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Observer-based adaptive fuzzy backstepping control of uncertain nonlinear pure-feedback systems

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Abstract In this paper, a new fuzzy adaptive control approach is developed for a class of SISO uncertain pure-feedback nonlinear systems with immeasurable states. Fuzzy logic systems are utilized to approximate the unknown nonlinear functions; and the filtered signals are introduced to circumvent algebraic loop systems encountered in the implementation of the controller, and a fuzzy state adaptive observer is designed to estimate the immeasurable states. By combining the adaptive backstepping technique, an adaptive fuzzy output feedback control scheme is developed. It is proven that the proposed control approach can guarantee that all the signals of the resulting closed-loop system are semi-globally uniformly ultimately bounded (SGUUB), and the observer and tracking errors converge to a small neighborhood of the origin by appropriate choice of the design parameters. Simulation studies are included to illustrate the effectiveness of the proposed approach.

Keywords nonlinear pure-feedback systems, adaptive fuzzy control, backstepping technique, state observer, stability

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1 Introduction

In recent years, adaptive backstepping control for uncertain nonlinear strict-feedback systems has been widely studied using fuzzy logic systems (FLSs) and neural networks (NNs), and many significant developments have been achieved (see for example, [1–13] and the references therein). The adaptive fuzzy and neural backstepping control approaches in [1–6] are developed for single-input and single-output (SISO) uncertain nonlinear strict-feedback systems, and those in [7–9] are for multiple-input and multiple-output (MIMO) uncertain nonlinear strict-feedback systems, while [10–13] are for the uncertain SISO nonlinear strict-feedback systems with immeasurable states. In general, these adaptive fuzzy or NN backstepping controllers can provide a systematic methodology of solving tracking or regulation control problems,

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where FLSs or NNs are used to approximate unknown nonlinear functions and typically adaptive fuzzy or NN controllers are constructed recursively in the framework of backstepping design technique. The main features of these adaptive approaches include (i) they can deal with those nonlinear systems without satisfying the matching conditions, and (ii) they do not require that the unknown nonlinear functions be linearly parameterized.

Although many important results have been achieved for the adaptive fuzzy and NN backstepping control for uncertain nonlinear strict-feedback systems, only a few results in the literatures can be available for solving the problem of adaptive control of uncertain nonlinear pure-feedback systems. As shown in [14–16], the pure-feedback system represents a more general class of triangular systems which have no affine appearance of the variables to be used as virtual controls. In practice, there are many systems falling into this category, such as mechanical systems, aircraft flight control systems, biochemical process, Duffing oscillator, etc. It is quite restrictive and difficult to find the explicit virtual controls to stabilize the pure-feedback system by using the existing backstepping technique. To solve the above control problem, adaptive NN control schemes in [15,16] are proposed for a class of pure-feedback systems, where the last equation of the controlled system is affine nonlinear to avoid the algebraic loop problem. Direct adaptive NN or fuzzy control approaches in [17–20] are developed for some classes of uncertain pure-feedback systems without or with time delays, where an implicit theorem is exploited to assert the existence of continuous desired feedback controllers, and NNs or FLSs are utilized to approximate these desired feedback controllers. However, the bounds and the signs of the derivatives of the nonlinear functions for all the variables are assumed to be known. Therefore, a priori knowledge of the plant dynamics was required to determine these signs and bounds, which are difficult to acquire in practical applications as indicated by [21]. More recently, an adaptive fuzzy backstepping control approach in [21] are developed for a class of uncertain nonlinear pure-feedback systems by using FLSs. The proposed approach not only relaxed the restrictive conditions in [17–20] that the bounds and the signs of the derivatives of the nonlinear functions for all the variables are assumed to be known, but also avoided the algebraic loop problem in [15,16] by introducing the filtered signals into the backstepping control design.

Despite these efforts in adaptive fuzzy and NN backstepping control, the above mentioned fuzzy or NN adaptive control methods are all based on the assumption that the states of the controlled systems are available for measurement. It is well known that state variables are often immeasurable for many practical nonlinear systems. In such a case, observer-based control schemes should be required. It is worth mentioning that in the case of linear systems, output-feedback control problems can be solved by combining state-feedback controllers with a state observer. However, the separation principle doses not hold for nonlinear systems; thus the observer-based adaptive output feedback backstepping control design and stability analysis are more complex and difficult than the counterpart of state feedback. To our best knowledge, to date, few attempts have been made on adaptive fuzzy or neural network backstepping controllers for uncertain SISO nonlinear pure-feedback systems with immeasurable states, which are important and more practical. This has motivated us to make this study.

Starting with the aforementioned observations, in this paper, an adaptive fuzzy backstepping output feedback control approach is proposed for a class of SISO nonlinear pure-feedback systems with immeasurable states. FLSs are first used to approximate the unknown nonlinear functions, and then a fuzzy state observer is designed for estimating the immeasurable states. Based on the backstepping design technique, an observer-based adaptive fuzzy output feedback approach is developed. It is proven that the proposed control approach can guarantee that all the signals of the resulting closed-loop system are SGUUB, and the observer and tracking errors converge to a small neighborhood of the origin by appropriate choice of the design parameters. The main advantages of the proposed control approach are as follows: (i) by designing a fuzzy state observer, the proposed adaptive control method removes the restrictive assumption in [15–21] that all the states of the system are available for measurement, (ii) by incorporating the filtered signals into the controllers designs, the proposed adaptive control method can avoid the algebraic loop problem in [15,16], and relax some restrictive conditions in [17–20] that the bounds and the signs of the derivatives of the nonlinear functions for all the variables are assumed to be known.

2 Problem formulations and preliminaries

2.1 System descriptions control problems

Consider the following uncertain SISO pure-feedback nonlinear system:

$$\begin{cases} \dot{x}_i = f_i(\underline{x}_i, x_{i+1}) + x_{i+1}, \quad i = 1, 2, \ldots, n-1, \\ \dot{x}_n = f_n(\underline{x}_n, u) + u, \\ y = x_1, \end{cases} \tag{1}$$

where $\underline{x}_i = (x_1, x_2, \ldots, x_i)^{\mathrm{T}}$, $i = 1, 2, \ldots, n$ are the state vectors of the system, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are system input and output, respectively; $f_i(\cdot)$, $i = 1, 2, \ldots, n$ are smooth unknown nonlinear functions. In this paper, it is assumed that only output y is available for measurement.

Let $f_i'(\underline{x}_i, x_{i+1}) = f_i(\underline{x}_i, x_{i+1}) + x_{i+1}$, $i = 1, 2, \ldots, n-1$ and $f_n'(\underline{x}_n, u) = f_n(\underline{x}_n, u) + u$. Then the nonlinear pure-feedback system (1) can be transformed into the following pure-feedback nonlinear systems considered in [17,20]:

$$\begin{cases} \dot{x}_i = f_i'(\underline{x}_i, x_{i+1}), \quad i = 1, 2, \ldots, n-1, \\ \dot{x}_n = f_i'(\underline{x}_n, u), \\ y = x_1. \end{cases} \tag{2}$$

Remark 1. It should be mentioned that if the states are available for measurement, then system (1) is the one considered by [21]. If the states are available for measurement and $\dot{x}_n = f_n(\underline{x}_n) + \gamma(\underline{x}_n)u$, with $\gamma(\underline{x}_n)$ being a known function, then system (1) is the model studied by [15,16]. If the states are available for measurement and the bounds and signs of $\partial f_i'(\underline{x}_i, x_{i+1})/\partial x_{i+1}$ and $\partial f_i'(\underline{x}_i, u)/\partial u$ are known, then system (2) becomes the models in [17,20]. In this paper, the states are not required to be available for feedback; thus the control design and stability analysis are more difficult than in [15–20].

Assumption 1. There exists a set of known constants m_i, $i = 1, 2, \ldots, n$, such that $\forall X_1, X_2 \in \mathbb{R}^i$ the following inequality holds: $|f_i(X_1) - f_i(X_2)| \leqslant m_i \|X_1 - X_2\|$, where $\|X\|$ denotes the 2-norm of a vector X.

Our control objective is to design an adaptive fuzzy output feedback control scheme by using FLSs so that all the signals involved in the resulting closed-loop system are SGUUB, and the state observer and tracking errors are as small as desired.

