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Universal approximation of polygonal fuzzy neural networks in sense of K-integral norms

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Abstract In this paper, we introduce polygonal fuzzy numbers to overcome the operational complexity of ordinary fuzzy numbers, and obtain two important inequalities by taking advantage of their fine properties. By presenting an actual example, we demonstrate that the approximation capability of polygonal fuzzy numbers is efficient. Furthermore, the concepts of K-quasi-additive integrals and K-integral norms are introduced. Whenever the polygonal fuzzy numbers space satisfies separability, the density problems for several functions spaces can be studied, by means of fuzzy-valued simple functions and fuzzy-valued Bernstein polynomials. We establish that the class of the integrally-bounded fuzzy-valued functions spans a complete and separable metric space in the K-integral norms. Finally, in the sense of K-integral norms, the universal approximation of four-layer regular polygonal fuzzy neural networks for fuzzy-valued simple functions is discussed. Furthermore, we show that this type of networks also possesses universal approximation for the class of integrally-bounded fuzzy-valued functions. This result indicates that the approximation capability which regular polygonal fuzzy neural networks for continuous fuzzy systems can be extended as for general integrable systems.

Keywords polygonal fuzzy numbers, K-quasi-additive integrals, K-integral norms, polygonal fuzzy neural networks, universal approximations

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1 Introduction

A fuzzy neural network is an organic combination of an artificial neural network and fuzzy techniques, that form a hybrid intelligent system with both intelligent information processing and adaptability. As a particular type of pure fuzzy systems, fuzzy neural networks can effectively handle natural language messages. In the real world, there are more data messages of digital type than language messages. Thus, we may obtain data messages with corresponding input-output relationship of a fuzzy system by measurement date and transmission. In studying the universal approximation of regular fuzzy neural networks in 1994, Buckley [1] conjectured that a regular fuzzy neural network is a universal approximator of a continuously-increasing fuzzy function class. Later, from the point of view of system approximations and learning algorithms, this class of networks was thoroughly and systematically studied by many scholars both domestically and internationally [2–5]. In China, the Professor Liu Puyin [6–10] later

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developed a great deal of useful work in regard to the above two aspects. He negated Buckley's conjecture by providing a counterexample, and introduced various concepts associated with Bernstein polynomials, closure fuzzy mappings and integral norms, and subsequently, three classes of fuzzy functions in which multiple-layer regular fuzzy neural networks constituting a universal approximator were given. In recent years, the universal approximation of four-layer regular fuzzy neural networks for a class of fuzzy-valued functions was investigated based on Lebesgue's and Sugeno's integral norm [11, 12]. In applications, all of these results have important value for fuzzy inference, fuzzy control, and image restoration techniques.

In 1987, beginning with quasi-addition and quasi-multiplication, Sugeno et al. [13] developed definitions of quasi-additive measures and integrals. In 1993, the Kt and tK integrals were defined by Jiang in [14] by taking advantage of special operators K and t. With this foundation, induced operators were introduced in 1998, and K-quasi-additive integrals were proposed in [15]. Furthermore, their convergence properties were investigated, and some useful results [15–18] were obtained. Liu [7] introduced for the first time in 2002 the two concepts, polygonal fuzzy numbers and polygonal fuzzy neural networks. In this paper, we provide a definition of the K-integral norm using K-quasi-additive integrals, and study completeness properties and separability of the spaces of $\hat{\mu}$-integrable-bounded fuzzy-valued functions in the sense of this integral norm. Later, we discuss the universal approximation of polygonal fuzzy neural networks for the class of $\hat{\mu}$-integrable-bounded fuzzy-valued functions in K-integral norm by means of the integral transformation theorem and the operation properties with respect to polygonal fuzzy numbers.

2 Fuzzy numbers

Let $\mathbb{R}^+ = [0, +\infty), \mathbb{R}^d$ a d-dimensional Euclidean space, $\| \cdot \|$ be a norm in $\mathbb{R}^d$, and $\mathbb{N}$ the set of natural numbers. For arbitrary $\forall A, B \subset \mathbb{R}^d$, define

$$d_H(A, B) = \max\{ \bigvee_{x \in A} \bigwedge_{y \in B} \| x - y \|, \bigvee_{y \in B} \bigwedge_{x \in A} \| x - y \| \}.$$

From [23], we know that $d_H(A, B)$ is an Hausdorff distance between A and B. In particular, $d_H(A, B) = | a - c | \vee | b - d |$ whenever $A = [a, b]$ and $B = [c, d] \subset \mathbb{R}$. If $[a_1, b_1] \subset [a_2, b_2] \subset [a_3, b_3] \subset \mathbb{R}$, then is the following can easily verified

$$d_H([a_1, b_1], [a_2, b_2]) \vee d_H([a_2, b_2], [a_3, b_3]) \leqslant d_H([a_1, b_1], [a_3, b_3]).$$

