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A fractal and scale-free model of complex networks with hub attraction behaviors

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Abstract It is widely believed that fractality of complex networks originate from hub repulsion behaviors (anticorrelation or disassortativity), which means that large degree nodes tend to connect with small degree nodes. This hypothesis was demonstrated by a dynamical growth model, which evolves as the inverse renormalization procedure, proposed by Song et al. Now we find that the dynamical growth model is based on the assumption that all the cross-box links have the same probability e to link to the most connected nodes inside each box. Therefore, we modify the growth model by adopting the flexible probability e, which makes hubs to have higher probability to connect with hubs than non-hubs. With this model, we find that some fractal and scale-free networks have hub attraction behaviors (correlation or assortativity). The results are the counter-examples of former beliefs. Actually, the real-world collaboration network of movie actors also is fractal and shows assortative mixing.

Keywords scale-free, fractal network, self-similarity, fractal dimension, hub attraction

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1 Introduction

Fractal Geometry theory was proposed by Mandelbrot to explain the mathematical set which has a fractal dimension beyond its topological dimension [1,2]. As to the fractality, it means the patterns that exhibit self-similarity at different length scales. Nowadays, more and more researchers try to explore the fractality of complex networks. As we know, many complex networks are scale-free networks [3–5], that is, the probability $P(k)$ (k is the connections to a node) fulfills the power-law relation as in (1), which is quite similar to the definition of fractality:

$$P(k) \approx k^{-\gamma}. \tag{1}$$

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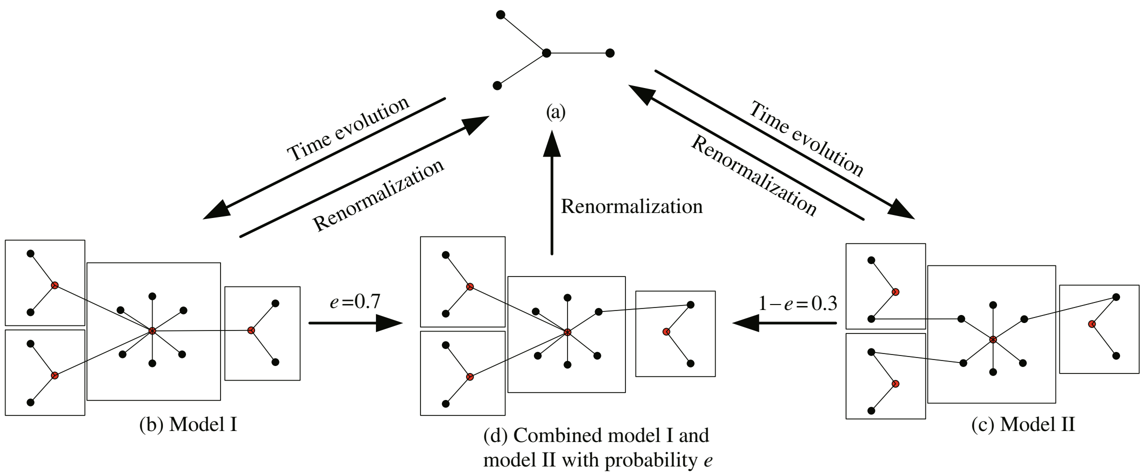

Figure 1 Dynamical growth networks with parameters: offspring nodes parameter $m = 2$ and box size $\ell_B = 3$. The dynamical growth process can be seen as the inverse renormalization procedure. (a) Four nodes at $t = 0$ that could grow to networks in (b)–(d) through time evolution; (b) mode I alone, where most connected nodes in each box connect directly to each other; (c) mode II alone, where boxes connect by less connected nodes; (d) combination of modes I and II with probability $e = 0.7$.

Thus, it is a question whether complex networks have the fractal property [6–10] in topological space or not. Song et al. [6], therefore, proposed a renormalization procedure to explore the fractal property in complex networks. The renormalization procedure is implemented by the box-covering method, revised from the classic "box counting" method. This box-covering method actually tiles the networks into boxes with a given box size as shown in Figure 1. The box size ℓ_B is the upper bound of shortest paths between any pair of nodes in each box. The nodes in each box would be merged to one node, and the process is iterated until only one single node is left. Finally, they found that many real world networks, such as the World Wide Web, the Protein-Protein Interaction networks, and the Cellular networks have scale-invariance and fractal properties. The scale-invariance means the degree distribution of the renormalized networks still follows the power-law under different length scale renormalizations, and the fractality means the number of boxes N_B, which is needed to cover the whole network, is approximate to power $-d_B$ of the box size ℓ_B. This can be defined as