2.2 Fuzzy logic system

A fuzzy logic system (FLS) consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine, and the defuzzifier. The knowledge base is composed of a collection of fuzzy If-then rules of the following form:

R^l: If x_1 is F_1^l and x_2 is F_2^l and … and x_n is F_n^l, then y is G^l, $l = 1, 2, \ldots, N$, where $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ and y are the FLS input and output, respectively. Fuzzy sets F_i^l and G^l are associated with the fuzzy membership functions $\mu_{F_i^l}(x_i)$ and $\mu_{G^l}(y)$, respectively. N is the rule number of IF-THEN.

Through singleton fuzzifier, center average defuzzification and product inference [22], the FLS can be expressed as

$$y(x) = \frac{\sum_{l=1}^{N} \bar{y}_l \prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{N} [\prod_{i=1}^{n} \mu_{F_i^l}(x_i)]}, \tag{3}$$

where $\bar{y}_l = \max_{y \in \mathbb{R}} \mu_{G^l}(y)$.

Define the fuzzy basis functions as

$$\varphi_l = \frac{\prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{N} (\prod_{i=1}^{n} \mu_{F_i^l}(x_i))}. \tag{4}$$

Denote $\theta^{\mathrm{T}} = [\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_N] = [\theta_1, \theta_2, \ldots, \theta_N]$ and $\varphi(x) = [\varphi_1(x), \ldots, \varphi_N(x)]^{\mathrm{T}}$. Then fuzzy logic system (3) can be rewritten as

$$y(x) = \theta^{\mathrm{T}}\varphi(x). \tag{5}$$

Lemma 1 ([22])**.** Let $f(x)$ be a continuous function defined on a compact set Ω. Then for any constant $\varepsilon > 0$, there exists a fuzzy logic system (5) such that $\sup_{x\in\Omega}\left|f(x) - \theta^T\varphi(x)\right| \leqslant \varepsilon$.

3 State observer design

Note that the states $x_2, x_3, \ldots, x_{n-1}$ and x_n in system (1) are not available for feedback; therefore, a state observer should be established to estimate the unmeasured states, and then fuzzy adaptive output feedback control scheme is investigated based on the designed state observer.

To begin with, rewrite (1) as

$$\begin{cases} \dot{x}_i = f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) + x_{i+1} + \Delta f_i, \quad i = 1, 2, \ldots, n-1, \\ \dot{x}_n = f_n(\underline{\hat{x}}_n, u_f) + u + \Delta f_n, \\ y = x_1, \end{cases} \tag{6}$$

where $\Delta f_i = f_i(\underline{x}_i, x_{i+1}) - f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f})$, $i = 1, 2, \ldots, n-1$; $\Delta f_n = f_n(\underline{x}_n, u) - f_n(\underline{\hat{x}}_n, u_f)$; $\underline{\hat{x}}_i$ is the estimate of the state vectors $\underline{x}_i$, which can be obtained by the state observer designed later. $\hat{x}_{i,f}$ and u_f are the filtered signals defined by [21,23]

$$\hat{x}_{i,f} = H_L(s)\hat{x}_i, \quad u_f = H_L(s)u, \tag{7}$$

where $H_L(s)$ is a Butterworth low-pass filter (LPF), the corresponding filter parameters of Butterworth filters with the cutoff frequency $\omega_C = 1$ rad/s for different values of can be obtained in [21].

Remark 2. As stated by [21,23], the filtered signals used in (7) are to circumvent the algebraic loop problem, and the replacements $\hat{x}_i \approx \hat{x}_{i,f}$ and $u \approx u_f$ are reasonable because most actuators have low-pass properties.

Denote $\hat{x}_{n+1,f} = u_f$, and rewrite (6) in the state space form

$$\begin{aligned} \underline{\dot{x}}_n &= A\underline{x}_n + Ky + \sum_{i=1}^{n-1} B_i[f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) + \Delta f_i] + B_n[f_n(\underline{\hat{x}}_n, u_f) + u + \Delta f_n] \\ &= A\underline{x}_n + Ky + \sum_{i=1}^{n} B_i[f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) + \Delta f_i] + B_n u, \end{aligned} \tag{8}$$

where

$$A = \begin{bmatrix} -k_1 & & \\ \vdots & & I \\ -k_n & 0 \ldots 0 & \end{bmatrix}, \; K = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}, \; B_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \; B_i = \begin{bmatrix} 0 \cdots 1 \cdots 0 \end{bmatrix}^{\mathrm{T}}.$$

.

Choose the vector K to make matrix A a strict Hurwitz matrix. Thus, given a matrix $Q = Q^{\mathrm{T}} > 0$, there exists a matrix $P = P^{\mathrm{T}} > 0$ satisfying

$$A^{\mathrm{T}}P + PA = -2Q. \tag{9}$$

By Lemma 1, the fuzzy logic system is a universal approximator, i.e., it can approximate any smooth function on a compact space; thus it can be assumed that the nonlinear functions $f_i(\cdot)$ in (8) can be approximated by the following fuzzy logic systems: $\hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i) = \theta_i^{\mathrm{T}}\varphi_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f})$, $1 \leqslant i \leqslant n$. The optimal parameter vector θ_i^* is defined as $\theta_i^* = \arg\min_{\theta_i\in\Omega_i}[\sup_{(\underline{\hat{x}}_i, \hat{x}_{i+1,f})\in U_i}|\hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i) - f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f})|]$,

$1 \leqslant i \leqslant n$ where Ω_i and U_i are bounded compact sets for θ_i and $(\underline{\hat{x}}_i, \hat{x}_{i+1,f})$, respectively. Also the fuzzy minimum approximation error ε_i and fuzzy approximation error δ_i are defined by

$$\varepsilon_i = f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i^*),\ \delta_i = f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i). \tag{10}$$

Assumption 2. There exist unknown constants ε_i^*, δ_i^* and $\tau_{i,0}$ $(\tau_{1,0} = 0)$ such that $|\varepsilon_i| \leqslant \varepsilon_i^*$, $|\delta_i| \leqslant \delta_i^*$, and $|\hat{x}_{i+1} - \hat{x}_{i+1,f}| \leqslant \tau_{i,0}$, $i = 1, 2, \ldots, n$. Denote $w_i = \varepsilon_i - \delta_i$, $i = 2, \ldots, n$, by Assumption 2, there is an unknown constant $w_i^* > 0$ such that $|w_i| \leqslant w_i^* = \varepsilon_i^* + \delta_i^*$.

Remark 3. By Lemma 1, a fuzzy logic system has the approximation capability for any continuous smooth function. Thus, it is generally assumed that the fuzzy minimum approximation errors ε_i and approximation errors δ_i are bounded by their known supper bounds ε_i^* and δ_i^*, for example, [1–7,12,13] and the references therein. However, in practice, it is difficult to determine the upper bounds ε_i^* and δ_i^*. In this paper, an approach to estimating them online via adaptation laws is proposed. Also, according to [21,23], the filtered signals $\hat{x}_{i,f}$ satisfy $\hat{x}_{i,f} = H_L(s)\hat{x}_i \approx \hat{x}_i$, $i = 1, 2, \ldots, n+1$; therefore, it is reasonable to assume that $|x_{i+1} - x_{i+1,f}| \leqslant \tau_{i,0}$, with $\tau_{i,0}$ being a known constant.

Design a fuzzy state observer as

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + k_i(y - \hat{x}_1) + \hat{f}_i(\hat{x}_i, \hat{x}_{i+1,f} \,|\theta_i), \quad i = 1, 2, \ldots, n-1, \\ \dot{\hat{x}}_n = k_n(y - \hat{x}_1) + \hat{f}_n(\underline{\hat{x}}_n, \hat{x}_{n+1,f} \,|\theta_n) + u, \\ \hat{y} = \hat{x}_1. \end{cases} \tag{11}$$

Rewrite (11) as

$$\begin{cases} \dot{\underline{\hat{x}}}_n = A\underline{\hat{x}}_n + Ky + \sum_{i=1}^n B_i\hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i) + B_n u, \\ \hat{y} = C\underline{\hat{x}}_n, \end{cases} \tag{12}$$

where $C = [\,1\ \cdots\ 0\ \cdots\ 0\,]$.