Definition 2.1. Let $\widetilde{A} : \mathbb{R} \longrightarrow [0, 1]$ be a mapping, then $\widetilde{A}$ is called a fuzzy number, if the following conditions (1) and (2) are satisfied: (1) $\mathrm{Ker}(\widetilde{A}) = \{x \in \mathbb{R} \mid \widetilde{A}(x) = 1\} \neq \emptyset$; (2) the cut set $\widetilde{A}_\lambda = \{x \in \mathbb{R} \mid \widetilde{A}(x) \geqslant \lambda\}$ is a bounded closed interval, for arbitrary $\lambda \in (0, 1]$.

Let $F_0(\mathbb{R})$ denote the family of all fuzzy numbers on $\mathbb{R}$. For each $a \in \mathbb{R}$, define $\widetilde{a}(a) = 1$; $\widetilde{a}(x) = 0, x \neq a$. Obviously, the real number a is a special fuzzy number, and for all $\lambda \in (0, 1]$, $\widetilde{a}_\lambda = \{a\} = [a, a]$.

In fact, for classical sets, only single point sets and bounded closed units intervals constitute fuzzy numbers. This is Because (2) in Definition 2.1 is very hard to satisfy for other types of classical sets. For example, let $A = \{1, 2, 3\}$ and $B = (1, 2]$, then for arbitrary $\lambda \in [0, 1]$, both $A_\lambda = \{1, 2, 3\}$ and $B_\lambda = (1, 2]$ do not constitute closed intervals, and thus A and B are not fuzzy numbers. For the order, operations and limits with respect to fuzzy numbers, the reader is referred to [19].

In this paper, on space $F_0(\mathbb{R})$ of fuzzy numbers, we introduce following [19] a Hausdorff metric to define $D(\widetilde{A}, \widetilde{B}) = \vee_{\lambda \in [0,1]} d_H(\widetilde{A}_\lambda, \widetilde{B}_\lambda)$, for arbitrary $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$. In light of [19], it follows that $(F_0(\mathbb{R}), D)$ constitute a complete metric space.

3 Polygonal fuzzy numbers

The application of fuzzy numbers poses significant problems for fuzzy theory. Unfortunately, fuzzy arithmetic operations are nonlinear and extremely complex, even for the simplest cases, the triangular and ladder fuzzy numbers. The reason is that the four arithmetic operations in Zadeh's extension principle do not satisfy closeness. Therefore, a significant question of interest is for general fuzzy numbers how to develop these nonlinear operations in some approximation scheme. One such approximation was proposed in [7], involves the n-symmetric polygonal fuzzy numbers (simply called polygonal fuzzy numbers) which have excellent linear properties that simplifies their operations. In this section, we summarize some of these properties to develop the important Theorem 3.2. This theorem lays the theoretical foundation for discussing a universal approximation of fuzzy neural networks.

Definition 3.1 [7]. Let $\widetilde{A} \in F_0(\mathbb{R})$, for given $n \in \mathbb{N}$, divide the closed interval [0,1] along the y-axis into n equi-sized closed intervals bounded by points $x_i = \frac{i}{n}, i = 1, 2, \ldots, n-1$. If there exists a set of ordered real numbers: $a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2 \in \mathbb{R}$ with $a_0^1 \leqslant a_1^1 \leqslant \cdots \leqslant a_n^1 \leqslant a_n^2 \leqslant \cdots \leqslant a_1^2 \leqslant a_0^2$ such that $\widetilde{A}(a_i^q) = \frac{i}{n}, q = 1, 2$ and $\widetilde{A}(x)$ is defined below, takes straight lines in $[a_{i-1}^1, a_i^1]$ and $[a_i^2, a_{i-1}^2]$, where $i = 1, 2, \ldots, n$, (see Figure 1), i.e., for any $x \in \mathbb{R}$,

$$\widetilde{A}(x) = \begin{cases} \frac{i-1}{n} + \frac{(x-a_{i-1}^1)}{n(a_i^1 - a_{i-1}^1)}, & x \in [a_{i-1}^1, a_i^1], i = 1, 2, \ldots, n, \\ 1, & x \in [a_n^1, a_n^2], \\ \frac{i-1}{n} + \frac{(a_{i-1}^2 - x)}{n(a_{i-1}^2 - a_i^2)}, & x \in [a_i^2, a_{i-1}^2], i = 1, 2, \ldots, n, \\ 0, & \text{otherwise}, \end{cases}$$

where we define $\frac{0}{0} = 0$. Then $\widetilde{A} = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$, or more simply $\widetilde{A}$, is called an n-polygonal fuzzy number.

