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# R-calculus without the cut rule

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**Abstract** **R**-calculus is an inference system for deducing all possible changes when a theory is refuted by the facts. In this paper, we try to eliminate the cut rule in **R**-calculus by modifying the existing rules and by introducing new rules. The result is the **R**-calculus without the cut rule, which still preserves the reachability, soundness and completeness as **R**-calculus does. **R**-calculus without the cut rule is a formal inference system of logical connective symbols and quantifier symbols solely. It can serve as the theoretical foundation of the automation of revision calculus.

**Keywords** revision, **R**-calculus, cut rule

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## 1 Introduction

The concept of revision is first formally defined by Gädenfors et al. in their AGM theory [1]. AGM theory contains a set of rational postulates which revision must satisfy. It is a framework for belief revision.

Unlike AGM theory, **R**-calculus focuses on the revision problem in scientific discovery and software testing [2, 3]. **R**-calculus is concerned with two kinds of objects in scientific discovery. One is the facts supported by experiments or observations. The other is theories that need to be revised. **R**-calculus provides a formal inference system to deduce all maximal contractions, which are the maximal subsets for a theory to be consistent with the facts [4, 5].

**R**-calculus contains a very important rule, the **R**-cut rule. The **R**-cut rule is powerful and very easy to be used manually, but it also has some disadvantages. First, it depends on first-order inference system, such as the Gentzen system [6], which compromises the independence of **R**-calculus. Second, it is not a rule about logical connective symbols or quantifier symbols and thus it is neither an inference system of logical connective symbols and quantifier symbols solely. Hence, it is difficult to implement **R**-calculus in software. One possible way to deal with these problems is eliminating the cut rule from **R**-calculus.

In this paper, we try to eliminate the cut rule in **R**-calculus, and formalize an **R**-calculus without the cut rule which still has the reachability, soundness and completeness as **R**-calculus. Since **R**-calculus without the cut rule is an inference system of logical connective symbols and quantifier symbols solely, like the Gentzen system without the cut rule, it can serve as the theoretical foundation for both automatic theorem proving and automatic revision calculus. The equivalence problem between **R**-calculus and **R**-calculus without the cut rule is also discussed. **R**-calculus without the cut rule is weaker than the original **R**-calculus, that is, all rules in **R**-calculus without the cut rule can be derived from **R**-calculus, but **R**-cut rule cannot be derived from **R**-calculus without the cut rule.

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## 2 R-calculus

**R**-calculus proposed by Li [2], the first revision calculus, turns the revision into part of the study of first-order logic. **R**-calculus is designed to deduce all possible changes, when a theory is refuted by facts. Let us first review some concepts of **R**-calculus.

**Definition 1** (**R**-refutation and **R**-contraction)**.** Let $\Gamma$ be a finite formula set and $\Delta$ be a formal theory consisting of finitely many atomic formulas or negations of atomic formulas. If $\Delta$ together with $\Gamma$ is inconsistent, then we call $\Delta$ an **R**-refutation of $\Gamma$.

We call a formal theory $\Lambda$ an **R**-contraction of $\Gamma$ with respect to an **R**-refutation $\Delta$ if $\Lambda$ is the maximal subset of $\Gamma$ that is consistent with $\Delta$.

**Definition 2** (Model of **R**-refutation)**.** Let $\Delta = \{A_1, \ldots, A_n\}$ be a formal theory consisting of finitely many atomic formulas or negations of atomic formulas such that $\Gamma \models \neg A_1 \vee \cdots \vee \neg A_n$. If there is a model $\mathbf{M}$ such that $\mathbf{M} \models \Delta$, then we call $\mathbf{M}$ the model of refutation by facts of $\Gamma$ with respect to $\Delta$, or say that $\Gamma$ is refuted by $\mathbf{M}$ with respect to $\Delta$. Let

$$\Gamma_{\mathbf{M}(\Delta)} = \{B \mid B \in \Gamma, \mathbf{M} \models B, \mathbf{M} \models \Delta\}.$$

We call $\mathbf{M}$ a model of **R**-refutation of $\Gamma$ with respect to $\Delta$ if $\Gamma_{\mathbf{M}(\Delta)}$ is maximal; that is, there does not exist another model $\mathbf{M}'$ of refutation by facts with respect to $\Delta$ such that $\Gamma_{\mathbf{M}(\Delta)} \subset \Gamma_{\mathbf{M}'(\Delta)}$.

**Definition 3** (**R**-configuration)**.** Let $\Gamma$ be a finite formula set and $\Delta$ a finite formal theory consisting of atomic formulas or negations of atomic formulas. We call $\Delta \mid \Gamma$ an **R**-configuration.

If $\Delta$ is an **R**-refutation of $\Gamma$, then we call $\Delta \mid \Gamma$ an inconsistent **R**-configuration. If $\Delta$ is consistent with $\Gamma$, then we call $\Delta \mid \Gamma$ an **R**-termination.

**Definition 4** (**R**-transition)**.**

$$\Delta \mid \Gamma \Longrightarrow \Delta' \mid \Gamma'$$

is called an **R**-transition. It transforms the **R**-configuration $\Delta \mid \Gamma$ into the **R**-configuration $\Delta' \mid \Gamma'$.

In particular, the **R**-transition

$$\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma$$

denotes the transformation of the **R**-configuration $\Delta \mid A, \Gamma$ into $\Delta \mid \Gamma$. As a result, $A$ in the formula set $A, \Gamma$ on the right-hand side of "|" is deleted.

**Definition 5** (**R**-calculus)**.** **R**-calculus is a formal inference system on **R**-configurations. It consists of the following rules: the axiom, the rules of logical connective symbols, the rules of quantifier symbols, and the cut rule.

**R**-axiom:

$$A, \Delta \mid \neg A, \Gamma \Longrightarrow A, \Delta \mid \Gamma,$$

**R**-$\wedge$ rule:

$$\frac{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma}, \qquad \frac{\Delta \mid B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R**-$\vee$ rule:

$$\frac{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma \qquad \Delta \mid B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \vee B, \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R**-$\rightarrow$ rule:

$$\frac{\Delta \mid \neg A, \Gamma \Longrightarrow \Delta \mid \Gamma \qquad \Delta \mid B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \rightarrow B, \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R**-$\forall$ rule:

$$\frac{\Delta \mid A[t/x], \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid \forall x A(x), \Gamma \Longrightarrow \Delta \mid \Gamma},$$

where $t$ is a term.