Let $e = \underline{x}_n - \underline{\hat{x}}_n$ be observer error. Then from (8) and (12), one has

$$\dot{e} = Ae + \sum_{i=1}^n B_i[(f_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \hat{f}_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f} \,|\theta_i)) + \Delta f_i] = Ae + \sum_{i=1}^n B_i[\delta_i + \Delta f_i] = Ae + \delta + \Delta F, \tag{13}$$

where $\delta = [\delta_1, \ldots, \delta_n]^{\mathrm{T}}$ and $\Delta F = [\Delta f_1, \ldots, \Delta f_n]^{\mathrm{T}}$.

Consider the Lyapunov function candidate V_0 as

$$V_0 = \frac{1}{2}e^{\mathrm{T}}Pe. \tag{14}$$

The time derivation of V_0 is

$$\dot{V}_0 = \frac{1}{2}\dot{e}^{\mathrm{T}}Pe + \frac{1}{2}e^{\mathrm{T}}P\dot{e}. \tag{15}$$

Using (9) and substituting (13) into (15) results in

$$\dot{V}_0 \leqslant -\lambda_{\min}(Q)\|e\|^2 + e^{\mathrm{T}}P(\delta + \Delta F), \tag{16}$$

where $\lambda_{\min}(Q)$ is the largest eigenvalue of the matrix Q.

By the Young's inequality $2ab \leqslant a^2 + b^2$, and by Assumptions 1 and 2, one can obtain the following inequalities:

$$e^{\mathrm{T}}P\delta \leqslant \frac{1}{2}\|e\|^2 + \frac{1}{2}\|P\|^2\|\delta\|^2 \leqslant \frac{1}{2}\|e\|^2 + \frac{1}{2}\|P\|^2\|\delta^*\|^2, \tag{17}$$

$$\begin{aligned} \left|e^{\mathrm{T}}P\Delta F\right| &\leqslant \frac{1}{2}\|e\|^2 + \frac{1}{2}\|P\|^2\|\Delta F\|^2 \leqslant \frac{1}{2}\|e\|^2 + \frac{1}{2}\|P\|^2(|\Delta f_1|^2 + \cdots + |\Delta f_n|^2) \\ &\leqslant \frac{1}{2}\|e\|^2 + \|P\|^2\sum_{i=1}^n m_i^2\|e\|^2 + \|P\|^2\sum_{i=1}^n m_i^2\tau_{i,0}^2, \end{aligned} \tag{18}$$

where $\delta^* = [\delta_1^*, \ldots, \delta_n^*]^{\mathrm{T}}$.

Substituting (17) and (18) into (16) yields

$$\dot{V}_0 \leqslant -r\|e\|^2 + M, \tag{19}$$

where $r = \lambda_{\min}(Q) - 1 - \|P\|^2 \sum_{i=1}^n m_i^2$ and $M = \frac{1}{2}\|P\|^2\|\delta^*\|^2 + \|P\|^2 \sum_{i=1}^n m_i^2 \tau_{i,0}^2$.

Remark 4. By the designed state observer (11) can guarantee the convergences of the observer errors if we choose $r > 0$. Thus the designed state observer of this paper is reasonable.

4 Fuzzy adaptive output feedback control design and stability analysis

In this section, a fuzzy adaptive output feedback controller will be developed by using the above fuzzy state observer in the framework of the backstepping technique. The stability of the closed-loop system will be given below.

The n-steps fuzzy adaptive output feedback backstepping design is based on the change of coordinates:

$$\chi_1 = y - y_r, \quad \chi_i = \hat{x}_i - \alpha_{i-1}, \quad i = 2, \ldots, n, \tag{20}$$

where α_{i-1} is an intermediate control, and u is designated in the last step.

Step 1: Expressing x_2 in terms of its estimate as $x_2 = \hat{x}_2 + e_2$, we obtain

$$\begin{aligned}\dot{\chi}_1 &= \dot{x}_1 - \dot{y}_r = x_2 + f_1(x_1, x_2) - \dot{y}_r = \hat{x}_2 + f_1(\hat{x}_1, \hat{x}_{2,f}) - \dot{y}_r + e_2 + \Delta f_1 \\ &= \hat{x}_2 + \theta_1^{\mathrm{T}} \varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \dot{y}_r + e_2 + \tilde{\theta}_1^{\mathrm{T}} \varphi_1(\hat{x}_1, \hat{x}_{2,f}) + \varepsilon_1 + \Delta f_1,\end{aligned} \tag{21}$$

where $\tilde{\theta}_1 = \theta_1^* - \theta_1$ is the parameter error vector.

Taking $\hat{x}_2$ as a virtual control, from (20) and (21), one has

$$\dot{\chi}_1 = \chi_2 + \alpha_1 + \theta_1^{\mathrm{T}} \varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \dot{y}_r + e_2 + \tilde{\theta}_1^{\mathrm{T}} \varphi_1(\hat{x}_1, \hat{x}_{2,f}) + \varepsilon_1 + \Delta f_1. \tag{22}$$

Consider the following Lyapunov function candidate:

$$V_1 = V_0 + \frac{1}{2}\chi_1^2 + \frac{1}{2\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\tilde{\theta}_1 + \frac{1}{2\bar{\gamma}_1}\tilde{\varepsilon}_1^2, \tag{23}$$

where $\gamma_1 > 0$ and $\bar{\gamma}_1 > 0$ are design constants. $\hat{\varepsilon}_1$ is the estimate of ε_1^*, and $\tilde{\varepsilon}_1 = \varepsilon_1^* - \hat{\varepsilon}_1$.

The time derivative of V_1 along with (19) and (22) is

$$\begin{aligned}\dot{V}_1 &= \dot{V}_0 + \chi_1\dot{\chi}_1 + \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\dot{\tilde{\theta}}_1 + \frac{1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\dot{\tilde{\varepsilon}}_1 \\ &= \dot{V}_0 + \chi_1[\chi_2 + \alpha_1 + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \dot{y}_r + e_2 + \tilde{\theta}_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) + \varepsilon_1 + \Delta f_1] + \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\dot{\tilde{\theta}}_1 + \frac{1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\dot{\tilde{\varepsilon}}_1 \\ &\leqslant -r\|e\|^2 + \chi_1[\chi_2 + \alpha_1 + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \dot{y}_r] + |\chi_1\Delta f_1| + |\chi_1|\,\varepsilon_1^* \\ &\quad + e_2\chi_1 + \tilde{\theta}_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 + \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\dot{\tilde{\theta}}_1 + \frac{1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\dot{\tilde{\varepsilon}}_1 + M.\end{aligned} \tag{24}$$

By Young's inequality $2ab \leqslant a^2 + b^2$ and Assumption 1, one has

$$e_2\chi_1 \leqslant \frac{1}{2}|e_2|^2 + \frac{1}{2}{\chi_1}^2 \leqslant \frac{1}{2}\|e\|^2 + \frac{1}{2}{\chi_1}^2, \tag{25}$$

$$|\chi_1\Delta f_1| \leqslant \frac{1}{2}\chi_1^2 + \frac{1}{2}|\Delta f_1|^2 \leqslant \frac{1}{2}\chi_1^2 + m_1^2\|e\|^2 + m_2^2\tau_{2,0}^2. \tag{26}$$

Substituting (25) and (26) into (24) yields

$$\begin{aligned}\dot{V}_1 &\leqslant -(r - \frac{1}{2} - m_1^2)\|e\|^2 + \chi_1[\chi_2 + \chi_1 + \alpha_1 - \dot{y}_r + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f})] \\ &\quad + |\chi_1|\,\varepsilon_1^* + \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}(\gamma_1\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 - \dot{\theta}_1) + M + m_2^2\tau_{2,0}^2.\end{aligned} \tag{27}$$

For the convenience of the subsequent derivations, we cite the following Lemma 2.

Lemma 2 ([24])**.** For any $\kappa > 0$, $|x| - x\tanh(x/_{\kappa}) \leqslant 0.2785\kappa = \kappa'$ is satisfied.