For given $n \in \mathbb{R}$, let $Z_n(F_0(\mathbb{R}))$ denote the family of all n-polygonal fuzzy numbers on $F_0(\mathbb{R})$. Obviously, if $n = 1$, a 1-polygonal fuzzy number $\widetilde{A}$ reduces to a ladder fuzzy number or a trigonometric fuzzy number whenever. In addition, by $\widetilde{A}(a_i^q) = \frac{i}{n}$, it is clear to see that

$$\widetilde{A}(a_i^q) - \widetilde{A}(a_{i-1}^q) = \frac{1}{n}, \quad i = 1, 2, \cdots, n; \quad q = 1, 2.$$

From Definition 3.1, we easily find that the properties of polygonal fuzzy numbers are similar to those of either ladder or trigonometric fuzzy numbers. For given $n \in \mathbb{N}$, the n-polygonal fuzzy number of a $\widetilde{A}$ can be completely determined by the finite number of points $a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2$ on $\mathbb{R}$. Therefore, for each fuzzy number in $F_0(\mathbb{R})$ determines a unique n-polygonal fuzzy number. The explicit construction is as follows:

For fixed $n \in \mathbb{N}$, let $Z_n : F_0(\mathbb{R}) \to Z_n(F_0(\mathbb{R}))$ be a mapping where Z_n is said to be an n-polygonal operator. For $\widetilde{A} \in F_0(\mathbb{R})$, divide the unit closed interval $[0, 1]$ on y-axis into n equal parts hat is, insert $n-1$ partitioning points $\lambda_i = \frac{i}{n}, i = 1, 2, \ldots, n-1$. For arbitrary $\lambda \in [0, 1]$, let $\widetilde{A}(x) \geqslant \lambda_1 = \frac{1}{n}$; from Definition 2.1, we know that this inequality has a unique solution on Supp$\widetilde{A}$ and solve for x such that $a_1^1 \leqslant x \leqslant a_1^2$. Let $\widetilde{A}(x) \geqslant \lambda_i = \frac{i}{n}, i = 1, 2, \ldots, n-1$; similarly, we can solve for x that satisfies $a_i^1 \leqslant x \leqslant a_i^2$, and $[a_n^1, a_n^2] \subset [a_{n-1}^1, a_{n-1}^2] \subset \cdots \subset [a_1^1, a_1^2] \subset [a_0^1, a_0^2]$.

Thus, we obtain a set of real numbers $a_i^q, i = 0, 1, 2, \ldots, n; q = 1, 2$ with $a_0^1 \leqslant a_1^1 \leqslant \cdots \leqslant a_n^1 \leqslant a_n^2 \leqslant \cdots \leqslant a_1^2 \leqslant a_0^2$, that is to say that $\widetilde{A}$ can be changed into an n-polygonal fuzzy number, denoted as $Z_n(\widetilde{A}) = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2) \in Z_n(F_0(\mathbb{R}))$.

Alternatively, let $\widetilde{A}_{\frac{i}{n}} = [a_i^1, a_i^2]$ where $i = 0, 1, 2, \ldots, n$, connect the knot points $(a_0^1, 0), (a_1^1, \frac{1}{n}), (a_2^1, \frac{2}{n})$, $\ldots, (a_n^1, 1), (a_n^2, 1), \ldots, (a_2^2, \frac{2}{n}), (a_1^2, \frac{1}{n}), (a_0^2, 0)$ which are the points on the curve of membership function $\widetilde{A}(x)$ with straight line segments in order. Consequently, we get one ladder polygonal with continuity from the right whenever $x < a_n^1$, and continuity from the left whenever $x > a_n^2$. Obviously, it is not hard to see that

$$\mathrm{Ker}(Z_n(\widetilde{A})) = \mathrm{Ker}\widetilde{A} = [a_n^1, a_n^2], \quad \mathrm{Supp}(Z_n(\widetilde{A})) = \mathrm{Supp}\widetilde{A} = [a_0^1, a_0^2];$$

$$(Z_n(\widetilde{A}))_{\frac{i}{n}} = \widetilde{A}_{\frac{i}{n}} = [a_i^1, a_i^2], \quad i = 0, 1, 2, \ldots, n.$$

Note 1. It is clear that polygonal fuzzy numbers are a special type of fuzzy numbers, i.e., $Z_n(F_0(\mathbb{R})) \subset F_0(\mathbb{R})$. As for a given fuzzy number, its corresponding polygonal fuzzy number depends on the selection of n; the larger the value of n is, the more knots there are in the polygonal representation. Consequently,