**R**-∃ rule:

$$\frac{\Delta \mid A[y/x], \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid \exists x A(x), \Gamma \Longrightarrow \Delta \mid \Gamma},$$

where $y$ is either $x$ or an arbitrary eigen-variable; that is, the variable $y$ is different from all the variables in the denominator of the **R**-∃ rule.

**R**-cut rule:

$$\frac{\Gamma_1, A, \Gamma_2 \vdash C \qquad A \mapsto_{\mathcal{T}} C \qquad \Delta \mid C, \Gamma_2 \Longrightarrow \Delta \mid \Gamma_2}{\Delta \mid \Gamma_1, A, \Gamma_2 \Longrightarrow \Delta \mid \Gamma_1, \Gamma_2}.$$

The numerator of the **R**-cut rule specifies the following conditions.

1. $\Gamma_1, A, \Gamma_2 \vdash C$ is provable. This indicates that the formula $C$ is a formal consequence of $\Gamma_1, A, \Gamma_2$.

2. The condition $A \mapsto_{\mathcal{T}} C$ holds. This indicates that $A$ is a necessary antecedent of $C$ with respect to the proof tree $\mathcal{T}$ of $\Gamma_1, A, \Gamma_2 \vdash C$; that is, $A$ is necessary in constructing proof tree $\mathcal{T}$.

3. The **R**-transition $\Delta \mid C, \Gamma_2 \Longrightarrow \Delta \mid \Gamma_2$ holds. This indicates that the formal consequence $C$ of $\Gamma_1, A, \Gamma_2$ is refuted by $\Delta$ and hence must be deleted.

The **R**-cut rule shows that, when the conditions in the numerator are satisfied, $A$ must be deleted from the right-hand side of the **R**-configuration $\Delta \mid \Gamma_1, A, \Gamma_2$ in the denominator.

With the above rules, we can derive the following rule about logical connective symbol ¬.

**Lemma 1** (**R**-¬ derived rule)**.**

$$\frac{\Delta \mid A', \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma}$$

holds, where $A$ and $A'$ are specified by the following table:

| $A$ | $\neg(B \wedge C)$ | $\neg(B \vee C)$ | $\neg\neg B$ | $\neg(B \to C)$ | $\neg\forall x B(x)$ | $\neg\exists x B(x)$ |
|---|---|---|---|---|---|---|
| $A'$ | $\neg B \vee \neg C$ | $\neg B \wedge \neg C$ | $B$ | $B \wedge \neg C$ | $\exists x \neg B(x)$ | $\forall x \neg B(x)$ |

Like the Gentzen system that has soundness and completeness, **R**-calculus has the concepts of reachability, soundness and completeness.

**Definition 6** (**R**-reachability)**.** If for any given inconsistent **R**-configuration $\Delta \mid \Gamma$ and an arbitrary **R**-contraction $\Gamma'$ of $\Gamma$ with respect to $\Delta$, there always exists an **R**-transition sequence such that

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$$

is provable, and $\Delta \mid \Gamma'$ is an **R**-termination, then we say that **R**-calculus is **R**-reachable.

**Definition 7** (**R**-soundness)**.** Let $\Delta \mid \Gamma$ be an inconsistent **R**-configuration and $\Gamma'$ be an **R**-contraction of $\Gamma$ with respect to $\Delta$. That is, there exists a provable **R**-transition sequence

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'.$$

If there exists a model $\mathbf{M}$ of **R**-refutation such that both $\mathbf{M} \models \Delta$ and $\Gamma_{\mathbf{M}(\Delta)} = \Gamma'$ hold, then we say that **R**-calculus is **R**-sound.

**Definition 8** (**R**-completeness)**.** If for an arbitrary inconsistent **R**-configuration $\Delta \mid \Gamma$ and an arbitrary model $\mathbf{M}$ of **R**-refutation of $\Gamma$ with respect to $\Delta$, there always exists a provable **R**-transition sequence

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma_{\mathbf{M}(\Delta)},$$

then we say that **R**-calculus is **R**-complete.

**R**-calculus has been proved to be **R**-reachable, **R**-sound and **R**-complete [4]. Thus, when we try to eliminate the cut rule from **R**-calculus, the newly proposed calculus should still have the reachability, soundness and completeness as **R**-calculus.

## 3 Case study of the R-cut rule

In this section, we will analyze the scenarios in which we have to use the **R**-cut rule, and try to find a way to avoid using the **R**-cut rule.

**Example 1** (**R**-$\wedge$ rule). Let $\Delta = \{\neg C\}$ and $\Gamma = \{A \wedge B, (A \wedge B) \to C\}$. It is easy to verify that $\Delta$ is an **R**-refutation of $\Gamma$. Using **G** system, we can prove that $A \wedge B, A \to (B \to C) \vdash C$, and $A \wedge B$ is a necessary antecedent of $C$. According to the **R**-axiom, $\neg C \mid C \Longrightarrow \neg C \mid \varnothing$ is provable. Thus, by applying the **R**-cut rule directly,

$$\neg C \mid A \wedge B, (A \wedge B) \to C \Longrightarrow \neg C \mid (A \wedge B) \to C$$

is provable.

But if we apply the **R**-$\wedge$ rule on $A \wedge B$ first, then we have to prove either

$$\neg C \mid A, (A \wedge B) \to C \Longrightarrow \neg C \mid (A \wedge B) \to C \text{ or } \neg C \mid B, (A \wedge B) \to C \Longrightarrow \neg C \mid (A \wedge B) \to C.$$

However, both of them are unprovable. This indicates that the **R**-$\wedge$ rule does not reflect the semantics of logical connective symbol $\wedge$ exactly. The **R**-$\wedge$ rule must be adapted to be useful in any case.

According to the semantics of $\wedge$ and by comparing with the $\wedge$-L rule in **G** system, we conjecture that the **R**-$\wedge$ rule should be of form

$$\frac{\Delta \mid A, B, \Gamma \Longrightarrow \Delta \mid B, \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma}, \qquad \frac{\Delta \mid A, B, \Gamma \Longrightarrow \Delta \mid A, \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

**Example 2** (**R**-$\forall$ rule). Consider the **R**-transition

$$\neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C.$$

According to the **G** system, $\forall x A(x), (A(a) \wedge A(b)) \to C \vdash C$ is provable, and $\forall x A(x)$ is a necessary antecedent of $C$. According to the **R**-axiom, $\neg C \mid C \Longrightarrow \neg C \mid \emptyset$ is provable. So the above **R**-transition can be obtained by applying the **R**-cut rule on $\forall x A(x)$ directly.