By using Lemma 2, Eq. (27) can be rewritten as

$$\begin{aligned}
\dot{V}_1 \leqslant& -(r - \frac{1}{2} - m_1^2)\|e\|^2 + \chi_1[\chi_2 + \chi_1 + \alpha_1 - \dot{y}_r + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f})] \\
&+ |\chi_1|\,\varepsilon_1^* - \chi_1\varepsilon_1^*\tanh(\chi_1/_{\kappa_1}) + \chi_1\varepsilon_1^*\tanh(\chi_1/_{\kappa_1}) \\
&+ \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}(\gamma_1\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 - \dot{\theta}_1)^2 + M + m_2^2\tau_{2,0}^2 + \frac{1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\dot{\hat{\varepsilon}}_1 \\
\leqslant& -(r - \frac{1}{2} - m_1^2)\|e\|^2 + \chi_1[\chi_2 + \chi_1 + \alpha_1 - \dot{y}_r + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) + \hat{\varepsilon}_1\tanh(\chi_1/_{\kappa_1})] + \varepsilon_1^*\kappa' \\
&+ \frac{1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}(\gamma_1\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 - \dot{\theta}_1) + \frac{1}{\bar{\gamma}_1}\tilde{\varepsilon}_1(\bar{\gamma}_1\chi_1\tanh(\chi_1/_{\kappa_1}) - \dot{\hat{\varepsilon}}_1) + M + m_2^2\tau_{2,0}^2.
\end{aligned} \tag{28}$$

Since variables χ_1, x_1 and $\hat{x}_{2,f}$ are available, the intermediate control function α_1, and the adaptation laws for θ_1 and $\hat{\varepsilon}_1$ are chosen as

$$\alpha_1 = -c_1\chi_1 - \chi_1 - \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \hat{\varepsilon}_1\tanh(\chi_1/_{\kappa_1}) + \dot{y}_r, \tag{29}$$

$$\dot{\theta}_1 = \gamma_1\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 - \sigma_1\theta_1, \tag{30}$$

$$\dot{\hat{\varepsilon}}_1 = \bar{\gamma}_1\chi_1\tanh(\chi_1/_{\kappa_1}) - \bar{\sigma}_1\hat{\varepsilon}_1, \tag{31}$$

where $c_1 > 0$, $\kappa_1 > 0$, $\sigma_1 > 0$ and $\bar{\sigma}_1 > 0$ are design parameters.

Substituting (29)–(31) into (28) and applying Lemma 2, one has

$$\dot{V}_1 \leqslant -r_1\|e\|^2 - c_1\chi_1^2 + \chi_1\chi_2 + \frac{\sigma_1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\theta_1 + \frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 + M_1, \tag{32}$$

where $r_1 = r - 1/2 - m_1^2$ and $M_1 = \varepsilon_1^*\kappa_1' + M + m_2^2\tau_{2,0}^2$.

Step 2: Differentiating (20) yields

$$\begin{aligned}
\dot{\chi}_2 =& \dot{\hat{x}}_2 - \dot{\alpha}_1 = \hat{x}_3 + k_2e_1 + \theta_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) + \tilde{\theta}_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) + w_2 \\
&- \frac{\partial\alpha_1}{\partial\hat{x}_1}\dot{\hat{x}}_1 - \frac{\partial\alpha_1}{\partial\hat{\varepsilon}_1}\dot{\hat{\varepsilon}}_1 - \frac{\partial\alpha_1}{\partial\theta_1}\dot{\theta}_1 - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\dot{y}_r}\ddot{y}_r - \frac{\partial\alpha_1}{\partial y}\dot{y} \\
=& \hat{x}_3 + k_2e_1 + \theta_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) + \tilde{\theta}_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) + w_2 \\
&- \frac{\partial\alpha_1}{\partial\hat{x}_1}[\hat{x}_2 + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) + k_1e_1] - \frac{\partial\alpha_1}{\partial\hat{\varepsilon}_1}\dot{\hat{\varepsilon}}_1 - \frac{\partial\alpha_1}{\partial\theta_1}\dot{\theta}_1 - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r \\
&- \frac{\partial\alpha_1}{\partial\dot{y}_r}\ddot{y}_r - \frac{\partial\alpha_1}{\partial y}[\hat{x}_2 + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) + e_2 + \delta_1 + \Delta f_1] \\
=& \hat{x}_3 + H_2 - \frac{\partial\alpha_1}{\partial y}e_2 - \frac{\partial\alpha_1}{\partial y}(\Delta f_1 + \delta_1) + \tilde{\theta}_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) + w_2,
\end{aligned} \tag{33}$$

where $H_2 = \theta_2^{\mathrm{T}}\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) - \frac{\partial\alpha_1}{\partial\hat{x}_1}[\hat{x}_2 + \theta_1^{\mathrm{T}}\varphi_1(x_1, \hat{x}_{2,f}) - k_1e_1] - \frac{\partial\alpha_1}{\partial\hat{\varepsilon}_1}\dot{\hat{\varepsilon}}_1 - \frac{\partial\alpha_1}{\partial\theta_1}\dot{\theta}_1 - \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial\alpha_1}{\partial\dot{y}_r}\ddot{y}_r + k_2e_1 - \frac{\partial\alpha_1}{\partial y}[\hat{x}_2 + \theta_1^{\mathrm{T}}\varphi_1(x_1, \hat{x}_{2,f})]$. Consider the following Lyapunov function candidate: $V_2 = V_1 + \frac{1}{2}\chi_2^2 + \frac{1}{2\gamma_2}\tilde{\theta}_2^{\mathrm{T}}\tilde{\theta}_2 + \frac{1}{2\bar{\gamma}_2}\tilde{w}_2^2$, where $\gamma_2 > 0$ and $\bar{\gamma}_2 > 0$ are design constants; $\tilde{\theta}_2 = \theta_2^* - \theta_2$; $\hat{w}_i$ is the estimate of w_i^*, and $\tilde{w}_i = w_i^* - \hat{w}_i$, $(i = 2, \ldots, n)$.

The time derivative of V_2 along with (33) is

$$\begin{aligned}
\dot{V}_2 =& \dot{V}_1 + \chi_2\dot{\chi}_2 + \frac{1}{\gamma_2}\tilde{\theta}_2^{\mathrm{T}}\dot{\tilde{\theta}}_2 + \frac{1}{\bar{\gamma}_2}\tilde{w}_2\dot{\tilde{w}}_2 \\
\leqslant& -r_1\|e\|^2 - c_1\chi_1^2 + \chi_1\chi_2 + \frac{\sigma_1}{\gamma_1}\tilde{\theta}_1^{\mathrm{T}}\theta_1 + \frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 + M_1 \\
&+ \chi_2\left[\hat{x}_3 + H_2 - \frac{\partial\alpha_1}{\partial y}e_2 - \frac{\partial\alpha_1}{\partial y}(\delta_1 + \Delta f_1) + \hat{w}_2\tanh(\chi_2/_{\kappa_2})\right] + |\chi_2|\,w_2^* - w_2^*\chi_2\tanh(\chi_2/_{\kappa_2}) \\
&+ \frac{1}{\gamma_2}\tilde{\theta}_2^{\mathrm{T}}(\gamma_2\chi_2\varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) - \dot{\theta}_2) + \frac{1}{\bar{\gamma}_2}\tilde{w}_2\left(\bar{\gamma}_2\chi_2\tanh(\chi_2/_{\kappa_2}) - \dot{\hat{w}}_2\right).
\end{aligned} \tag{34}$$

By using the Young's inequality, one has

$$-\chi_2\frac{\partial\alpha_1}{\partial y}e_2-\chi_2\frac{\partial\alpha_1}{\partial y}(\delta_1+\Delta f_1)\leqslant\frac{3}{2}(\partial\alpha_1/\partial y)^2\chi_2^2+\|e\|^2+\frac{1}{2}\delta_1^{*2}+\frac{1}{2}|\Delta f_1|^2$$
$$\leqslant\frac{3}{2}(\partial\alpha_1/\partial y)^2\chi_2^2+\frac{1}{2}\delta_1^{*2}+(m_1^2+1)\|e\|^2+m_2^2\tau_{2,0}^2. \quad (35)$$

By using Lemma 2 and substituting (35) into (34) yields

$$\begin{aligned}\dot V_2 &= \dot V_1+\chi_2\dot\chi_2+\frac{1}{\gamma_2}\tilde\theta_2^{\mathrm T}\dot{\hat\theta}_2\\ &\leqslant -r_2\|e\|^2-c_1\chi_1^2+\chi_1\chi_2+\frac{\sigma_1}{\gamma_1}\tilde\theta_1^{\mathrm T}\theta_1+\frac{\bar\sigma_1}{\bar\gamma_1}\tilde\varepsilon_1\hat\varepsilon_1+M_2+\chi_2[\hat x_3+H_2+\frac{3}{2}(\frac{\partial\alpha_1}{\partial y})^2\chi_2+\hat w_2\tanh(\chi_2/_{\kappa_2})]\\ &\quad+\frac{1}{\gamma_2}\tilde\theta_2^{\mathrm T}(\gamma_2\chi_2\varphi_2(\underline{\hat x}_2,\hat x_{3,f})-\dot\theta_2)+\frac{1}{\bar\gamma_2}\tilde w_2(\bar\gamma_2\chi_2\tanh(\chi_2/_{\kappa_2})-\dot{\hat w}_2),\end{aligned}\quad (36)$$

where $r_2=r_1-1-m_1^2$ and $M_2=M_1+\frac{1}{2}\delta_1^{*2}+m_2^2\tau_{2,0}^2+w_2^*\kappa_2'$.