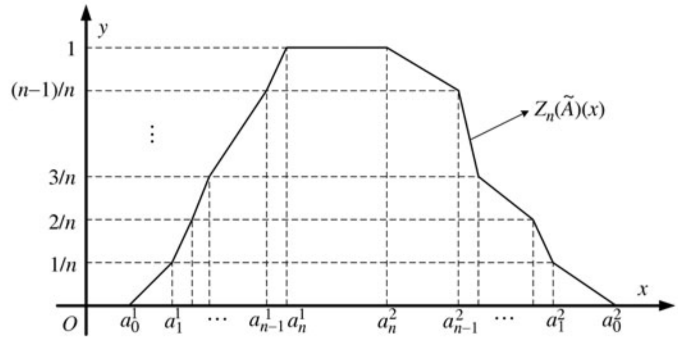


Figure 1 n-polygonal fuzzy number.

the approximation capability of large n-polygonal fuzzy numbers of a given fuzzy numbers is much stronger, at the moment, they are becoming more complex.

Definition 3.2. For given $n \in \mathbb{N}$, let $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, and $Z_n(\widetilde{A}) = (a_0^1, a_1^1, \ldots, a_n^1, a_n^2, \ldots, a_1^2, a_0^2)$, $Z_n(\widetilde{B}) = (b_0^1, b_1^1, \ldots, b_n^1, b_n^2, \ldots, b_1^2, b_0^2) \in Z_n(F_0(\mathbb{R}))$, where $a_i^q, b_i^q \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n; q = 1, 2$, we define addition, subtraction, multiplication, and scalar multiplication, as follows:

(1) $Z_n(\widetilde{A}) + Z_n(\widetilde{B}) = (a_0^1 + b_0^1, a_1^1 + b_1^1, \ldots,\ a_n^1 + b_n^1, a_n^2 + b_n^2, \ldots,\ a_1^2 + b_1^2, a_0^2 + b_0^2)$;

(2) $Z_n(\widetilde{A}) - Z_n(\widetilde{B}) = (a_0^1 - b_0^2, a_1^1 - b_1^2, \ldots,\ a_n^1 - b_n^2, a_n^2 - b_n^1, \ldots,\ a_1^2 - b_1^1, a_0^2 - b_0^1)$;

(3) $Z_n(\widetilde{A}) \cdot Z_n(\widetilde{B}) = (c_0^1, c_1^1, \ldots, c_n^1, c_n^2, \ldots, c_1^2, c_0^2)$ where $c_i^1 = a_i^1 b_i^1 \wedge a_i^1 b_i^2 \wedge a_i^2 b_i^1 \wedge a_i^2 b_i^2$ and $c_i^2 = a_i^1 b_i^1 \vee a_i^1 b_i^2 \vee a_i^2 b_i^1 \vee a_i^2 b_i^2,\ i = 0, 1, 2, \ldots, n$;

(4) $k \cdot Z_n(\widetilde{A}) = (ka_0^1, ka_1^1, \ldots, ka_n^1, ka_n^2, \ldots, ka_1^2, ka_0^2)$ where $k \geqslant 0$.

Note 2. Definition 2.1 implies that $[a, b]_\lambda = [a, b]$ for any $\lambda \in (0, 1]$ whenever $\widetilde{A}$ reduces to a closed interval $[a, b]$. In particular, $\{a\}_\lambda = \{a\} = [a, a]$. Hence, the single point set $\{a\}$ constitutes a fuzzy number defined by $Z_n(\{a\}) = (a, a, \ldots, a, a, \ldots, a, a)$ for arbitrary $a \in \mathbb{R}$ with $Z_n(a) = Z_n(\{a\})$. In general though, $Z_n(\widetilde{A})$ has no significance whenever $\widetilde{A}$ does not constitute a fuzzy number; for example $Z_n(\{1, 2, 3\})$.

Theorem 3.1 [7]**.** If $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, for given $n \in \mathbb{N}$, then the following properties (1) and (2) hold

(1) $Z_n(\widetilde{A} \pm \widetilde{B}) = Z_n(\widetilde{A}) \pm Z_n(\widetilde{B})$, $Z_n(\widetilde{A} \cdot \widetilde{B}) = Z_n(\widetilde{A}) \cdot Z_n(\widetilde{B})$;

(2) $Z_n(Z_n(\widetilde{A})) = Z_n(\widetilde{A})$; $Z_n(k \cdot \widetilde{A}) = k \cdot Z_n(\widetilde{A})$, $k \geqslant 0$ where $k \geqslant 0$ and k can be regarded as $\{k\}$.

Evidently, the space $Z_n(F_0(\mathbb{R}))$ of polygonal fuzzy numbers is closed with respect to the linear operations, its extension operations are simpler than the corresponding operations in Zadeh's extension principle, and possess excellent properties, all of which contribute to the success of polygonal fuzzy numbers.