$$N_B(\ell_B) \approx \ell_B{}^{-d_B}. \tag{2}$$

Song et al. later analyzed the origin of the fractality on complex networks [7]. They proposed the dynamical growth model (DGM) from the perspective of networks evolution based on the ideas of the Barabási-Albert model [4]. Actually, this model is the inverse of the renormalization procedure as shown in Figure 1. In this model, the network evolves as time step t increases; every node at step $t-1$ with degree $k(t-1)$ will evolve to a visual box with $mk(t-1)$ offspring nodes (black nodes) generated inside each box, where m controls the number of offspring nodes; and initial nodes (red nodes) connect to all the offspring nodes inside each box. They proposed two kinds of models, which have different methods of generating cross-box links. In Model I, most connected nodes in each box connect directly to each other as shown in Figure 1(b), whereas in Model II, cross-box links connect to less connected nodes as shown in Figure 1(c). The extensive growth model is the combination of Model I and Model II with a stationary probability e throughout the whole network as shown in Figure 1(d), where e is defined as the measurement of the level of "hub" attraction. Their results have shown that networks with higher probability of e tend to be non-fractal, yet networks with lower probability of e are inclined to be fractal. Therefore, they concluded that "hub" repulsion is the cause of fractality.

Similar researches claimed that fractal scale-free networks are disassortative mixing [11], and the fractal networks have to fulfill a criticality condition [12]. The criticality condition says that the skeleton, which has a branching tree structure, grows from the root node (most connected node in network) perpetually with offsprings neither flourishing nor dying out.

However, the above assertions are based on experiments rather than theoretical proofs. In this paper, we get different results by simply modifying the DGM.

As we know, hubs are defined as the most connected nodes in the whole network [3]. We noticed that Song et al. presumed the most connected nodes in each box as the hubs of whole network. In fact, because of the power-law degree distribution, most of the highest degree nodes in boxes have much lower degrees compared with the real hubs in the network. Therefore, we apply flexible probability e mechanism on cross-box links. By this mechanism, we makes real hubs have higher probability to connect with each other, while highest degree nodes in boxes but non-hubs have lower probability of direct connections. By applying this mechanism, we can achieve fractal and scale-free networks with strong hub attraction. As expected, we found that this phenomenon could be observed in social networks, such as the collaboration network of movie actors [4]. The actor network has proved to be a fractal network [6]. Meanwhile, it also shows assortative mixing, because high degree actors have high probability of collaboration. Besides, we found relative research work supporting our statement. Some optimization networks exhibit both fractal and assortative mixing properties [13]. Overall, our research gives a fundamental challenge to the former researches on the origin of fractality on complex networks. Hence, we reinvestigate the origin of fractality on complex networks. We found there was a structure equilibrium in the fractal networks. The hub attraction and boundary growth are a pair of opposing forces to maintain the structural equilibrium.

This paper is organized as follows. In Section 2, we introduce the hub attraction dynamical growth model (HADGM). The two subsections present the flexible probability e mechanism and within-box link-growth method, respectively. In Section 3, we analyze the properties of the HADGM networks, such as fractality, scale-free, and correlations. We also reinvestigate the origin of fractality in complex networks by proposing a structure equilibrium theory. In Section 4, we introduce the fractal optimization network and compare the assortativity of the synthetic and real-world networks. Finally, in Section 5, we summarize our works and discuss some researches of fractal dimensions.

2 Hub attraction dynamical growth model

The hub attraction dynamical growth model is based on the dynamical growth framework of DGM. Two methods are applied to modify the DGM as shown in Figure 2. Firstly, we make probability e flexible (hubs can have higher probability to connect with each other than non-hubs). This mechanism alone could make hubs connect together, while persevering the fractal property. Secondly, we find that DGM networks are only spinning trees without loops, which are not similar with real-world networks. Thus, we propose the within-box link-growth method, which increases the clustering coefficient of the HADGM networks.

2.1 Flexible probability e mechanism

In the dynamical growth processes, each edge δ_{ij} at step $t-1$ will become cross-box link $\delta_{ij}^{'}$ at step t. The parameter e determines the probability of high-degree nodes attraction on cross-box link $\delta_{ij}^{'}$. We define probability e as a piecewise function $e = f(\delta_{ij}^{'})$, which depends on the degrees of nodes i and j that are attached on both sides of the edge δ_{ij} at step $t-1$. As shown as

