} \\ \neg\varphi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \neg\varphi_2 & (S^\wedge) \\ \neg\varphi_1 \vee \neg\varphi_2|\varphi_1 \wedge \varphi_2 \Rightarrow \neg\varphi_1 \vee \neg\varphi_2 & (T^\vee). \end{array}$$

If $\varphi = \varphi_1 \vee \varphi_2$ then $\neg\varphi|\varphi \equiv \neg\varphi_1 \wedge \neg\varphi_2|\varphi_1 \vee \varphi_2$, and

$$
\begin{array}{lr}
\neg\varphi_1|\varphi_1 \Rightarrow \neg\varphi_1 & \text{(induction assumption)} \\
\neg\varphi_1 \wedge \neg\varphi_2|\varphi_1 \Rightarrow \neg\varphi_1 \wedge \neg\varphi_2 & (T^{\wedge}) \\
\neg\varphi_2|\varphi_2 \Rightarrow \neg\varphi_2 & \text{(induction assumption)} \\
\neg\varphi_1 \wedge \neg\varphi_2|\varphi_2 \Rightarrow \neg\varphi_1 \wedge \neg\varphi_2 & (T^{\wedge}) \\
\neg\varphi_1 \wedge \neg\varphi_2|\varphi_1 \vee \varphi_2 \Rightarrow \neg\varphi_1 \wedge \neg\varphi_2 & (S^{\vee}).
\end{array}
$$

Similar for $\varphi = \varphi_1 \to \varphi_2$.

We extend Theorems 11 and 12 for a single formula φ to a theory Δ as follows.

Theorem 13. For any consistent theories Δ, Γ and Θ, if $\Delta|\Gamma \Rightarrow \Theta$ is provable in **T** then Θ is a maximal consistent set of Γ by Δ.

Theorem 14. For any consistent theories Δ, Γ and Θ, if Θ is a maximal consistent set of Γ by Δ then $\Delta|\Gamma \Rightarrow \Theta$ is provable in **T**.

6 Conclusion

This paper gave two **R**-calculi **S** and **T**, which both are sound and complete with respect to the maximal consistent sets, given that **T** is a set of axioms without $(S^{\neg})$.

Further work is to reduce (S^{con}) into a set of deduction rules. In **S**, firstly it is to be decided whether φ is consistent with Γ, and if not then we use other deduction rules to reduce φ in $\Delta|\varphi$ into the empty string so that $\Delta|\varphi \Rightarrow \Delta|\lambda$, i.e., $\Delta|\varphi \Rightarrow \Delta$ is provable.

This question seems difficult, owing to the asymmetrical properties of consistence and inconsistence.

Let con(Γ) and incon(Γ) denote that Γ is consistent and inconsistent, respectively. Then, for the consistence,

$$
\begin{array}{llll}
(\text{con}\wedge_1^+) & \dfrac{\text{con}(\varphi_1 \wedge \varphi_2, \Gamma)}{\text{con}(\varphi_1, \Gamma)}, & (\text{con}\wedge_2^+) & \dfrac{\text{con}(\varphi_1 \wedge \varphi_2, \Gamma)}{\text{con}(\varphi_2, \Gamma)}, \\
(\times\text{con}\wedge^-) & \dfrac{\text{con}(\varphi_1, \Gamma) \quad \text{con}(\varphi_2, \Gamma)}{\text{con}(\varphi_1 \wedge \varphi_2, \Gamma)}, & & \\
(\text{con}\vee^+) & \dfrac{\text{con}(\varphi_1, \Gamma)}{\text{con}(\varphi_1 \vee \varphi_2, \Gamma)}, & (\text{con}\vee^-) & \dfrac{\text{con}(\varphi_2, \Gamma)}{\text{con}(\varphi_1 \vee \varphi_2, \Gamma)}, \\
(\text{con} \to^+) & \dfrac{\text{con}(\neg\varphi_1, \Gamma)}{\text{con}(\varphi_1 \to \varphi_2, \Gamma)}, & (\text{con} \to^-) & \dfrac{\text{con}(\varphi_2, \Gamma)}{\text{con}(\varphi_1 \to \varphi_2, \Gamma)}, \\
(\text{con}\neg^-) & \dfrac{\Gamma \nvdash \neg\varphi_1}{\text{con}(\varphi_1, \Gamma)}, & (\neg^+) & \dfrac{\text{con}(\varphi_1, \Gamma)}{\Gamma \nvdash \neg\varphi_1},
\end{array}
$$

and for the inconsistence,

$$
\begin{array}{llll}
(\wedge_1^+) & \dfrac{\text{incon}(\varphi_1, \Gamma)}{\text{incon}(\varphi_1 \wedge \varphi_2, \Gamma)}, & (\wedge_2^+) & \dfrac{\text{incon}(\varphi_2, \Gamma)}{\text{incon}(\varphi_1 \wedge \varphi_2, \Gamma)}, \\
(\times\wedge^-) & \dfrac{\text{incon}(\varphi_1 \wedge \varphi_2, \Gamma)}{\text{incon}(\varphi_1, \Gamma) \text{ or } \text{incon}(\varphi_2, \Gamma)}, & (\wedge^-) & \dfrac{\text{incon}(\varphi_1 \wedge \varphi_2, \Gamma)}{\text{incon}(\varphi_1, \Gamma) \text{ or } \text{incon}(\varphi_2, \Gamma \cup \{\varphi_1\})}, \\
(\vee^+) & \dfrac{\text{incon}(\varphi_1, \Gamma) \ \text{incon}(\varphi_2, \Gamma)}{\text{incon}(\varphi_1 \vee \varphi_2, \Gamma)}, & (\vee^-) & \dfrac{\text{incon}(\varphi_1 \vee \varphi_2, \Gamma)}{\text{incon}(\varphi_1, \Gamma) \quad \text{incon}(\varphi_2, \Gamma)}, \\
(\to^+) & \dfrac{\text{incon}(\neg\varphi_1, \Gamma) \ \text{incon}(\varphi_2, \Gamma)}{\text{incon}(\varphi_1 \to \varphi_2, \Gamma)}, & (\to^-) & \dfrac{\text{incon}(\varphi_1 \to \varphi_2, \Gamma)}{\text{incon}(\neg\varphi_1, \Gamma) \quad \text{incon}(\varphi_2, \Gamma)}, \\
(\neg^-) & \dfrac{\Gamma \vdash \neg\varphi_1}{\text{incon}(\varphi_1, \Gamma)}, & (\neg^+) & \dfrac{\text{incon}(\varphi_1, \Gamma)}{\Gamma \vdash \neg\varphi_1}.
\end{array}
$$

From these rules, we see that the rules for the consistence are not dual to those for the inconsistence; $(\times\wedge^-)$ should be replaced by $(\wedge^-)$; and each other rule has an inverse rule.

For the asymmetry of the rules for inconsistence, we should make the rules for the revision asymmetrical.

Acknowledgements

This work was supported by Open Fund of the State Key Laboratory of Software Development Environment, Beihang University (Grant No. SKLSDE-2010KF-06), and National Basic Research Program of China (973 Program) (Grant No. 2005CB321901).

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