Note 3. For given $n \in \mathbb{N}$ and for any $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, we find from [7]

$$D(Z_n(\widetilde{A}), Z_n(\widetilde{B})) = \mathop{\vee}_{i=0}^{n} d_H((Z_n(\widetilde{A}))_{\frac{i}{n}}, (Z_n(\widetilde{B}))_{\frac{i}{n}}) = \mathop{\vee}_{i=0}^{n} (\mid a_i^1 - b_i^1 \mid \vee \mid a_i^2 - b_i^2 \mid),$$

where D is a metric in $F_0(\mathbb{R})$. Specifically, whenever $\widetilde{B} = \widetilde{0}$, we define the norm of a polygonal fuzzy number $Z_n(\widetilde{A})$, i.e., $\parallel Z_n(\widetilde{A}) \parallel = D(Z_n(\widetilde{A}), Z_n(\{0\}))$. In addition, $(Z_n(\widetilde{A}))_{\frac{i}{n}} = [a_i^1, a_i^2]$ with $(Z_n(\{0\}))_{\frac{i}{n}} = [0, 0]$, and therefore, $\parallel Z_n(\widetilde{A}) \parallel = \vee_{i=0}^n (\mid a_i^1 \mid \vee \mid a_i^2 \mid)$ satisfying the inequalities

$$\mid a_i^q \mid \leqslant \parallel Z_n(\widetilde{A}) \parallel,\ \ \mid a_i^q - b_i^q \mid \leqslant D(Z_n(\widetilde{A}), Z_n(\widetilde{B})),\ q = 1, 2;\ i = 0, 1, 2, \ldots, n.$$

Lemma 1 [7]**.** For any $a_i, b_i \in \mathbb{R}$, if there exists a real number $\beta > 0$ such that $\mid a_i - b_i \mid \leqslant \beta$, where $i = 1, 2, \ldots, n$, then $\mid \wedge_{i=0}^n a_i - \wedge_{i=0}^n b_i \mid \leqslant \beta$, where $\wedge$ =inf.

Lemma 2. Let $a_{ij} > 0$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ be a set of positive real numbers, then

(1)$(\sum_{j=1}^m a_{ij}) \vee (\sum_{j=1}^m b_{ij}) \leqslant \sum_{j=1}^m (a_{ij} \vee b_{ij})$; (2) $\vee_{i=0}^n (\sum_{j=1}^m a_{ij}) \leqslant \sum_{j=1}^m (\vee_{i=0}^n a_{ij})$.

Proof. (1) For any $i \in \{1, 2, \ldots, n\}$, then $\sum_{j=1}^m a_{ij} \leqslant \sum_{j=1}^m (a_{ij} \vee b_{ij})$ and $\sum_{j=1}^m b_{ij} \leqslant \sum_{j=1}^m (a_{ij} \vee b_{ij})$ are obvious. Hence, the (1) holds.

(2) For any $j = 1, 2, \ldots, m$, we have $a_{ij} \leqslant \vee_{i=0}^{n} a_{ij} \Rightarrow \sum_{j=1}^{m} a_{ij} \leqslant \sum_{j=1}^{m}(\vee_{i=0}^{n} a_{ij})$; whereas the left hand side depends on i, the right side is independent of i and j. Taking the maximum with respect to $i \in \{1, 2, \ldots, n\}$, then we can prove that the inequalities hold. Applying Lemma 1 and Lemma 2, we next give the following important Theorem 3.2 for this paper.

Theorem 3.2. Let $\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3 \in F_0(\mathbb{R}), \widetilde{B}_k, \widetilde{C}_k \in F_0(\mathbb{R})$ for $k = 1, 2, \ldots, m$, for given $n \in \mathbb{N}$, then the following conclusions (1)–(2) hold

(1) $D\ (Z_n(\widetilde{A}_1 \cdot \widetilde{A}_2), Z_n(\widetilde{A}_1 \cdot \widetilde{A}_3)) \leqslant \| Z_n(\widetilde{A}_1) \| \cdot D\ (Z_n(\widetilde{A}_1), Z_n(\widetilde{A}_3))$;

(2) $D\ (Z_n(\sum_{k=1}^{m} \widetilde{B}_k), Z_n(\sum_{k=1}^{m} \widetilde{C}_k)) \leqslant \sum_{k=1}^{m} D\ (Z_n(\widetilde{B}_k), Z_n(\widetilde{C}_k))$.