Take $\hat x_3$ as a virtual control, and choose intermediate control function α_2, the adaptation laws for θ_2 and $\hat w_2$ as

$$\alpha_2=-\chi_1-c_2\chi_2-\frac{3}{2}(\partial\alpha_1/\partial y)^2\chi_2-H_2-\hat w_2\tanh(\chi_2/_{\kappa_2}), \quad (37)$$
$$\dot\theta_2=\gamma_2\chi_2\varphi_2(\underline{\hat x}_2,\hat x_{3,f})-\sigma_2\theta_2, \quad (38)$$
$$\dot{\hat w}_2=\bar\gamma_2\chi_2\tanh(\chi_2/_{\kappa_2})-\bar\sigma_2\hat w_2, \quad (39)$$

where $\sigma_2>0$ and $\bar\sigma_2>0$ are design parameters.

From (20), (36)–(39), (36) can be rewritten as

$$\dot V_2\leqslant -r_2\|e\|^2+M_2-\sum_{k=1}^{2}c_k\chi_k^2+\chi_2\chi_3+\sum_{k=1}^{2}\frac{\sigma_i}{\gamma_i}\tilde\theta_k^{\mathrm T}\theta_k+\frac{\bar\sigma_1}{\bar\gamma_1}\tilde\varepsilon_1\hat\varepsilon_1+\frac{\bar\sigma_2}{\bar\gamma_2}\tilde w_2\hat w_2. \quad (40)$$

Step i ($3\leqslant i\leqslant n-1$): The similar procedures to step 2 are employed recursively at step i. The time derivative of χ_i is

$$\begin{aligned}\dot\chi_i &= \dot{\hat x}_i-\dot\alpha_{i-1}\\ &= \hat x_{i+1}+k_ie_1+\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})+\tilde\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})+w_i-\frac{\partial\alpha_{i-1}}{\partial y}\dot y-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat x_k}\dot{\hat x}_k\\ &\quad-\frac{\partial\alpha_{i-1}}{\partial\hat\varepsilon_1}\dot{\hat\varepsilon}_1-\sum_{k=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat w_k}\dot{\hat w}_k-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\theta_k}\dot\theta_k-\sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\\ &= \hat x_{i+1}+k_ie_1+\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})+\tilde\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})+w_i-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat x_k}[\hat x_{k+1}+\theta_k^{\mathrm T}\varphi_i(\underline{\hat x}_k,\hat x_{k+1,f})]\\ &\quad-\frac{\partial\alpha_{i-1}}{\partial\hat\varepsilon_1}\dot{\hat\varepsilon}_1-\sum_{k=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat w_k}\dot{\hat w}_k-\sum_{j=1}^{i-1}k_j\frac{\partial\alpha_{i-1}}{\partial\hat x_j}e_1-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\theta_k}\dot\theta_k-\sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}\\ &\quad-\frac{\partial\alpha_{i-1}}{\partial y}[\hat x_2+\theta_1^{\mathrm T}\varphi_1(\hat x_1,\hat x_{2,f})+e_2+\delta_1+\Delta f_1]\\ &= \hat x_{i+1}+H_i+\tilde\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})+w_i-\frac{\partial\alpha_{i-1}}{\partial y}e_2-\frac{\partial\alpha_{i-1}}{\partial y}(\delta_1+\Delta f_1),\end{aligned}\quad (41)$$

where $H_i=k_ie_1+\theta_i^{\mathrm T}\varphi_i(\underline{\hat x}_i,\hat x_{i+1,f})-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat x_k}[\hat x_{k+1}+\theta_k^{\mathrm T}\varphi_k(\underline{\hat x}_k,\hat x_{k+1,f})]-\frac{\partial\alpha_{i-1}}{\partial\hat\varepsilon_1}\dot{\hat\varepsilon}_1-\sum_{k=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat w_k}\dot{\hat w}_k-\sum_{j=1}^{i-1}k_j\frac{\partial\alpha_{i-1}}{\partial\hat x_j}e_1-\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\theta_k}\dot\theta_k-\sum_{k=1}^{i}\frac{\partial\alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)}-\frac{\partial\alpha_{i-1}}{\partial y}[\hat x_2+\theta_1^{\mathrm T}\varphi_1(\hat x_1,\hat x_{2,f})]$. Consider the following Lyapunov function candidate:

$$V_i=V_{i-1}+\frac{1}{2}\chi_i^2+\frac{1}{2\gamma_i}\tilde\theta_i^{\mathrm T}\tilde\theta_i+\frac{1}{2\bar\gamma_i}\tilde w_i^2, \quad (42)$$

where $\gamma_i > 0$ and $\bar{\gamma}_i > 0$ are design constants and $\tilde{\theta}_i = \theta_i^* - \theta_i$.

The time derivative of V_i along with (41) is

$$\dot{V}_i \leqslant \dot{V}_{i-1} + \chi_i[\hat{x}_{i+1} + H_i + w_i - \frac{\partial \alpha_{i-1}}{\partial y} e_2 - \frac{\partial \alpha_{i-1}}{\partial y}(\delta_1 + \Delta f_1) + \hat{w}_i \tanh({}^{\chi_i}/_{\kappa_i})] + |\chi_i| w_i^* \\ - w_i^* \chi_i \tanh({}^{\chi_i}/_{\kappa_i}) + \frac{1}{\gamma_i}\tilde{\theta}_i^{\mathrm{T}}(\gamma_i \chi_i \varphi_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \dot{\theta}_i) + \frac{1}{\bar{\gamma}_i}\tilde{w}_i(\bar{\gamma}_i \chi_i \tanh({}^{\chi_i}/_{\kappa_i}) - \dot{\hat{w}}_i). \tag{43}$$

Again by using Young's inequality, we have

$$-\chi_i \frac{\partial \alpha_{i-1}}{\partial y} e_2 - \chi_i \frac{\partial \alpha_{i-1}}{\partial y}(\delta_1 + \Delta f_1) \leqslant \frac{3}{2}(\frac{\partial \alpha_{i-1}}{\partial y})^2 \chi_i^2 + \frac{1}{2}\delta_1^{*2} + (m_1^2 + 1)\|e\|^2 + m_2^2 \tau_{2,0}^2. \tag{44}$$

By using Lemma 2 and substituting (44) into (43), (43) becomes

$$\dot{V}_i \leqslant -r_i\|e\|^2 - \sum_{k=1}^{i-1} c_k \chi_k^2 + \chi_{i-1}\chi_i + \sum_{k=1}^{i-1} \frac{\sigma_k}{\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\theta_k + \frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 + \sum_{k=2}^{i-1}\frac{\bar{\sigma}_k}{\bar{\gamma}_k}\tilde{w}_k\hat{w}_k + M_i \\ + \chi_i\left[\hat{x}_{i+1} + H_i + \frac{3}{2}\left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \chi_i + \hat{w}_i \tanh({}^{\chi_i}/_{\kappa})\right] + \frac{1}{\gamma_i}\tilde{\theta}_i^{\mathrm{T}}(\gamma_i\chi_i\varphi_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \dot{\theta}_i) \\ + \frac{1}{\bar{\gamma}_i}\tilde{w}_i(\bar{\gamma}_i\chi_i \tanh({}^{\chi_i}/_{\kappa}) - \dot{\hat{w}}_i), \tag{45}$$

where $r_i = r_1 - (i-1)(1+m_1^2)$ and $M_i = M_1 + (i-1)(\frac{1}{2}\delta_1^{*2} + m_2^2\tau_{2,0}^2) + \sum_{j=2}^{i} w_j^*\kappa_j'$.