However, if we apply the **R**-$\forall$ rule on $\forall x A(x)$ first, then we will get

$$\neg C \mid A[t/x], (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C.$$

No matter what $t$ stands for, the above **R**-transition is unprovable. The reason is that the **R**-$\forall$ rule is only allowed to be applied on $\forall x A(x)$ once and generates a single instance. So the **R**-$\forall$ rule should also be adapted to allow multiple instantiation of quantified formulas.

According to the semantics of $\forall$ and by comparing with the $\forall$-L rule in **G** system, we conjecture that the **R**-$\forall$ rule should be of form

$$\frac{\Delta \mid A[t/x], \forall x A(x), \Gamma \Longrightarrow \Delta \mid \forall x A(x), \Gamma}{\Delta \mid \forall x A(x), \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

In **R**-calculus, besides the **R**-cut rule, the others are all rules about logical connective and quantifier symbols in the formula to be deleted. There is no corresponding rules for the logical connective and quantifier symbols in the formulas which will not be deleted, for example, the formula $A \to B$ in **R**-transition $\neg B \mid A, A \to B \Longrightarrow \neg B \mid A \to B$. Thus, when the deletion of a formula involves other formulas, the only option is to use the **R**-cut rule. In other words, if we want to eliminate the **R**-cut rule, we must guarantee that there is a rule for each logical connective or quantifier symbol, i.e., we must add new rules for logical connective and quantifier symbols in the formulas which will not be deleted. In the following, we will demonstrate which form these new rules should have.

**Example 3** (**R**$'$-axiom). Consider the **R**-transition

$$\neg B \mid A, A \to B \Longrightarrow \neg B \mid A.$$

This **R**-transition can be proved by applying the **R**-cut rule on $A \to B$.

To prove the above **R**-transition without the **R**-cut rule, the only option is to apply the **R**-$\to$ rule on $A \to B$. The proof tree is as follows:

$$\frac{\neg B \mid A, \neg A \Longrightarrow \neg B \mid A \qquad \neg B \mid A, B \Longrightarrow \neg B \mid A}{\neg B \mid A, A \to B \Longrightarrow \neg B \mid A}.$$

This means that we have to prove both $\neg B \mid A, \neg A \Longrightarrow \neg B \mid A$ and $\neg B \mid A, B \Longrightarrow \neg B \mid A$ in order to prove the above **R**-transition.

According to the **R**-axiom, $\neg B \mid A, B \Longrightarrow \neg B \mid A$ is provable. But to prove $\neg B \mid A, \neg A \Longrightarrow \neg B \mid A$, we must introduce new axioms which allow us to delete one of $A$ and $\neg A$. According to this example and the semantics of $\neg$, we proposed axioms with the following form:

$$\Delta \mid A, \neg A, \Gamma \Longrightarrow \Delta \mid A, \Gamma, \qquad \Delta \mid A, \neg A, \Gamma \Longrightarrow \Delta \mid \neg A, \Gamma,$$

where $A$ is an atomic formula, $A \notin \Delta$ and $\neg A \notin \Delta$. We call these two new axioms the $\mathbf{R}'$-axiom.

**Example 4** ($\mathbf{R}'$-$\to$ rule)**.** Consider the **R**-transition

$$\neg B \mid A, A \to B \Longrightarrow \neg B \mid A \to B.$$

This **R**-transition can be proved by applying the **R**-cut rule on $A$, i.e., $A$ must be deleted.

Without the **R**-cut rule, we cannot apply any **R**-calculus rule to $A$. This means we need not only the **R**-calculus rules for the formula to be deleted but also the **R**-calculus rules for the formulas that will not be deleted. For this specific example, we need an addition rule for the logical connective symbol $\to$ in $A \to B$. According to the semantics of $\to$, we propose the following rule:

$$\frac{\neg B \mid A, \neg A \Longrightarrow \neg B \mid \neg A \qquad \neg B \mid A, B \Longrightarrow \neg B \mid A}{\neg B \mid A, A \to B \Longrightarrow \neg B \mid A \to B}.$$

According to the **R**-axiom and the newly introduced $\mathbf{R}'$-axiom in Example 3, the two **R**-transitions in the numerator of the above rule are both provable. Thus, the **R**-transition in the denominator is also provable by the above rule as we expected.

According to this example, we can extract the following two symmetric rules for the logical connective symbol $\to$ in the formula that will not be deleted in this rule.

$$\frac{\Delta \mid C, \neg A, \Gamma \Longrightarrow \Delta \mid \neg A, \Gamma \qquad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid C, \Gamma}{\Delta \mid C, A \to B, \Gamma \Longrightarrow \Delta \mid A \to B, \Gamma},$$
$$\frac{\Delta \mid C, \neg A, \Gamma \Longrightarrow \Delta \mid C, \Gamma \qquad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid B, \Gamma}{\Delta \mid C, A \to B, \Gamma \Longrightarrow \Delta \mid A \to B, \Gamma}.$$

We call these two new rules the $\mathbf{R}'$-$\to$ rule.

Similarly, we can also find and introduce rules for logical connective and quantifier symbols $\wedge$, $\vee$, $\neg$, $\forall$ and $\exists$ in the formula that will not be deleted, i.e., the $\mathbf{R}'$-$\wedge$ rule, the $\mathbf{R}'$-$\vee$ rule, the $\mathbf{R}'$-$\neg$ rule, the $\mathbf{R}'$-$\forall$ rule and the $\mathbf{R}'$-$\exists$ rule.

## 4 R-calculus without the cut rule

Based on the discussion of the above section, we can propose **R**-calculus without the cut rule as two sets of rules. One is the axiom and rules about logical connective and quantifier symbols of **R**-calculus, where the **R**-$\wedge$ rule and **R**-$\forall$ rule have been changed according to the discussion in previous section, denoted by **R**-rules. They are rules about logical connective and quantifier symbols in formulas that are to be deleted. The other is the newly introduced axiom and rules about logical connective and quantifier symbols in formulas that will not be deleted, denoted by $\mathbf{R}'$-rules.

The axioms of **R**-calculus without the cut rule are as follows.

**Definition 9** (Axioms)**.**

**R**-axiom:

$$A, \Delta \mid \neg A, \Gamma \Longrightarrow A, \Delta \mid \Gamma.$$

**R$'$**-axiom:

$$\Delta \mid A, \neg A, \Gamma \Longrightarrow \Delta \mid A, \Gamma, \qquad \Delta \mid A, \neg A, \Gamma \Longrightarrow \Delta \mid \neg A, \Gamma,$$

where $A$ is an atomic formula, $A \notin \Delta$ and $\neg A \notin \Delta$.

The **R**-rules and **R$'$**-rules of logical connective symbols and quantifier symbols are the following.