Proof. (1) For fixed $n \in \mathbb{N}$, let $Z_n(\widetilde{A}_i) = (a_{i0}^1, a_{i1}^1, \ldots, a_{in}^1, a_{in}^2, \ldots, a_{i1}^2, a_{i0}^2) \in Z_n(F_0(\mathbb{R}))$ where $i = 0, 1, 2, \ldots, n$, in light of Theorem 3.1 and Definition 3.2, it follows that

$$Z_n(\widetilde{A}_1 \cdot \widetilde{A}_2) = Z_n(\widetilde{A}_1) \cdot Z_n(\widetilde{A}_2) = (c_0^1, c_1^1, \ldots, c_n^1, c_n^2, \ldots, c_1^2, c_0^2),$$

where $c_j^1 = a_{1j}^1 a_{2j}^1 \wedge a_{1j}^1 a_{2j}^2 \wedge a_{ij}^2 a_{2j}^1 \wedge a_{ij}^2 a_{2j}^2$ and $c_j^2 = a_{1j}^1 a_{2j}^1 \vee a_{1j}^1 a_{2j}^2 \vee a_{ij}^2 a_{2j}^1 \vee a_{ij}^2 a_{2j}^2$ with $j = 0, 1, 2, \ldots, n$.

Analogously,

$$Z_n(\widetilde{A}_1 \cdot \widetilde{A}_3) = Z_n(\widetilde{A}_1) \cdot Z_n(\widetilde{A}_3) = (d_0^1, d_1^1, \ldots, d_n^1, d_n^2, \ldots, d_1^2, d_0^2),$$

where $d_j^1 = a_{1j}^1 a_{3j}^1 \wedge a_{1j}^1 a_{3j}^2 \wedge a_{ij}^2 a_{3j}^1 \wedge a_{ij}^2 a_{3j}^2$ and $d_j^2 = a_{1j}^1 a_{3j}^1 \vee a_{1j}^1 a_{3j}^2 \vee a_{ij}^2 a_{3j}^1 \vee a_{ij}^2 a_{3j}^2$.

Substituting the corresponding terms c_j^1 and d_j^1, combined with the definition of the norm $\| Z_n(\cdot) \|$, we derive from the above expressions

$$| a_{1j}^p a_{2j}^q - a_{1j}^p a_{3j}^q | = | a_{ij}^p | \cdot | a_{2j}^q - a_{3j}^q | \leqslant \| Z_n(\widetilde{A}_1) \| \cdot D\ (Z_n(\widetilde{A}_2), Z_n(\widetilde{A}_3)),\ \ p, q = 1, 2.$$

In accordance with Lemma 1, we can obtain

$$| c_j^1 - d_j^1 | \leqslant \| Z_n(\widetilde{A}_1) \| \cdot D\ (Z_n(\widetilde{A}_2), Z_n(\widetilde{A}_3)).$$

Similarly,

$$| c_j^2 - d_j^2 | \leqslant \| Z_n(\widetilde{A}_1) \| \cdot D\ (Z_n(\widetilde{A}_2), Z_n(\widetilde{A}_3)).$$

Thus, by Note 3, we immediately have

$$D\ (Z_n(\widetilde{A}_1 \cdot \widetilde{A}_2), Z_n(\widetilde{A}_1 \cdot \widetilde{A}_3)) \leqslant \| Z_n(\widetilde{A}_1) \| \cdot D\ (Z_n(\widetilde{A}_2), Z_n(\widetilde{A}_3)).$$

(2) Let $Z_n(\widetilde{B}_k) = (b_{k0}^1, b_{k1}^1, \ldots, b_{kn}^1, a_{kn}^2, \ldots, b_{k1}^2, b_{k0}^2)$ and $Z_n(\widetilde{C}_k) = (c_{k0}^1, c_{k1}^1, \ldots, c_{kn}^1, c_{kn}^2, \ldots, c_{k1}^2, c_{k0}^2)$ for $k = 1, 2, \ldots, n$. Taking advantage of Theorem 3.1(1) and Definition 3.2(1), it follows that

$$Z_n\left(\sum_{k=1}^{m} \widetilde{B}_k\right) = \sum_{k=1}^{m} Z_n(\widetilde{B}_k) = \left(\sum_{k=1}^{m} b_{k0}^1, \sum_{k=1}^{m} b_{k1}^1, \ldots, \sum_{k=1}^{m} b_{kn}^1, \sum_{k=1}^{m} b_{kn}^2, \ldots, \sum_{k=1}^{m} b_{k1}^2, \sum_{k=1}^{m} b_{k0}^2\right);$$

$$Z_n\left(\sum_{k=1}^{m} \widetilde{C}_k\right) = \left(\sum_{k=1}^{m} c_{k0}^1, \sum_{k=1}^{m} c_{k1}^1, \ldots, \sum_{k=1}^{m} c_{kn}^1, \sum_{k=1}^{m} c_{kn}^2, \ldots, \sum_{k=1}^{m} c_{k1}^2, \sum_{k=1}^{m} c_{k0}^2\right).$$