Choose intermediate control function α_i, adaptation laws for θ_i and $\hat{w}_i$ as

$$\alpha_i = -\chi_{i-1} - c_i\chi_i - \frac{3}{2}(\frac{\partial \alpha_{i-1}}{\partial y})^2\chi_i - H_i - \hat{w}_i \tanh({}^{\chi_i}/_{\kappa_i}), \tag{46}$$

$$\dot{\theta}_i = \gamma_i\chi_i\varphi_i(\underline{\hat{x}}_i, \hat{x}_{i+1,f}) - \sigma_i\theta_i, \tag{47}$$

$$\dot{\hat{w}}_i = \bar{\gamma}_i\chi_i \tanh({}^{\chi_i}/_{\kappa_i}) - \bar{\sigma}_i\hat{w}_i, \tag{48}$$

where $\sigma_i > 0$ and $\bar{\sigma}_i > 0$ are design parameters.

Substituting (46)–(48) into (45), we have

$$\dot{V}_i \leqslant -r_i\|e\|^2 - \sum_{k=1}^{i} c_k\chi_k^2 + \chi_i\chi_{i+1} + \sum_{k=1}^{i}\frac{\sigma_k}{\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\theta_k + \frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 + \sum_{k=2}^{i}\frac{\bar{\sigma}_k}{\bar{\gamma}_k}\tilde{w}_k\hat{w}_k + M_i. \tag{49}$$

Step n: In the final design step, the actual control input u appears.

The time derivative of χ_n is

$$\dot{\chi}_n = \dot{\hat{x}}_n - \dot{\alpha}_{n-1} = u + H_n + \tilde{\theta}_n^{\mathrm{T}}\varphi_n(\underline{\hat{x}}_n, u_f) + w_n - \frac{\partial \alpha_{n-1}}{\partial y}e_2 - \frac{\partial \alpha_{n-1}}{\partial y}(\delta_1 + \Delta f_1), \tag{50}$$

where $H_n = k_n e_1 + \theta_n^{\mathrm{T}}\varphi_n(\underline{\hat{x}}_n, u_f) - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_k}[\hat{x}_{k+1} + \theta_k^{\mathrm{T}}\varphi_k(\underline{\hat{x}}_k, \hat{x}_{k+1,f})] - \frac{\partial \alpha_{n-1}}{\partial \hat{\varepsilon}_1}\dot{\hat{\varepsilon}}_1 - \sum_{k=2}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{w}_k}\dot{\hat{w}}_k - \sum_{j=1}^{n-1}k_j\frac{\partial \alpha_{n-1}}{\partial \hat{x}_j}e_1 - \sum_{k=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \theta_k}\dot{\theta}_k - \sum_{k=1}^{n}\frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}}y_r^{(k)} - \frac{\partial \alpha_{n-1}}{\partial y}[\hat{x}_2 + \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f})]$. We consider the overall Lyapunov function candidate as

$$V_n = V_{n-1} + \frac{1}{2}\chi_n^2 + \frac{1}{2\gamma_n}\tilde{\theta}_n^{\mathrm{T}}\tilde{\theta}_n + \frac{1}{2\bar{\gamma}_n}\tilde{w}_n^2, \tag{51}$$

where $\gamma_n > 0$ and $\bar{\gamma}_n > 0$ are design constants.

By setting $i = n$, the control u and adaptation laws for θ_n and $\hat{w}_n$ are described by

$$u = -\chi_{n-1} - c_n\chi_n - H_n - \frac{3}{2}(\frac{\partial \alpha_{n-1}}{\partial y})^2\chi_n - \hat{w}_n \tanh({}^{\chi_n}/_{\kappa_n}), \tag{52}$$

$$\dot{\theta}_n = \gamma_n\varphi_n(\underline{\hat{x}}_n, u_f)\chi_n - \sigma_n\theta_n, \tag{53}$$

$$\dot{\hat{w}}_n = \bar{\gamma}_n \chi_n \tanh(\chi_n/_{\kappa_n}) - \bar{\sigma}_n \hat{w}_n. \tag{54}$$

By (51)–(54), and in the same procedures done previously in step i, one can obtain

$$\dot{V}_n \leqslant -r_n\|e\|^2 - \sum_{k=1}^{n} c_k\chi_k^2 + \sum_{k=1}^{n}\frac{\sigma_k}{\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\theta_k + \frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 + \sum_{k=2}^{n}\frac{\bar{\sigma}_k}{\bar{\gamma}_k}\tilde{w}_k\hat{w}_k + M_n, \tag{55}$$

where $r_n = r_1 - (n-1)(1+m_1^2)$ and $M_n = M_1 + (n-1)(\frac{1}{2}\delta_1^{*2} + m_2^2\tau_{2,0}^2) + \sum_{j=2}^{n} w_j^*\kappa_j'$.

By using Young's inequality, one has the following inequalities:

$$\frac{\sigma_k}{\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\theta_k = \frac{\sigma_k}{\gamma_k}\tilde{\theta}_k^{\mathrm{T}}(-\tilde{\theta}_k + \theta_k^*) \leqslant -\frac{\sigma_k}{2\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\tilde{\theta}_k + \frac{\sigma_k}{2\gamma_k}|\theta_k^*|^2, \tag{56}$$

$$\frac{\bar{\sigma}_1}{\bar{\gamma}_1}\tilde{\varepsilon}_1\hat{\varepsilon}_1 \leqslant -\frac{\bar{\sigma}_1}{2\bar{\gamma}_1}\tilde{\varepsilon}_1^2 + \frac{\bar{\sigma}_1}{2\bar{\gamma}_1}\varepsilon_1^{*2}, \tag{57}$$

$$\frac{\bar{\sigma}_k}{\bar{\gamma}_k}\tilde{w}_k\hat{w}_k \leqslant -\frac{\bar{\sigma}_k}{2\bar{\gamma}_k}\tilde{w}_k^2 + \frac{\bar{\sigma}_k}{2\bar{\gamma}_k}w_k^{*2}. \tag{58}$$

Substituting (56)–(58) into (55) results in

$$\begin{aligned}\dot{V}_n \leqslant{}& -r_n\|e\|^2 - \sum_{k=1}^{n} c_k\chi_k^2 - \sum_{k=1}^{n}\frac{\sigma_k}{2\gamma_k}\tilde{\theta}_k^{\mathrm{T}}\tilde{\theta}_k - \frac{\bar{\sigma}_1}{2\bar{\gamma}_1}\tilde{\varepsilon}_1^2 - \sum_{k=2}^{n}\frac{\bar{\sigma}_k}{2\bar{\gamma}_k}\tilde{w}_k^2 + M_n \\ &+ \sum_{k=1}^{n}\frac{\sigma_k}{2\gamma_k}|\theta_k^*|^2 + \frac{\bar{\sigma}_1}{2\bar{\gamma}_1}\varepsilon_1^{*2} + \sum_{k=2}^{n}\frac{\bar{\sigma}_k}{2\bar{\gamma}_k}w_k^{*2}.\end{aligned} \tag{59}$$

Let

$$r_n > 0, \tag{60}$$

$$c = \min\{2r_n/\lambda_{\min}(P), 2c_i, \sigma_i, \bar{\sigma}_i; i = 1,\ldots,n\}, \tag{61}$$

$$\lambda = M_n + \sum_{k=1}^{n}\frac{\sigma_k}{2\gamma_k}|\theta_k^*|^2 + \frac{\bar{\sigma}_1}{2\bar{\gamma}_1}\varepsilon_1^{*2} + \sum_{k=2}^{n}\frac{\bar{\sigma}_k}{2\bar{\gamma}_k}w_k^{*2}. \tag{62}$$

Then Eq. (59) becomes

$$\dot{V} \leqslant -cV + \lambda. \tag{63}$$

Eq. (63) can be further rewritten as

$$V(t) \leqslant V(0)\mathrm{e}^{-ct} + \frac{\lambda}{c}. \tag{64}$$

From (64), and in the same proof as [1–13] it can be shown that for each $i = 1, 2, \ldots, n$, the signals $x_i(t)$, $\hat{x}_i(t)$, $e(t)$, θ_i, $\hat{\varepsilon}_1$, $\hat{w}_i$ and $u(t)$ are SGUUB, and there exists a time T such that for all $t \geqslant T$, the state observer and tracking errors satisfy $|e_i(t)| \leqslant \mu$ and $|y(t) - y_r(t)| \leqslant \mu$, where $\mu > (2\lambda/c)^{1/2}$.