**Definition 10** ($\wedge$ rules)**.**

**R**-$\wedge$ rule:

$$\frac{\Delta \mid A, B, \Gamma \Longrightarrow \Delta \mid B, \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma}, \qquad \frac{\Delta \mid A, B, \Gamma \Longrightarrow \Delta \mid A, \Gamma}{\Delta \mid A \wedge B, \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

**R$'$**-$\wedge$ rule:

$$\frac{\Delta \mid C, A, B, \Gamma \Longrightarrow \Delta \mid A, B, \Gamma}{\Delta \mid C, A \wedge B, \Gamma \Longrightarrow \Delta \mid A \wedge B, \Gamma}.$$

**Definition 11** ($\vee$ rules)**.**

**R**-$\vee$ rule:

$$\frac{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma \quad \Delta \mid B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \vee B, \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

**R$'$**-$\vee$ rule:

$$\frac{\Delta \mid C, A, \Gamma \Longrightarrow \Delta \mid A, \Gamma \quad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid C, \Gamma}{\Delta \mid C, A \vee B, \Gamma \Longrightarrow \Delta \mid A \vee B, \Gamma},$$
$$\frac{\Delta \mid C, A, \Gamma \Longrightarrow \Delta \mid C, \Gamma \quad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid B, \Gamma}{\Delta \mid C, A \vee B, \Gamma \Longrightarrow \Delta \mid A \vee B, \Gamma}.$$

**Definition 12** ($\rightarrow$ rules)**.**

**R**-$\rightarrow$ rule:

$$\frac{\Delta \mid \neg A, \Gamma \Longrightarrow \Delta \mid \Gamma \quad \Delta \mid B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A \rightarrow B, \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

**R$'$**-$\rightarrow$ rule:

$$\frac{\Delta \mid C, \neg A, \Gamma \Longrightarrow \Delta \mid \neg A, \Gamma \quad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid C, \Gamma}{\Delta \mid C, A \rightarrow B, \Gamma \Longrightarrow \Delta \mid A \rightarrow B, \Gamma},$$
$$\frac{\Delta \mid C, \neg A, \Gamma \Longrightarrow \Delta \mid C, \Gamma \quad \Delta \mid C, B, \Gamma \Longrightarrow \Delta \mid B, \Gamma}{\Delta \mid C, A \rightarrow B, \Gamma \Longrightarrow \Delta \mid A \rightarrow B, \Gamma}.$$

**Definition 13** ($\forall$ rules)**.**

**R**-$\forall$ rule:

$$\frac{\Delta \mid A[t/x], \forall x A(x), \Gamma \Longrightarrow \Delta \mid \forall x A(x), \Gamma}{\Delta \mid \forall x A(x), \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R$'$**-$\forall$ rule:

$$\frac{\Delta \mid C, A[t/x], \forall x A(x), \Gamma \Longrightarrow \Delta \mid A[t/x], \forall x A(x), \Gamma}{\Delta \mid C, \forall x A(x), \Gamma \Longrightarrow \Delta \mid \forall x A(x), \Gamma},$$

where $t$ is a term.

**Definition 14** ($\exists$ rules)**.**

**R**-$\exists$ rule:

$$\frac{\Delta \mid A[y/x], \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid \exists x A(x), \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R$'$**-$\exists$ rule:

$$\frac{\Delta \mid C, A[y/x], \Gamma \Longrightarrow \Delta \mid A[y/x], \Gamma}{\Delta \mid C, \exists x A(x), \Gamma \Longrightarrow \Delta \mid \exists x A(x), \Gamma},$$

where $y$ is either $x$ or an eigen-variable; that is, the variable $y$ is different from all the variables in the denominator of the rule.

**Definition 15** ($\neg$ rules)**.**

**R**-$\neg$ rule:

$$\frac{\Delta \mid A', \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma},$$

**R**$'$-$\neg$ rule:

$$\frac{\Delta \mid C, A', \Gamma \Longrightarrow \Delta \mid A', \Gamma}{\Delta \mid C, A, \Gamma \Longrightarrow \Delta \mid A, \Gamma},$$

where $A$ and $A'$ are specified by the following table:

| $A$ | $\neg(D \wedge E)$ | $\neg(D \vee E)$ | $\neg\neg D$ | $\neg(D \to E)$ | $\neg\forall x D(x)$ | $\neg\exists x D(x)$ |
|---|---|---|---|---|---|---|
| $A'$ | $\neg D \vee \neg E$ | $\neg D \wedge \neg E$ | $D$ | $D \wedge \neg E$ | $\exists x \neg D(x)$ | $\forall x \neg D(x)$ |

**Definition 16** (**R**-calculus without the cut rule)**.** **R**-calculus without the cut rule consists of the axioms, the $\wedge$ rules, the $\vee$ rules, the $\to$ rules, the $\neg$ rules, the $\forall$ rules and the $\exists$ rules.

From the definition, we know that every rule of this calculus is a formal inference rule about a logical connective symbol or a quantifier symbol. **R**-rules are inference rules about logical connective symbols and quantifier symbols of the formula to be deleted from $\Gamma$, where **R**$'$-rules are inference rules about logical connective symbols and quantifier symbols of the formulas in $\Gamma$ not to be deleted. Both sets of rules make the whole calculus system work properly.

The definition of inference tree, proof tree and provability in **R**-calculus without the cut rule are almost the same as in **R**-calculus. The only difference is that the rules used to construct inference tree and proof tree are rules of **R**-calculus without the cut rule instead.

We can prove that all rules of **R**-calculus without the cut rule can be derived from **R**-calculus.

## 5 Examples

We will now give a few examples to demonstrate the usage of **R**-calculus without the cut rule and show some interesting properties.

**Example 5.** The following is a proof tree of the **R**-transition in Example 1.

$$\dfrac{\dfrac{\neg C \mid A, B, \neg A \Longrightarrow \neg C \mid B, \neg A \qquad \dfrac{\neg C \mid A, B, \neg B \Longrightarrow \neg C \mid A, B \qquad \neg C \mid A, B, C \Longrightarrow \neg C \mid A, B}{\neg C \mid A, B, B \to C \Longrightarrow \neg C \mid A, B}}{\neg C \mid A, B, A \to (B \to C) \Longrightarrow \neg C \mid B, A \to (B \to C)}}{\neg C \mid A \wedge B, A \to (B \to C) \Longrightarrow \neg C \mid A \to (B \to C)}.$$

It can be obtained in the following steps.