Obviously, $| \sum_{k=1}^{m} b_{ki}^q - \sum_{k=1}^{m} c_{ki}^q | \leqslant \sum_{k=1}^{m} | b_{ki}^q - c_{ki}^q |$ for $q = 1, 2$ and $i = 0, 1, 2, \ldots, n$. By Note 3 and Lemma 1, we find

$$\begin{aligned} D\left(Z_n\left(\sum_{k=1}^{m} \widetilde{B}_k\right), Z_n\left(\sum_{k=1}^{m} \widetilde{C}_k\right)\right) &= \mathop{\vee}_{i=0}^{n} \left(\left|\sum_{k=1}^{m} b_{ki}^1 - \sum_{k=1}^{m} c_{ki}^1\right| \vee \left|\sum_{k=1}^{m} b_{ki}^2 - \sum_{k=1}^{m} c_{ki}^2\right|\right) \\ &\leqslant \mathop{\vee}_{i=0}^{n} \left(\sum_{k=1}^{m} | b_{ki}^1 - c_{ki}^1 | \vee \sum_{k=1}^{m} | b_{ki}^2 - c_{ki}^2 |\right) \\ &\leqslant \mathop{\vee}_{i=0}^{n} \sum_{k=1}^{m} (| b_{ki}^1 - c_{ki}^1 | \vee | b_{ki}^2 - c_{ki}^2 |) \\ &\leqslant \sum_{k=1}^{m} \mathop{\vee}_{i=0}^{n} (| b_{ki}^1 - c_{ki}^1 | \vee | b_{ki}^2 - c_{ki}^2 |) \end{aligned}$$

$$= \sum_{k=1}^{m} D\ (Z_n(\widetilde{B}_k), Z_n(\widetilde{C}_k)).$$

Lemma 3 [7]. The $(Z_n(F_0(\mathbb{R})), D)$ constitutes a completely separable metric space.

Lemma 4 [7]. Let $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, for arbitrary $n \in \mathbb{N}$, then $D\ (Z_n(\widetilde{A}), Z_n(\widetilde{B})) \leqslant D\ (\widetilde{A}, \widetilde{B})$ and satisfies $\lim_{n\to\infty} D\ (\widetilde{A}, Z_n(\widetilde{A})) = 0$.

Example 1. Let fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ satisfy

$$\widetilde{A}(x) = \begin{cases} \sqrt{x+1} - 1, & 0 \leqslant x < 3, \\ 1, & 3 \leqslant x \leqslant 4, \\ 3 - \sqrt{x}, & 4 < x \leqslant 9, \\ 0, & \text{otherwise}, \end{cases} \qquad \widetilde{B}(x) = \begin{cases} 2x - x^2, & 0 \leqslant x < 1, \\ 1, & 1 \leqslant x \leqslant 2, \\ \dfrac{4x - x^2}{4}, & 2 < x \leqslant 4, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, for the fuzzy number $\widetilde{A}$, Supp$\widetilde{A} = [0, 9]$, Ker$\widetilde{A} = [3, 4]$.

Putting $n = 3$, we choose divided points $\lambda_1 = \frac{1}{3},\ \lambda_2 = \frac{2}{3}$; whenever $x \in [0, 3)$, let $\widetilde{A}(x) \geqslant \frac{1}{3}, \frac{2}{3} \Rightarrow x \geqslant \frac{7}{9}, x \geqslant \frac{16}{9}$; whenever $x \in [4, 9]$, let $\widetilde{A}(x) \geqslant \frac{1}{3}, \frac{2}{3} \Rightarrow x \leqslant \frac{64}{9}, x \leqslant \frac{49}{9}$. Hence, we can obtain a 3-polygonal fuzzy number of $\widetilde{A}$, which plotted in Figure 2. It is easy to see that

$$Z_3(\widetilde{A}) = \left(0,\ \frac{7}{9},\ \frac{16}{9},\ 3,\ 4,\ \frac{49}{9},\ \frac{64}{9},\ 9\right).$$

Putting $n = 4$, we select divided points $\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{2}{4}, \lambda_3 = \frac{3}{4}$, then one obtains similarly a 4-polygonal fuzzy number for $\widetilde{A}$ in the form

$$Z_4(\widetilde{A}) = \left(0,\ \frac{9}{16},\ \frac{5}{4},\ \frac{33}{16},\ 3,\ 4, \frac{81}{16},\ \frac{25}{4},\ \frac{121}{16},\ 9\right).$$

Now, returning to $n = 3$, the coordinates of the knots of $\widetilde{A}$ are in increasing order $(0, 0), (\frac{7}{9}, \frac{1}{3}), (\frac{16}{9}, \frac{2}{3}), (3, 1), (4, 1), (\frac{49}{9}, \frac{2}{3}), (\frac{64}{9}, \frac{1}{3}), (9, 0)$. The membership function $Z_3(\widetilde{A})(x)$ of the 3-polygonal fuzzy number of $\widetilde{A}$ can be obtained and is given in Figure 2.