The above design and analysis are summarized in the following theorem.

Theorem 1. Suppose Assumptions 1 and 2 hold. Then the fuzzy adaptive output tracking design scheme described by the state observer (12), intermediate control functions (46), control law (52) and parameter adaptive laws (47) and (48) can guarantee that all the signals of the closed-loop system are SGUUB, and the observer and tracking errors converge to a small neighborhood of the origin by appropriate choice of the design parameters.

Remark 5. According to the definition of A and (9), the parameters k_i, $i = 1, 2, \ldots, n$ are chosen to make A a stable matrix, i.e., the all real parts a_l of eigenvalues of A (denoted by $\lambda_l = -a_l + b_l i$) satisfy that $a_l > 0$, $l = 1, 2, \ldots, n$. By (13), if the parameters k_i are chosen to make a_l larger, the observer error vector can be made smaller.

According to Refs. [4–9,11–13] and from (61) and (64), increasing the values of the design parameters c_i, γ_i, $\bar{\gamma}_i$, λ, σ_i and $\bar{\sigma}_i$, $i = 1, 2, \ldots, n$ and decreasing the value of κ_i can decrease the observer errors and tracking errors. However, if c_i, γ_i, $\bar{\gamma}_i$, λ, σ_i and $\bar{\sigma}_i$ are larger and κ_i is smaller, control energy will become larger. Therefore, in practical applications, the design parameters should be chosen suitably to achieve a better transient performance and control action.

Remark 6. The proposed adaptive control approach can be used for a large class of SISO nonlinear systems with immeasurable states, including the following class of uncertain SISO strict-feedback nonlinear systems:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \\ \dot{x}_2 &= f_2(\underline{x}_2) + g_2(\underline{x}_2)x_3, \\ &\cdots, \\ \dot{x}_{n-1} &= f_{n-1}(\underline{x}_{n-1}) + g_{n-1}(\underline{x}_{n-1})x_n, \\ \dot{x}_n &= f_n(\underline{x}_n) + g_n(\underline{x}_n)u, \\ y &= x_1. \end{aligned} \tag{65}$$

Let $f_i(\underline{x}_i, x_{i+1}) = \bar{f}_i(\underline{x}_i) + \bar{g}_i(\underline{x}_i)x_{i+1},\ i = 1, 2, \ldots, n-1$; $f_n(\underline{x}_n, u) = \bar{f}_n(\underline{x}_n) + \bar{g}_n(\underline{x}_n)u$. Then Eq. (65) can be expressed in the form of (2).

Remark 7. It is worth mentioning that some observer based adaptive fuzzy backstepping control approaches have been recently developed by [10]-[13] for a special class of uncertain SISO strict-feedback nonlinear system (65). They require that $f_i(\underline{x}_i, x_{i+1}) = \bar{f}_i(\underline{x}_i) + x_{i+1}$ and $f_n(\underline{x}_n, u) = \bar{f}_n(\underline{x}_n) + u$; that is, the virtual and control gains are known constants. Thus, they cannot be applied to the SISO strict-feedback nonlinear systems (65) with virtual and control gains being unknown continuous functions.

5 Simulation example

In this section, a simulation example is presented to show effectiveness of the proposed adaptive fuzzy control approach.

Example 1 ([17,21])**.** Consider the nonlinear system

$$\dot{x}_1 = x_1 + x_2 + x_2^3/5, \quad \dot{x}_2 = x_1 x_2 + u + u^3/7, \quad y = x_1. \tag{66}$$

The famous van der Pol oscillator is taken as the reference model:

$$\dot{x}_{d1} = x_{d2}, \quad \dot{x}_{d2} = -x_{d1} + \beta(1 - x_{d1}^2)x_{d2}, \quad y_r = x_{d1}, \tag{67}$$

which yields a limit cycle trajectory when $\beta > 0$ ($\beta = 0.2$ in the simulation) for initial states starting from points other than $(0,0)$. Let $x_{d1}(0) = 0.5$ and $x_{d2}(0) = 0.8$.

Remark 8. Note that the variable is assumed to be immeasurable; thus the adaptive NN and fuzzy approaches in [15–21] cannot be applied to control the nonlinear system (66). Meanwhile, system (66) is a nonlinear system in pure-feedback form; thus the control approaches in [10–13] cannot be applied to control this system.

The control objective is to make the output of system (66) follow the reference trajectory y_r generated from the van der Pol oscillator (67).

Choose fuzzy membership functions as $\mu_{F_2^l}(\hat{x}_1, \hat{x}_{2f}) = \exp[-\frac{(\hat{x}_1-6+2l)^2}{2}] \times \exp[-\frac{(\hat{x}_{2f}-3+l)^2}{4}]$, $\mu_{F_2^l}(\hat{x}_1, \hat{x}_2, u_f) = \exp[-\frac{(\hat{x}_1-6+2l)^2}{2}] \times \exp[-\frac{(\hat{x}_2-3+l)^2}{5}] \times \exp[-\frac{(u_f-9+3l)^2}{7}]$, $l = 1, \ldots, 5$. Construct fuzzy logic systems (5) according to (3) and (4). Choose the Butterworth low-pass filter as $H_L(s) = 1/(s^2 + 1.414s + 1)$.

The controller and parameter adaptive laws are given as follows:

$$\alpha_1 = -c_1\chi_1 - \chi_1 - \theta_1^{\mathrm{T}}\varphi_1(\hat{x}_1, \hat{x}_{2,f}) - \hat{\varepsilon}_1 \tanh(\chi_1/\kappa_1) + \dot{y}_r, \tag{68}$$

$$\dot{\theta}_1 = \gamma_1\varphi_1(\hat{x}_1, \hat{x}_{2,f})\chi_1 - \sigma_1\theta_1, \tag{69}$$

$$\dot{\hat{\varepsilon}}_1 = \bar{\gamma}_1\chi_1 \tanh(\chi_1/\kappa_1) - \bar{\sigma}_1\hat{\varepsilon}_1, \tag{70}$$

$$u = -\chi_1 - c_2\chi_2 - \frac{3}{2}\left(\frac{\partial\alpha_1}{\partial y}\right)^2 \chi_2 - H_2 - \hat{w}_2 \tanh(\chi_2/\kappa_2), \tag{71}$$

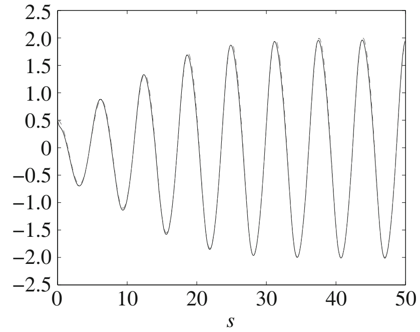

Figure 1 x_1 (solid line) and y_r (dash-dotted) of Case 1.

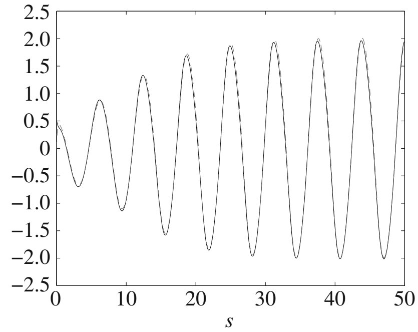

Figure 2 x_1 (solid line) and $\hat{x}_1$ (dash-dotted) of Case 1.

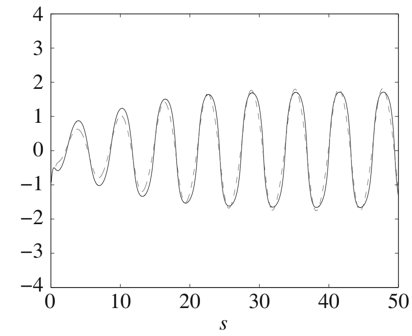

Figure 3 x_2 (solid line) and $\hat{x}_2$ (dash-dotted) of Case 1.