According to the **R**-axiom and **R**$'$-axiom, we can prove both

$$\neg C \mid A, B, \neg B \Longrightarrow \neg C \mid A, B \text{ and } \neg C \mid A, B, C \Longrightarrow \neg C \mid A, B.$$

According to the **R**-$\to$ rule,

$$\neg C \mid A, B, B \to C \Longrightarrow \neg C \mid A, B$$

is provable. By the **R**$'$-axiom,

$$\neg C \mid A, B, \neg A \Longrightarrow \neg C \mid B, \neg A$$

is also provable. According to the **R**$'$-$\to$ rule,

$$\neg C \mid A, B, A \to (B \to C) \Longrightarrow \neg C \mid B, A \to (B \to C)$$

is provable. Finally, by applying the **R**-$\wedge$ rule,

$$\neg C \mid A \wedge B, A \to (B \to C) \Longrightarrow \neg C \mid A \to (B \to C)$$

is provable.

The **R**-$\forall$ rule together with the **R**$'$-$\forall$ rule enables any universal quantified formula to be instantiated multiple times. Let us demonstrate their usage in the following example.

**Example 6.** Let us consider the **R**-transition

$$\neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C$$

in Example 2 again. We will try to prove it using **R**-calculus without the cut rule this time. A proof tree of it is as follows:

$$\dfrac{\dfrac{\neg C \mid A(a), A(b), \forall x A(x), \neg A(a) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg A(a) \quad \neg C \mid A(a), A(b), \forall x A(x), \neg A(b) \Longrightarrow \neg C \mid A(a), A(b), \forall x A(x)}{\neg C \mid A(a), A(b), \forall x A(x), \neg A(a) \vee \neg A(b) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg A(a) \vee \neg A(b)}(5),}{\neg C \mid A(a), A(b), \forall x A(x), \neg(A(a) \wedge A(b)) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg(A(a) \wedge A(b))}(4)$$

$$\nearrow$$

$$\dfrac{\dfrac{\dfrac{\neg C \mid A(a), A(b), \forall x A(x), \neg(A(a) \wedge A(b)) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg(A(a) \wedge A(b)) \quad \neg C \mid A(a), A(b), \forall x A(x), C \Longrightarrow \neg C \mid A(a), A(b), \forall x A(x)}{\neg C \mid A(a), A(b), \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid A(b), \forall x A(x), (A(a) \wedge A(b)) \to C}(3),}{\neg C \mid A(a), \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C}(2)}{\neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C}(1)$$

where (1) is an instance of the **R**-$\forall$ rule, (2) is an instance of the **R**$'$-$\forall$ rule, (3) is an instance of the **R**$'$-$\longrightarrow$ rule, (4) is an instance of the **R**$'$-$\neg$ rule, and (5) is an instance of the **R**$'$-$\vee$ rule. The leaf node in (3) is an instance of the **R**-axiom, while the leaf nodes in (5) are instances of the **R**$'$-axiom. Let us construct this proof tree backward step by step.

To prove $\neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C$, we should first generate an instance $A(a)$ of $\forall x A(x)$. According to the **R**-$\forall$ rule, we only need to prove

$$\neg C \mid A(a), \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C.$$

Then, we could instantiate $\forall x A(x)$ again and generate another instance $A(b)$. According to the **R**$'$-$\forall$ rule, we only need to prove

$$\neg C \mid A(a), A(b), \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid A(b), \forall x A(x), (A(a) \wedge A(b)) \to C.$$

By applying the **R**$'$-$\longrightarrow$ rule on $(A(a) \wedge A(b)) \to C$, we know that means we must prove

$$\neg C \mid A(a), A(b), \forall x A(x), \neg(A(a) \wedge A(b)) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg(A(a) \wedge A(b))$$

and

$$\neg C \mid A(a), A(b), \forall x A(x), C \Longrightarrow \neg C \mid A(a), A(b), \forall x A(x).$$

Because $\neg C \mid A(a), A(b), \forall x A(x), C \Longrightarrow \neg C \mid A(a), A(b), \forall x A(x)$ is an instance of the **R**-axiom, we only need to prove

$$\neg C \mid A(a), A(b), \forall x A(x), \neg(A(a) \wedge A(b)) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg(A(a) \wedge A(b)).$$

According to the **R**$'$-$\neg$ rule, we have to prove

$$\neg C \mid A(a), A(b), \forall x A(x), \neg A(a) \vee \neg A(b) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg A(a) \vee \neg A(b).$$

Finally, by applying the **R**$'$-$\vee$ rule on $\neg A(a) \vee \neg A(b)$, this means we must prove both

$$\neg C \mid A(a), A(b), \forall x A(x), \neg A(a) \Longrightarrow \neg C \mid A(b), \forall x A(x), \neg A(a)$$

and

$$\neg C \mid A(a), A(b), \forall x A(x), \neg A(b) \Longrightarrow \neg C \mid A(a), A(b), \forall x A(x).$$

Since they are both instances of the $\mathbf{R}'$-axiom, the proof tree is fully constructed. Thus, $\neg C \mid \forall x A(x), (A(a) \wedge A(b)) \to C \Longrightarrow \neg C \mid (A(a) \wedge A(b)) \to C$ is provable.

In the following, we will consider an **R**-transition with quantifier symbols $\forall$ and $\exists$, and discuss the order of applying the $\forall$ and $\exists$ rules in **R**-calculus without the cut rule.

**Example 7.** Consider the **R**-transition

$$\neg B \mid \forall x A(x), \exists x(A(x) \to B) \Longrightarrow \neg B \mid \exists x(A(x) \to B).$$

If we first use the $\forall$ rules, then we can construct the following inference tree:

$$\cfrac{\cfrac{\cfrac{\neg B \mid A[t/x], \forall x A(x), \neg A[y/x] \Longrightarrow \neg B \mid \forall x A(x), \neg A[y/x] \quad \neg B \mid A[t/x], \forall x A(x), B \Longrightarrow \neg B \mid A[t/x], \forall x A(x)}{\neg B \mid A[t/x], \forall x A(x), A[y/x] \to B \Longrightarrow \neg B \mid \forall x A(x), A[y/x] \to B}}{\neg B \mid A[t/x], \forall x A(x), \exists x(A(x) \to B) \Longrightarrow \neg B \mid \forall x A(x), \exists x(A(x) \to B)}}{\neg B \mid \forall x A(x), \exists x(A(x) \to B) \Longrightarrow \neg B \mid \exists x(A(x) \to B)},$$

where $t$ is a term, and $y$ is an eigen-variable different from $t$. Thus,

$$\neg B \mid A[t/x], \forall x A(x), \neg A[y/x] \Longrightarrow \neg B \mid \forall x A(x), \neg A[y/x]$$

is unprovable, and the above inference tree cannot be a proof tree.