$$e = f(\delta_{ij}^{'}) = f(k_i(t-1), k_j(t-1)) = \begin{cases} a, & \text{if } \dfrac{k_i(t-1)}{k_{\max}(t-1)} > T \text{ and } \dfrac{k_j(t-1)}{k_{\max}(t-1)} > T, \\ b, & \text{if } \dfrac{k_i(t-1)}{k_{\max}(t-1)} \leqslant T \text{ or } \dfrac{k_j(t-1)}{k_{\max}(t-1)} \leqslant T, \end{cases} \quad (3)$$

where T, a, and b are predefined parameters with domains as $0 \leqslant a \leqslant 1$, $0 \leqslant b \leqslant 1$, $0 < T \leqslant 1$ and $k_{\max}(t-1)$ is the maximal degree in the network at step $t-1$. Here, we choose $T = 0.5$ in all HADGM networks for simplicity. Therefore, if we define $a > b$, hubs in the network will have higher probability to connect with each other than non-hubs. We can even define $a = 1$ to force all hubs to be connected with each other as shown in Figure 2(a2) and Figure 2(b), where $m = 2$ and $b = 0.5$.

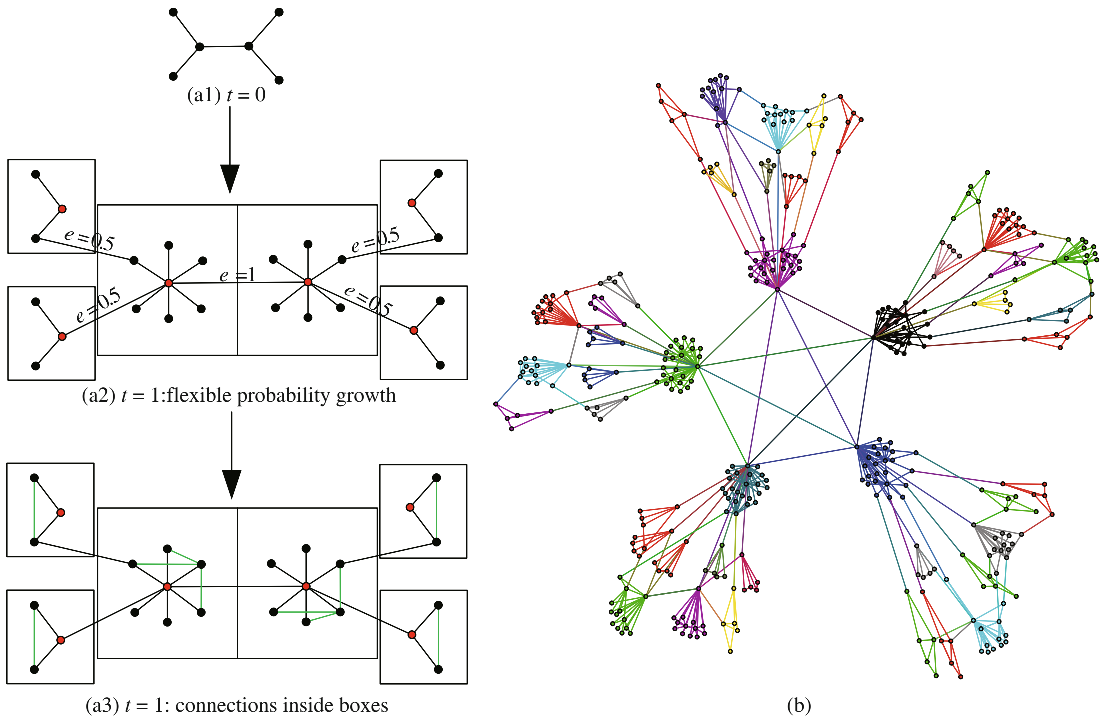

Figure 2 Hub attraction dynamical growth model. The demonstrated networks have $a = 1$, $b = 0.5$, and $m = 2$. (a1) Six nodes at step $t = 0$; (a2) dynamical growth network at $t = 1$ with flexible probability e, where $e = 1$ between high degree nodes ($k_i(0)/k_{\max}(0) > 0.5$ and $k_j(0)/k_{\max}(0) > 0.5$) and $e = 0.5$ between low degree nodes ($k_i(0)/k_{\max}(0) \leqslant 0.5$ or $k_j(0)/k_{\max}(0) \leqslant 0.5$); (a3) the method of generating connections inside box (green line). We randomly pick one offspring node in each box and link it to other $\tilde{k}(t-1)$ number of offspring nodes. (b) A generated HADGM network at step $t = 2$. Different colors represent the boxes with size $\ell_B = 3$.