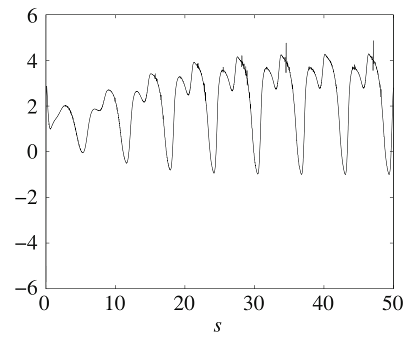

Figure 4 u of Case 1.

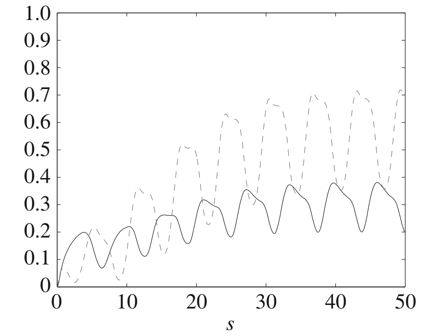

Figure 5 $\|\theta_1\|$ (solid line) and $\|\theta_2\|$ (dash-dotted) of Case 1.

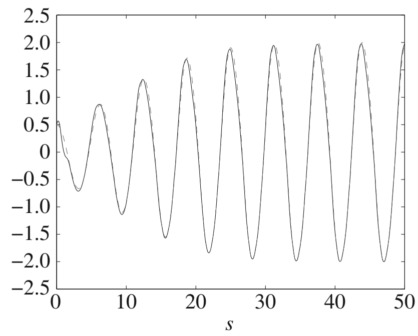

Figure 6 x_1 (solid line) and y_r (dash-dotted) of Case 2.

$$\dot{\theta}_2 = \gamma_2 \chi_2 \varphi_2(\underline{\hat{x}}_2, \hat{x}_{3,f}) - \sigma_2 \theta_2, \tag{72}$$

$$\dot{\hat{w}}_2 = \bar{\gamma}_2 \chi_2 \tanh(\chi_2/_{\kappa_2}) - \bar{\sigma}_2 \hat{w}_2. \tag{73}$$

To illustrate the effects of the main design parameters k_1, k_2, c_1, c_2, κ_1 and κ_2 on the control performances, in the simulation , two kinds of the design parameter selections are considered.

Case 1: Choose $k_1 = 10$, $k_2 = 12$, $c_1 = 7$, $c_2 = 5$, $\kappa_1 = \kappa_2 = 0.1$, $\gamma_1 = 2$, $\bar{\gamma}_1 = 4$, $\gamma_2 = 3$, $\bar{\gamma}_2 = 5$, $\sigma_1 = \sigma_2 = 0.12$, $\bar{\sigma}_1 = \bar{\sigma}_2 = 0.14$.

Case 2: Choose $k_1 = 5$, $k_2 = 6$, $c_1 = 4$, $c_2 = 3$, $\kappa_1 = \kappa_2 = 0.2$ and the other design parameters are chosen the same as case 1.

For the above two cases, the initial conditions are chosen as $x_1(0) = 0.5$, $x_2(0) = 0$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, $\hat{\varepsilon}_1(0) = 0$, $\hat{w}_2(0) = 0$, $\theta_1^{\mathrm{T}}(0) = [0, 0, 0, 0, 0]$, $\theta_2^{\mathrm{T}}(0) = [0, 0, 0, 0, 0]$. The simulation results for case 1 and case 2 are shown by Figures 1–5, and Figures 6–10, respectively, where Figures 1 and 6 show the trajectories of state y and y_r. Figures 2 and 7 show the trajectories of state x_1 and its estimate $\hat{x}_1$. Figures 3 and 8 show the trajectories of state x_2 and its estimate $\hat{x}_2$ and Figures 4 and 9 show the trajectory of input u. Figures 5 and 10 show the trajectories of both $\|\theta_1\|$ and $\|\theta_2\|$.

To further compare with the control performances between Case 1 and Case 2, define the performance

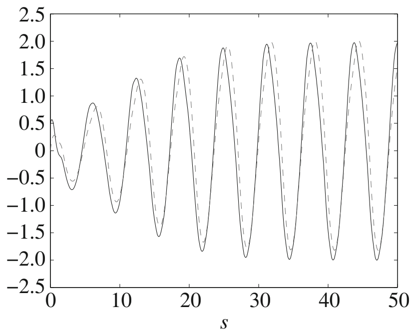

Figure 7 x_1 (solid line) and $\hat{x}_1$ (dash-dotted) of Case 2.

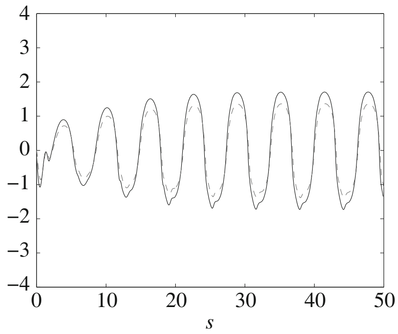

Figure 8 x_2 (solid line) and $\hat{x}_2$ (dash-dotted) of Case 2.

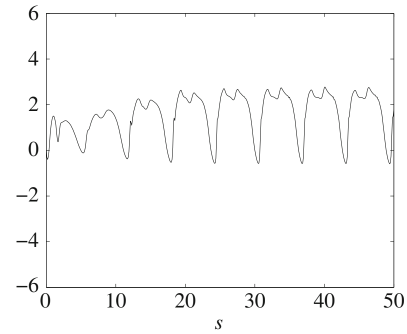

Figure 9 u of Case 2.

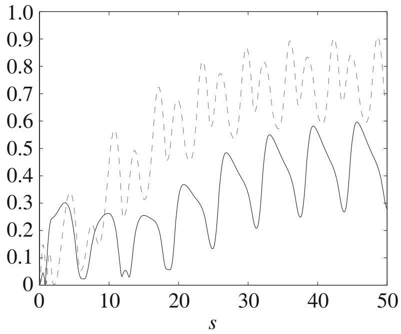

Figure 10 $\|\theta_1\|$ (solid line) and $\|\theta_2\|$ (dash-dotted) of Case 2.

Table 1 Performance comparisons between Case1 and Case 2 with the tracking error, observer errors and control indexes

Performance comparisons	Case 1	Case 2
I_1	2.141	15.632
I_2	3.375	12.589
I_3	1.811	2.070
I_4	81.342	41.132

indexes of the observer errors as $I_1 = \sum_{k=1}^{n} |x_1(k) - \hat{x}_1(k)|$ and $I_2 = \sum_{k=1}^{n} |x_2(k) - \hat{x}_2(k)|$. The tracking error and control indexes are defined as $I_3 = \sum_{k=1}^{n} |y(k) - y_r(k)|$ and $I_4 = \sum_{k=1}^{n} |u(k)|$, where n is the number of sampling data. The tracking error, observer errors and control indexes are calculated from 0 to 50s with a sampling period of 1s (note that the sampling operation is only adopted to obtain the tracking error, observer errors and control indexes).

From Figures 1–10 and Table 1, one can conclude that the larger the design parameters k_1, k_2, c_1 and c_2 are, and the smaller the design parameters κ_1 and κ_2 are, the faster the convergence rates of the tracking and the observer errors are. However, if k_1, k_2, c_1 and c_2 are chosen larger, and κ_1 and κ_2 are chosen smaller, the control energy will become larger. Therefore, in practice, to achieve satisfactory control performances, an appropriate choice of the design parameters is necessary.

6 Conclusion

In this paper, a fuzzy adaptive output feedback control approach has been developed for a class of SISO uncertain pure-feedback nonlinear systems with immeasurable states. Fuzzy logic systems are utilized to approximate the unknown nonlinear functions, and the filtered signals are introduced to circumvent algebraic loop systems encountered in the implementation of the controller. A fuzzy state adaptive observer is designed to estimate the immeasurable states. Based on the adaptive backstepping design technique, a fuzzy adaptive output feedback control is developed. It is proven that the proposed control approach can guarantee that all the signals of the resulting closed-loop system are SGUUB, and the observer and tracking errors converge to a small neighborhood of the origin by appropriate choice of the design parameters.

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