But if we use the $\exists$ rules first, then we can get the following proof tree

$$\cfrac{\cfrac{\cfrac{\neg B \mid A[y/x], \forall x A(x), \neg A[y/x] \Longrightarrow \neg B \mid \forall x A(x), \neg A[y/x]}{\neg B \mid \forall x A(x), \neg A[y/x] \Longrightarrow \neg B \mid \neg A[y/x]} \quad \neg B \mid \forall x A(x), B \Longrightarrow \neg B \mid \forall x A(x)}{\neg B \mid \forall x A(x), A[y/x] \to B \Longrightarrow \neg B \mid A[y/x] \to B}}{\neg B \mid \forall x A(x), \exists x(A(x) \to B) \Longrightarrow \neg B \mid \exists x(A(x) \to B)}.$$

This indicates that when using rules of **R**-calculus without the cut rule, $\exists$ rules should be used before $\forall$ rules. This is similar to the Gentzen system.

In the previous section, we say that the rules of **R**-calculus without the cut rule can be derived from **R**-calculus. But can we also derive all rules of **R**-calculus from **R**-calculus with the cut rule? The answer is negative. Let us see the following example.

**Example 8.** Suppose that $\Delta \mid A \vee B, \Gamma \Longrightarrow \Delta \mid \Gamma$ is provable. Since $A \vdash A \vee B$ and $A \mapsto A \vee B$, according to the **R**-cut rule, the following rule holds:

$$\frac{\Delta \mid A \vee B, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma}.$$

This rule is not a rule for logical connective symbol, nor does it hold in **R**-calculus without the cut rule. Therefore, the **R**-cut rule cannot be derived from **R**-calculus without the cut rule.

This gives us a counterexample of a derived rule of **R**-calculus that cannot be derived from **R**-calculus without the cut rule.

## 6 Reachability, soundness and completeness

As we demonstrated in the last section, **R**-calculus without the cut rule is not equivalent to **R**-calculus. We have to prove that **R**-calculus without the cut rule still has reachability, soundness and completeness to make it useful.

**Lemma 2** (Addition rule)**.** Let $\Theta$ be a set of atomic formulas and the negations of atomic formulas, and $\Theta$ is consistent with $\Delta$. Let $\Lambda$ be an arbitrary set of formulas. The following addition rule holds:

$$\frac{\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma}{\Delta, \Theta \mid A, \Gamma, \Lambda \Longrightarrow \Delta, \Theta \mid \Gamma, \Lambda}.$$

*Proof.* According to the precondition of this lemma, $\Delta, \Theta$ is a consistent set of atomic formulas and negation of atomic formulas. Since $\Delta$ in every rule of **R**-calculus without the cut rule can be replaced by any consistent set of atomic formulas and negations of atomic formulas and $\Gamma$ can be substituted by any set of formulas, the rules still hold after substituting every $\Delta$ and $\Gamma$ with $\Delta, \Theta$ and $\Gamma, \Lambda$ respectively. Because $\Delta \mid A, \Gamma \Longrightarrow \Delta \mid \Gamma$ is provable, without loss of generality, suppose that it has a proof tree $\mathcal{T}$. By adding $\Theta$ to the left of and $\Gamma$ to the right of **R**-configuration in every node of $\mathcal{T}$, we still get a proof tree, denoted by $\mathcal{T}'$. $\mathcal{T}'$ is a proof tree of $\Delta, \Theta \mid A, \Gamma, \Lambda \Longrightarrow \Delta, \Theta \mid \Gamma, \Lambda$. Thus, $\Delta, \Theta \mid A, \Gamma, \Lambda \Longrightarrow \Delta, \Theta \mid \Gamma, \Lambda$ is also provable. This lemma holds.

This derived rule indicates that by adding a set $\Theta$ which is consistent with $\Delta$ to the left-hand side of and a set $\Lambda$ to the right-hand side of the two **R**-configurations in a provable **R**-transition, the new **R**-transition is also provable.

**Lemma 3.** Let $\Gamma$ be a finite set of formulas. Suppose that $\Gamma'$ is a maximal subset of $\Gamma$ which is consistent with $\Delta$. If $B \in \Gamma - \Gamma'$, then

$$\Delta \mid B, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

is provable.

*Proof.* Let us first define an order on a pair of natural numbers $(k, l)$ as $(k, l) < (k', l')$ if $k < k'$, or $k = k'$ but $l < l'$. Based on this order, we can define the following rank on set of formulas $\Gamma = \{A_1, \dots, A_n\}$:

$$\text{rank}(\Gamma) = (k, l),$$

where $k = \max\{\text{rk}(A_1), \dots, \text{rk}(A_n)\}$, $l = |\{A_i \mid \text{rk}(A_i) = k\}|$, and rk is defined inductively as follows:

$$\begin{aligned}
\text{rk}(A) &= 1, \text{ where } A \text{ is a literal, i.e., atomic formula or negation of atomic formula;} \\
\text{rk}(A * B) &= \max\{\text{rk}(A), \text{rk}(B)\} + 1, \text{ where * is one of } \wedge, \vee, \rightarrow; \\
\text{rk}(\neg A) &= \text{rk}(A) + 1, \text{ where } A \text{ is not an atomic formula;} \\
\text{rk}(\forall x A) &= \text{rk}(A) + 1; \\
\text{rk}(\exists x A) &= \text{rk}(A) + 1.
\end{aligned}$$

We will prove this lemma by mathematical induction on the rank of $B, \Gamma'$.

1. Suppose $\text{rank}(B, \Gamma') = (1, l_0)$. Then $B, \Gamma'$ contains only literals. Let

$$\overline{B} = \begin{cases} \neg B, & \text{if } B \text{ is an atomic formula,} \\ A, & \text{if } B \text{ is } \neg A. \end{cases}$$

Since $\Delta, B, \Gamma'$ is inconsistent, $\overline{B}$ must be in $\Delta$ or $\Gamma'$. If $\overline{B} \in \Delta$, by the **R**-axiom, we have

$$\Delta \mid B, \Gamma' \Longrightarrow \Delta \mid \Gamma'.$$

If $\overline{B} \in \Gamma'$, by the **R**$'$-axiom, we have

$$\Delta \mid B, \Gamma' \Longrightarrow \Delta \mid \Gamma'.$$

2. Suppose the above lemma holds for $B, \Gamma'$ such that $\text{rank}(B, \Gamma') < (k, l)$. For those $B, \Gamma'$ has a rank $(k, l)$, we will prove the above lemma still holds for the following cases.