2.2 Within-box link-growth method

As most real-world networks have groups that cluster together with relatively high density of ties [14], we apply the within-box link-growth method to increase the clustering coefficient of the HADGM networks. As shown in Figure 2(a3), at each step of dynamical growth, we apply the within-box link-growth method after the flexible probability growth phase. By this method, at step t, we add $\tilde{k}(t-1)$ links in each box. Thus, we add $2\tilde{K}(t-1)$ links in the whole network at time step t, where $\tilde{K}(t)$ is the total number of links at time step t. Besides, this method does not affect the fractal and scale-free properties of the networks. We will prove this in Section 3.

3 Properties of HADGM networks

To measure the properties of HADGM networks, we apply a mathematic framework [for further deductions see the Appendix section]:

$$\begin{aligned} \tilde{N}(t) &\approx (2m+3)\tilde{N}(t-1), \quad \text{for} \;\; t>1, \\ \tilde{k}(t) &= (m+\bar{e})\tilde{k}(t-1), \\ \tilde{L}(t) &= (3-2\bar{e})\tilde{L}(t-1)+2\bar{e}, \end{aligned} \tag{4}$$

where $\tilde{N}(t)$ is the total number of nodes at step t, $\tilde{k}(t)$ is the maximal degree inside a box at step t, $\tilde{L}(t)$ is the diameter of the network at step t, and $\bar{e}$ is the average of flexible probability e. $\bar{e}$ can be written as

$$\bar{e} = aP\left(E \mid \frac{k_i}{k_{\max}} > T \cap \frac{k_j}{k_{\max}} > T\right) + bP\left(E \mid \frac{k_i}{k_{\max}} \leqslant T \cup \frac{k_j}{k_{\max}} \leqslant T\right), \tag{5}$$

where E is the edge that connecting a pair of nodes i and j with degrees k_i and k_j and $P(\cdot)$ is the probability of the event.

Fractality of HADGM networks. The HADGM networks evolve dynamically as the inverse renormalization procedure. Each node at step $t-1$ will grow into a virtual box at step t. Thus, based on Eq. (4), we have $\tilde{N}(t_1)/\tilde{N}(t_2) = N_B(\ell_B)/N = (2m+3)^{t_1-t_2}$. We also have $(\tilde{L}(t_2)+L_0)/(\tilde{L}(t_1)+L_0) = \ell_B + L_0 = (3-2\bar{e})^{t_2-t_1}$, where L_0 is the initial diameter [7]. Therefore, by inputting (2) and replacing the time interval t_2-t_1, we have the fractal dimension d_B as

$$d_B \approx \frac{\ln(2m+3)}{\ln(3-2\bar{e})}. \tag{6}$$

According to (4) and (6), d_B has to be a finite number to keep fractality, which means that $3-2\bar{e} > 1$. Thus, it requires the diameter $\tilde{L}(t)$ of the network to be the exponential growth with time evolution $(3-2\bar{e} > 1)$, rather than the linear growth $(3-2\bar{e} = 1)$. Therefore, to explain the exponential growth of $\tilde{L}(t)$, we believe that there is a structural equilibrium state in fractal networks.

Hub attraction and boundary growth are a pair of opposing forces to maintain the structural equilibrium in fractal network. Hub attraction will lead to $\tilde{L}(t)$ to drop dramatically, whereas the booming growth in boundary will result in $\tilde{L}(t)$ increasing sharply. In HADGM, hub attraction can be represented by parameter a, whereas the boundary growth can be represented by parameter b. If we apply the strongest hub attraction force as $a=1$, the transition from fractal to non-fractal will be dominated by boundary growth parameter b. On the contrary, if we apply the weakest boundary growth as $b=1$, the transition will be dominated by hub attraction force a.

To display the fractality, we apply the box-covering algorithm [15,16] to HADGM networks, which have different values of b while other parameters are the same. As shown in Figure 3(a), the results show that the HADGM networks are fractal for $b < 0.9$, and non-fractal for $b \geqslant 0.9$. As (6), if $\bar{e} \to 1$, then $d_B \to \infty$, which will make the HADGM networks become non-fractal. As here we apply $a=1$, according to (5), we may expect the HADGM networks become non-fractal when $b \to 1$. Surprisedly, our experiments show that the HADGM networks become non-fractal when $b \to 0.9$. Therefore, we analyze the dependency of fractal dimension d_B on b. As shown in Figure 3(d), the fractal dimensions d_B of HADGM networks (for both $m=2$ and $m=3$) increase smoothly on $b < 0.9$, after $b = 0.9$, the curves have a sudden rise. At this point, because of the strongest hub attraction $a=1$ and week boundary growth $b=0.9$, the structural equilibriums are broken on HADGM networks, which cause the transition from fractal to non-fractal. Also, we illustrate that the hub repulsion networks can be non-fractal because of the weakest boundary growth. As shown in Figure 3(b), the box number of the HADGM network with $a=0.9$, $b=1$, and $m=2$ shows exponential decay. To ensure the accuracy of results in Figure 3, we run the algorithm 20 times for every parameter setting.

Scale-free of HADGM networks. The recursive growth of node degree $\tilde{k}(t)$ and node number $\tilde{N}(t)$ leads to the power-law degree distributions of HADGM networks based on (4). The results are illustrated in Figure 3(c), the HADGM networks with different values of b clearly show the power-law degree distributions with fat tails. Because of the difference in average probability $\bar{e}$, the two HADGM networks have different slopes of the power-law curves.

Correlation and fractality. Previous research [7] asserted that fractal networks tend to show anticorrelation behaviors, whereas non-fractal networks exhibit correlation behaviors. The results are achieved by comparing the strengths of anticorrelation between different networks, such as non-fractal DGM networks with $e=1$ have stronger correlation than fractal DGM networks with $e=0.8$. However, our comparison results show that the fractal HADGM networks have even stronger correlations than non-fractal DGM networks with $e=1$. Therefore, our results are different from the assertion that the hub repulsion behaviors give rise to the fractal property.

The correlation is quantified by the equation: $R(k_1,k_2) = P(k_1,k_2)/P_r(k_1,k_2)$, which illustrates the correlation topological property of a network [17]. Given a network G, the $P(k_1,k_2)$ is defined as the joint probability of finding a node with degree k_1 linked to a node with degree k_2. $P_r(k_1,k_2)$ is defined as the joint probability of the null model G'. The null model G' is generated by randomly rewiring all