1) If $\text{rk}(B) = k$, it can be divided into the following cases.

a. $B$ is $C \vee D$. Since $\Delta, C \vee D, \Gamma'$ is inconsistent, both $C$ and $D$ are inconsistent with $\Delta, \Gamma'$. According to the induction hypothesis, both $\Delta \mid C, \Gamma' \Longrightarrow \Delta \mid \Gamma'$ and $\Delta \mid D, \Gamma' \Longrightarrow \Delta \mid \Gamma'$ hold. Thus,

$$\Delta \mid C \vee D, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\vee$ rule.

b. $B$ is $C \wedge D$. Since $\Delta, C \wedge D, \Gamma'$ is inconsistent, $\{C, D\}$ is inconsistent with $\Delta, \Gamma'$, i.e., either $D$ is inconsistent with $\Delta, \Gamma'$ or $D, \Delta, \Gamma'$ is consistent but $C$ is inconsistent with $D, \Delta, \Gamma'$. In the first case, $\Delta \mid C, D, \Gamma' \Longrightarrow \Delta \mid C, \Gamma'$ holds by the induction hypothesis. Thus,

$$\Delta \mid C \wedge D, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\wedge$ rule.

In the second case, $\Delta \mid C, D, \Gamma' \Longrightarrow \Delta \mid D, \Gamma'$ holds by the induction hypothesis. Thus,

$$\Delta \mid C \wedge D, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\wedge$ rule.

c. $B$ is $C \to D$. This is similar to the case where $B$ is $C \vee D$.

d. $B$ is $\forall x C(x)$. Since $\forall x C(x)$ is inconsistent with $\Delta, \Gamma'$, there exist some terms $t_1, \ldots, t_k$ such that $C[t_1/x], \ldots, C[t_k/x]$ are inconsistent with $\Delta, \Gamma'$.

If $k = 1$, then

$$\Delta \mid C[t_1/x], \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the induction hypothesis. By Lemma 2,

$$\Delta \mid C[t_1/x], \forall x C(x), \Gamma' \Longrightarrow \Delta \mid \forall x C(x), \Gamma'$$

holds. Thus,

$$\Delta \mid \forall x C(x), \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\forall$ rule.

If $k > 1$, then

$$\Delta \mid C[t_1/x], \ldots, C[t_k/x], \Gamma' \Longrightarrow \Delta \mid C[t_2/x], \ldots, C[t_k/x], \Gamma'$$

holds by the induction hypothesis. By Lemma 2,

$$\Delta \mid C[t_1/x], \ldots, C[t_k/x], \forall x C(x), \Gamma' \Longrightarrow \Delta \mid C[t_2/x], \ldots, C[t_k/x], \forall x C(x), \Gamma'$$

holds. By applying the **R**$'$-$\forall$ rule $k-1$ times on $C[t_2/x], \ldots, C[t_k/x]$, we have

$$\Delta \mid C[t_1/x], \forall x C(x), \Gamma' \Longrightarrow \Delta \mid \forall x C(x), \Gamma'.$$

Thus,

$$\Delta \mid \forall x C(x), \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\forall$ rule.

e. $B$ is $\exists x C(x)$. Since $\Delta, \exists x C(x), \Gamma'$ is inconsistent, $C[y/x]$ is inconsistent with $\Delta, \Gamma'$, where $y$ is a variable which does not occur in $B, \Delta, \Gamma$. So $\Delta \mid C[y/x], \Gamma' \Longrightarrow \Delta \mid \Gamma'$ holds by the induction hypothesis. Thus

$$\Delta \mid \exists x C(x), \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\exists$ rule.

f. $B$ is $\neg C$. According to the following table:

| $B$ | $\neg(D \wedge E)$ | $\neg(D \vee E)$ | $\neg\neg D$ | $\neg(D \to E)$ | $\neg\forall x D(x)$ | $\neg\exists x D(x)$ |
|---|---|---|---|---|---|---|
| $B'$ | $\neg D \vee \neg E$ | $\neg D \wedge \neg E$ | $D$ | $D \wedge \neg E$ | $\exists x \neg D(x)$ | $\forall x \neg D(x)$ |

this case can be reduced to case 1 or case a–f and we can prove that

$$\Delta \mid B', \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds. Thus,

$$\Delta \mid B, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds by the **R**-$\neg$ rule.

2) If $\mathrm{rk}(B) < k$, there must be a formula $F \in \Gamma$ such that $\mathrm{rk}(F) = k$. It can be divided into the following cases.

i. $F$ is $C \vee D$. Let $\Gamma'' = \Gamma' - \{C \vee D\}$. Since $\Delta, C \vee D, \Gamma''$ is consistent but $\Delta, B, C \vee D, \Gamma''$ is inconsistent, at least one of $C$ and $D$ is consistent with $\Delta, \Gamma''$ and both $\Delta, B, C, \Gamma''$ and $\Delta, B, D, \Gamma''$ are inconsistent. Without loss of generality, suppose that $C$ is consistent with $\Delta, \Gamma''$. Then,

$$\Delta \mid B, C, \Gamma'' \Longrightarrow \Delta \mid C, \Gamma''$$

holds by the induction hypothesis.

If $\Delta, B, \Gamma''$ is consistent, then

$$\Delta \mid B, D, \Gamma'' \Longrightarrow \Delta \mid B, \Gamma''$$

holds by the induction hypothesis. By the $\mathbf{R}'$-$\vee$ rule,

$$\Delta \mid B, C \vee D, \Gamma'' \Longrightarrow \Delta \mid C \vee D, \Gamma''$$

holds.

If $\Delta, B, \Gamma''$ is inconsistent, then

$$\Delta \mid B, \Gamma'' \Longrightarrow \Delta \mid \Gamma''$$

holds by the induction hypothesis. By Lemma 2,

$$\Delta \mid B, C \vee D, \Gamma'' \Longrightarrow \Delta \mid C \vee D, \Gamma''$$

holds.

ii. $F$ is $C \wedge D$. Let $\Gamma'' = \Gamma' - \{C \wedge D\}$. Since $B$ is inconsistent with $\Delta, C \wedge D, \Gamma''$ and $\Delta, C \wedge D, \Gamma''$ is consistent, $B$ is also inconsistent with $\Delta, C, D, \Gamma''$ and $\Delta, C, D, \Gamma''$ is consistent. Then,

$$\Delta \mid B, C, D, \Gamma'' \Longrightarrow \Delta \mid C, D, \Gamma''$$

holds by the induction hypothesis. Thus

$$\Delta \mid B, C \wedge D, \Gamma'' \Longrightarrow \Delta \mid C \wedge D, \Gamma''$$

holds by the $\mathbf{R}'$-$\wedge$ rules.