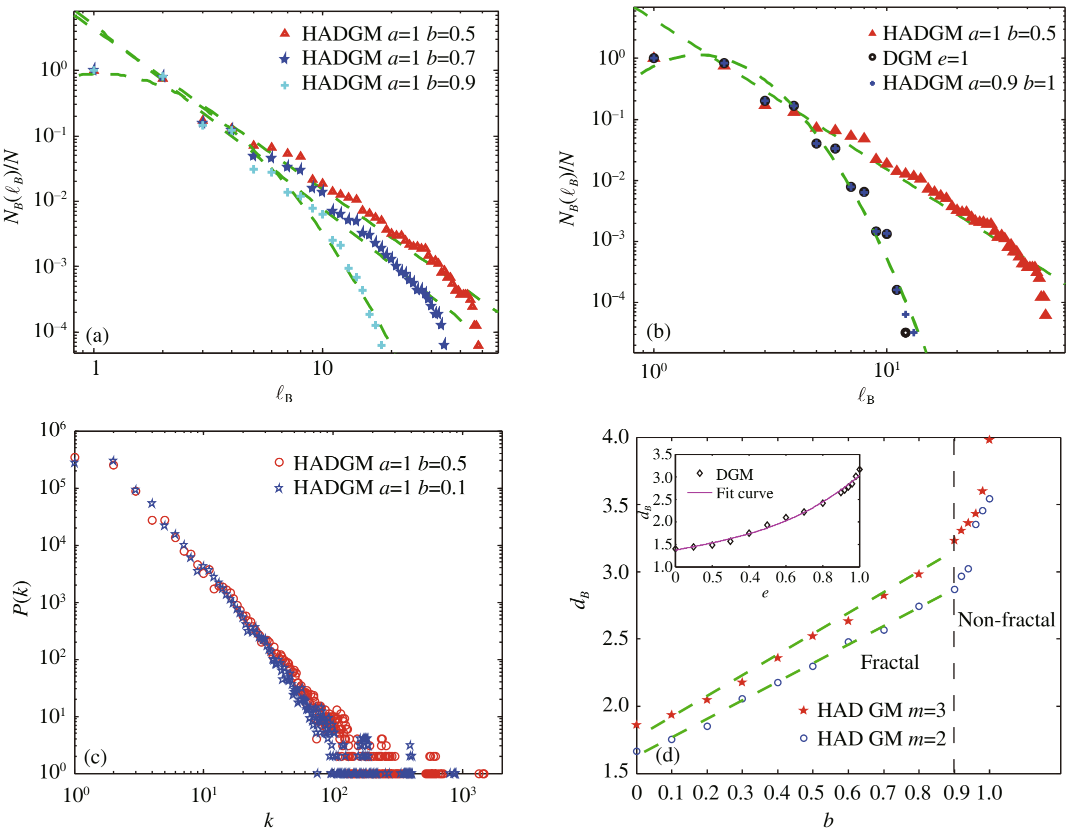

Figure 3 (a) Fractality of HADGM networks with different b. They show fractal properties on $b = 0.5$ and $b = 0.7$. As b increases to 0.9 the network becomes non-fractal. (b) Fractality of 3 networks. They are a fractal HADGM network ($a = 1$, $b = 0.5$), a non-fractal DGM network ($e = 1$), and a non-fractal HADGM network ($a = 0.9$, $b = 1$). (c) Degree distributions of HADGM networks as $b = 0.5$ (red circle) and $b = 0.1$ (blue star). They both have $a = 1$ and $m = 2$. (d) Dependency of fractal dimension on b. We test two groups of HADGM networks with $m = 2$ and $m = 3$, and both groups have $a = 1$. The inset shows that d_B grows exponentially with e on DGM networks.

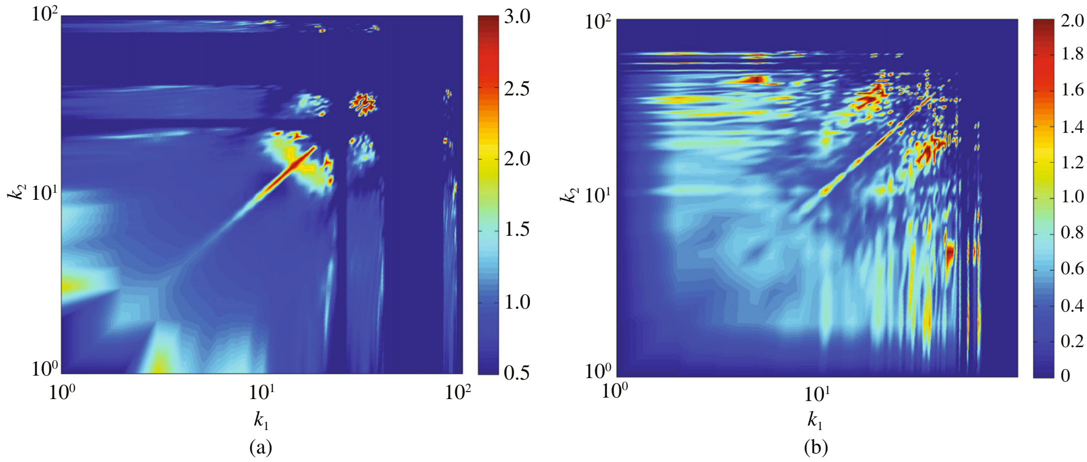

Figure 4 Correlation comparisons. Different colors relate to different strengths of correlations. Red in the plot represents strong correlation. The HADGM network shows stronger correlation between high-degree nodes than the other two networks. (a) Plot of $R_{\rm HADGM}(k_1, k_2)/R_{\rm DGM}(k_1, k_2)$ that compares the correlation of the HADGM network (fractal with $a = 1$, $b = 0.5$) with DGM network (non-fractal with $e = 1$); (b) plot of $R_{\rm HADGM}(k_1, k_2)/R_{\rm WWW}(k_1, k_2)$ that compares the correlation of the HADGM network (fractal with $a = 1$, $b = 0.5$) with the World Wide Web.

Table 1 Fractality and disassortativity

Network	N	r	Fractality	Disassortativity
DGM $e = 1$	781245	-0.0347	No	Yes
HADGM $a = 1, b = 0.5$	784325	-0.0344	Yes	Yes for $k < 100$
Optimization network	1500	0.8354	Yes	No
The Internet	22963	-0.198	No	Yes
Actor	520223	0.204	Yes	No

links of G, while keeping the degree distribution. We compare the correlation of a fractal HADGM network ($a = 1, b = 0.5$) with the non-fractal DGM network ($e = 1$) and the World Wide Web (WWW) [18]. As shown in Figure 4, the plots of $R_{\mathrm{HADGM}}(k_1, k_2)/R_{\mathrm{DGM}}(k_1, k_2)$ and $R_{\mathrm{HADGM}}(k_1, k_2)/R_{\mathrm{WWW}}(k_1, k_2)$ indicate that the fractal HADGM network ($a = 1, b = 0.5$) has stronger hub attraction behaviors than the fractal WWW network and non-fractal DGM network ($e = 1$).

4 Assortativity and fractality

The optimization model proposed by Zheng et al. supports our statement [13]. It can produce fractal scale-free networks with hubs aggregation together. This model has two optimization objectives: minimize the summation of the node degrees and maximize the summation of the edge degrees. And with constraint conditions that both the $\bar{\ell}$ (average shortest path) and $xmin$ (minimum degree of nodes throughout the entire network) are set as non-negative constants. The edge degree between nodes i and j is defined as: $D_{ij} = {k_i}^m {k_j}^n$, where k_i and k_j are the degrees of nodes i and j and m, n are non-negative constants. The optimization model can produce fractal scale-free networks by stretching $\bar{\ell}$. Here, we choose a optimization model network for comparison, which is a fractal and scale-free network and has node number $N = 1500$, $x_{\min} = 2$, $m = 0$, $n = 1$, and $\bar{\ell} = 40$.

Now we compare the assortativity of above synthetic networks and some real-world networks. The assortativity can be measured using the Pearson correlation coefficient r [19], which can be rewritten as

$$r = \frac{K^{-1}\sum_i j_i k_i - \left[K^{-1}\sum_i \frac{1}{2}(j_i + k_i)\right]^2}{K^{-1}\sum_i \frac{1}{2}({j_i}^2 + {k_i}^2) - \left[K^{-1}\sum_i \frac{1}{2}(j_i + k_i)^2\right]}, \tag{7}$$

where j_i, k_i are the degrees of the nodes at both sides of the ith edge, with $i = 1, \ldots, K$, and K is the total number of edges in a network. The positive or negative of r related to assortative or disassortative mixing, respectively. As shown in Table 1, the DGM network ($e = 1$) is non-fractal and disassortative, HADGM network ($a = 1, b = 0.5$) is fractal and disassortative, the optimization network is fractal [13] and assortative, the Internet of AS-level [20] is non-fractal and disassortative, and the actor collaboration network is assortative for $r = 0.204$; meanwhile, it has been proved to be fractal with fractal dimension $d_B = 6.3$ [6].

We also analyze the neighbor connectivity $\langle k_{nn} \rangle$ of the above networks. Neighbor connectivity is the average degree of neighbors of a node with degree k. It is defined as $\langle k_{nn} \rangle = \sum_{k'} k' P(k'|k)$, where $P(k'|k)$ is the conditional probability of an edge of node with degree k connected to a node with degree k' [21]. The network is assortative mixing if the slope of the $\langle k_{nn} \rangle$ curve is positive, which means high-degree nodes tend to connect with high-degree nodes. It is disassortative mixing if the slope of the $\langle k_{nn} \rangle$ curve is negative [19].

As shown in Figure 5(a), the fractal network produced by the optimization model is assortative and the non-fractal DGM network ($e = 1$) is disassortative. It also shows that the fractal HADGM network ($a = 1, b = 0.5$) is disassortative as $k < 100$, while it is assortative as $k > 100$. The Internet shows obvious disassortativity and non-fractal property as shown in Figure 5(b). And the actor network exhibits assortative mixing as shown in Figure 5(c). Therefore, the results of the DGM network ($e = 1$), optimization model, the Internet, and the actor network are served as counter examples of the conjecture