iii. $F$ is $C \to D$. This is similar to the case where $F$ is $C \vee D$.

iv. $F$ is $\forall x C(x)$. Let $\Gamma'' = \Gamma' - \{\forall x C(x)\}$. Since $\Delta, \forall x C(x), \Gamma''$ is consistent and $B$ is inconsistent with $\Delta, \forall x C(x), \Gamma''$, there exist some term $t_1, \ldots, t_k$ such that $\Delta, C[t_1/x], \ldots, C[t_k/x], \Gamma''$ is consistent but $B$ is inconsistent with $\Delta, C[t_1/x], \ldots, C[t_k/x], \Gamma''$. Then

$$\Delta \mid B, C[t_1/x], \ldots, C[t_k/x], \Gamma'' \Longrightarrow \Delta \mid C[t_1/x], \ldots, C[t_k/x], \Gamma''$$

holds by the induction hypothesis. By Lemma 2,

$$\Delta \mid B, C[t_1/x], \ldots, C[t_k/x], \forall x C(x), \Gamma'' \Longrightarrow \Delta \mid C[t_1/x], \ldots, C[t_k/x], \forall x C(x), \Gamma''$$

holds. Thus,

$$\Delta \mid B, \forall x C(x), \Gamma'' \Longrightarrow \Delta \mid \forall x C(x), \Gamma''$$

holds by applying the $\mathbf{R}'$-$\forall$ rule $k$ times on $C[t_1/x], \ldots, C[t_k/x]$ one by one.

v. $F$ is $\exists x C(x)$. Let $\Gamma'' = \Gamma' - \{\exists x C(x)\}$. Since $B$ is inconsistent with $\Delta, \exists x C(x), \Gamma''$ and $\Delta, \exists x C(x), \Gamma''$ is consistent, $B$ is also inconsistent with $\Delta, C[y/x], \Gamma''$ and $\Delta, C[y/x], \Gamma''$ is consistent, where $y$ is an eigen-variable. Then

$$\Delta \mid B, C[y/x], \Gamma'' \Longrightarrow \Delta \mid C[y/x], \Gamma''$$

holds by the induction hypothesis. Thus

$$\Delta \mid B, \exists x C(x), \Gamma'' \Longrightarrow \Delta \mid \exists x C(x), \Gamma''$$

holds by the $\mathbf{R}'$-$\exists$ rule.

vi. $F$ is $\neg C$. Let $\Gamma'' = \Gamma' - \{\neg C\}$. According to the following table:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\neg C$ | $\neg(D \wedge E)$ | $\neg(D \vee E)$ | $\neg\neg D$ | $\neg(D \to E)$ | $\neg\forall x D(x)$ | $\neg\exists x D(x)$ |
| $C'$ | $\neg D \vee \neg E$ | $\neg D \wedge \neg E$ | $D$ | $D \wedge \neg E$ | $\exists x \neg D(x)$ | $\forall x \neg D(x)$ |

this case can be reduced to case 1 or case i–vi and we can prove that

$$\Delta \mid B, C', \Gamma'' \Longrightarrow \Delta \mid C', \Gamma''$$

holds. Thus,

$$\Delta \mid B, \neg C, \Gamma'' \Longrightarrow \Delta \mid \neg C, \Gamma''$$

holds by the $\mathbf{R}'$-$\neg$ rule.

In summary, if $\Gamma'$ is a maximal subset of $\Gamma$ which is consistent with $\Delta$ and $B \in \Gamma - \Gamma'$, then

$$\Delta \mid B, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds.

Using Lemma 2 and Lemma 3, we can prove the following basic theorem of testing.

**Theorem 1** (Basic theorem of testing). Let $\Delta$ be a consistent set of atomic formulas and the negation of atomic formulas, and let $\Gamma$ be a finite set of formulas. Suppose that $\Gamma'$ be a maximal subset of $\Gamma$ which is consistent with $\Delta$, the **R**-transition sequence

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$$

is provable.

*Proof.* Since $\Gamma$ is a set of finitely many formulas, $\Gamma - \Gamma'$ also contains finitely many formulas. Suppose that $\Gamma - \Gamma' = \{B_1, \ldots, B_m\}$. According to Lemma 3, for each formula $B_i \in \Gamma - \Gamma'$,

$$\Delta \mid B_i, \Gamma' \Longrightarrow \Delta \mid \Gamma'$$

holds, where $1 \leqslant i \leqslant m$. According to Lemma 2,

$$\begin{gathered} \Delta \mid B_1, \ldots, B_m, \Gamma' \Longrightarrow \Delta \mid B_2, \ldots, B_m, \Gamma', \\ \Delta \mid B_2, \ldots, B_m, \Gamma' \Longrightarrow \Delta \mid B_3, \ldots, B_m, \Gamma', \\ \vdots \\ \Delta \mid B_m, \Gamma' \Longrightarrow \Delta \mid \Gamma' \end{gathered}$$

are all provable. Thus,

$$\Delta \mid \Gamma \Longrightarrow^* \Delta \mid \Gamma'$$

is provable.

We can easily deduce from this theorem that **R**-calculus without the cut rule is reachable.

**Theorem 2** (Reachability). **R**-calculus without the cut rule is **R**-reachable.

Similar to the proof of soundness and completeness of **R**-calculus, we can easily prove that **R**-calculus without the cut rule is also **R**-sound and **R**-complete [5].

**Theorem 3** (Soundness). **R**-calculus without the cut rule is **R**-sound.

**Theorem 4** (Completeness). **R**-calculus without the cut rule is **R**-complete.

## 7 Conclusions

In this paper, we eliminate the **R**-cut rule in **R**-calculus by modifying the **R**-$\wedge$ and **R**-$\forall$ rule in **R**-calculus and adding new rules of logical connective and quantifier symbols in formulas which are not to be deleted in **R**-transition. The result is a new formal inference system of logical connective symbols and quantifier symbols solely, named **R**-calculus without the cut rule. **R**-calculus without the cut rule is weaker than **R**-calculus, but we prove that **R**-calculus still preserves the reachability, soundness and completeness that **R**-calculus has. Hence, all **R**-contractions still can be derived by using **R**-calculus without the cut rule.

Since **R**-calculus without the cut rule is independent of the inference system of first-order logic and is a formal inference system of logical connective symbols and quantifier symbols solely, it would be possible to develop an interactive software tool to perform revision calculus semi-automatically like interactive theorem provers, such as Coq [7] and Isabelle [8].

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