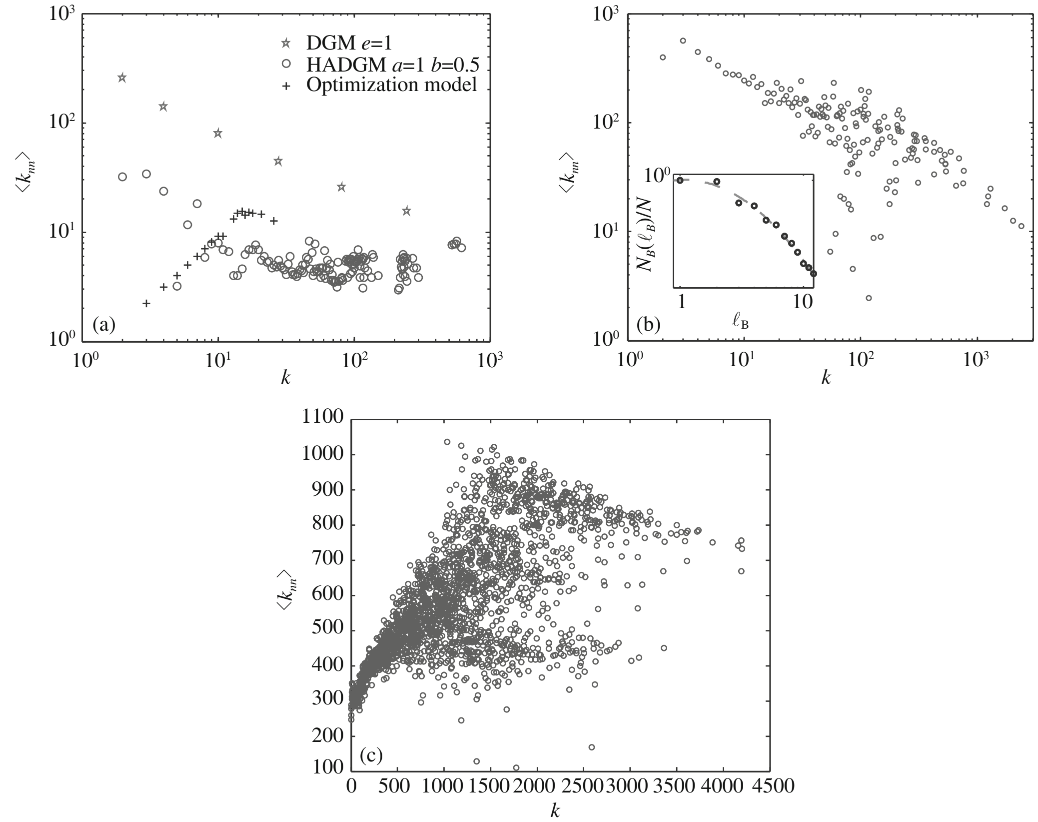

Figure 5 Neighbor connectivity of networks. (a) Log-log plot of the neighbor connectivity of the DGM network ($e = 1$) (star), the HADGM network (circle), and the optimization fractal network (cross). (b) Log-log plot of neighbor connectivity of the Internet. The inset shows the non-fractal of the Internet. (c) Linear plot of neighbor connectivity of the actor network, which shows assortative mixing.

that self-similar scale-free networks are disassortative mixing. Also, the HADGM network ($a = 1, b = 0.5$) has stronger correlation than the DGM network ($e = 1$), yet the DGM network ($e = 1$) is non-fractal and the HADGM network ($a = 1, b = 0.5$) is fractal. Overall, our results show that the fractal property is independent of the assortative mixing.

5 Conclusion and discussion

In this paper, we proposed a novel dynamical growth model, which can produce fractal networks with hub attraction behaviors. The model is based on the traditional DGM and modified by applying flexible probability of cross-box links mechanism and within-box link-growth method. More general models can also be proposed by applying different functions of flexible probability e or other kinds of link-growth methods.

There are several fractal dimensions that have been proposed, such as the correlation dimension based on the ergodic theory [22,23] and the fractal dimension represented by the average density [24,25]. The above definitions obtain similar results of the fractal dimensions comparing with box-covering dimension. Thus, our method should obtain similar results on the above fractal dimensions. Besides, there is an interesting research work using fluctuation analysis to qualify long-range correlation in complex networks [26]. It shows that the fluctuation functions of fractal networks display power-law behaviors. It is worth noticing that our definitions of a and b only control the probability of attraction or repulsion between neighbor nodes. To understand the origin of fractality, the long-range correlation needs to be taken into consideration for future work.

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Appendix A

According to the rule of HADGM, at step $t+1$ we generate $mk(t)$ offspring nodes for each node with degree $k(t)$ at step t. Therefore, we have

$$\tilde{N}(t+1) = \tilde{N}(t) + 2m\tilde{K}(t), \tag{A1}$$

where $\tilde{K}(t)$ is total number of links at step t. At step $t+1$, in the flexible probability growth phase, we have $(2m+1)\tilde{K}(t)$ edges in network, and then in the within-box link-growth phase, we add $2\tilde{K}(t)$ edges. Thus, we have

$$\tilde{K}(t+1) = (2m+3)\tilde{K}(t). \tag{A2}$$

Combining (A1) and (A2), we have

$$\tilde{N}(t) = \tilde{N}(0) + m\tilde{K}(0)\frac{(2m+3)^t - 1}{m+1}. \tag{A3}$$

Because $\tilde{N}(t) \gg \tilde{N}(0)$, we can acquire $\tilde{N}(t) \approx (2m+3)\tilde{N}(t-1)$ for $t > 1$.

Considering that the node degrees growth with time evolution, for Model I alone, we have $\tilde{k}(t) = (m+1)\tilde{k}(t-1)$ and for Model II alone, we have $\tilde{k}(t) = m\tilde{k}(t-1)$. Therefore, with probability $\bar{e}$ to combine Model I and Model II, we have $\tilde{k}(t) = (m+\bar{e})\tilde{k}(t-1)$.

Finally, we investigate the diameter growth with time evolution. For Model I alone, we have $\tilde{L}(t) = \tilde{L}(t-1)+2$ and for Model II alone, we have $\tilde{L}(t) = 3\tilde{L}(t-1)$. Therefore, with probability $\bar{e}$ to combine Model I and Model II, we have $\tilde{L}(t) = (3-2\bar{e})\tilde{L}(t-1) + 2\bar{e}$.