By utilizing the same method, the fuzzy number for $\widetilde{B}$ can be shown to be

$$Z_4(\widetilde{B}) = \left(0, 1 - \frac{\sqrt{3}}{2}, 1 - \frac{1}{\sqrt{2}}, \frac{1}{2},\ 1,\ 2,\ 3, 2 + \sqrt{2}, 2 + \sqrt{3},\ 4\right).$$

In light of Theorem 3.1(1) and Definition 3.2(1), we get

$$Z_4(\widetilde{A} + \widetilde{B}) = Z_4(\widetilde{A}) + Z_4(\widetilde{B}) = \left(0, \frac{25}{16} - \frac{\sqrt{3}}{2}, \frac{9}{4} - \frac{1}{\sqrt{2}}, \frac{41}{16},\ 4,\ 6, \frac{129}{16}, \frac{33}{4} + \sqrt{2}, \frac{153}{16} + \sqrt{3},\ 13\right).$$

Next, we use Lemma 4 to discuss how well the the n-polygonal fuzzy number $Z_n(\widetilde{A})$ approximates to $\widetilde{A}$.

In fact, for every $\lambda \in (0, 1]$, let $\widetilde{A}_\lambda = [\varphi_1(\lambda), \varphi_2(\lambda)]$, we can infer $\varphi_1(\lambda) = (1 + \lambda)^2 - 1$ and $\varphi_2(\lambda) = (3 - \lambda)^2$. The function $\varphi_1(\lambda)$ increases on [0,1], whereas $\varphi_2(\lambda)$ decreases on [0,1]. Moreover, for arbitrary $\lambda \in (0, 1]$ and $n \in \mathbb{N}$, there exists $i \in \{1, 2, \ldots, n\}$ such that $\lambda \in [\frac{i-1}{n}, \frac{i}{n}]$ with

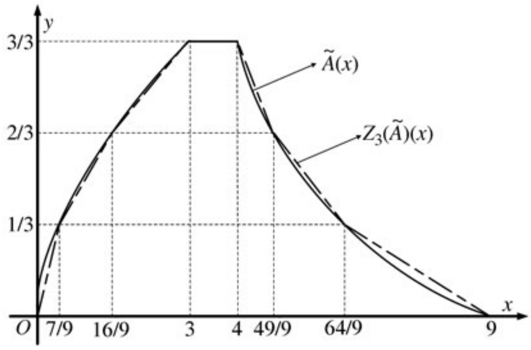
$$(Z_n(\widetilde{A}))_{\frac{i}{n}} = \widetilde{A}_{\frac{i}{n}} \subset \widetilde{A}_\lambda \subset \widetilde{A}_{\frac{i-1}{n}} = (Z_n(\widetilde{A}))_{\frac{i-1}{n}}.$$

Since d_{H} is a Hausdorff distance, for arbitrary $i \in \{1, 2, \ldots, n\}$, it is straightforward to see that

$$d_{\mathrm{H}}(\widetilde{A}_\lambda, \widetilde{A}_{\frac{i-1}{n}}) \leqslant d_{\mathrm{H}}(\widetilde{A}_{\frac{i}{n}}, \widetilde{A}_{\frac{i-1}{n}}),\quad d_{\mathrm{H}}(\widetilde{A}_{\frac{i-1}{n}}, (Z_n(\widetilde{A}))_{\frac{i-1}{n}}) = 0.$$

$$Z_3(\widetilde{A})(x) = \begin{cases} \frac{3}{7}x, & 0 \leqslant x \leqslant \frac{7}{9}, \\ \frac{1}{3}x + \frac{2}{27}, & \frac{7}{9} < x \leqslant \frac{16}{9}, \\ \frac{3}{11}x + \frac{2}{11}, & \frac{16}{9} < x \leqslant 3, \\ 1, & 3 < x \leqslant 4, \\ -\frac{3}{13}x + \frac{25}{13}, & 4 < x \leqslant \frac{49}{9}, \\ -\frac{1}{5}x + \frac{79}{45}, & \frac{49}{9} < x \leqslant \frac{64}{9}, \\ -\frac{3}{17}x + \frac{27}{17}, & \frac{64}{9} < x \leqslant 9. \end{cases}$$

(a)



(b)

Figure 2 (a) Analytic expressions of $Z_3(\tilde{A})(x)$; (b) graph of $\tilde{A}(x)$ and $Z_3(\tilde{A})(x)$.

Furthermore, we deduce that

$$\begin{aligned} d_{\mathrm{H}}(\widetilde{A}_\lambda, (Z_n(\widetilde{A}))_\lambda) &\leqslant d_{\mathrm{H}}(\widetilde{A}_\lambda, \widetilde{A}_{\frac{i-1}{n}}) + d_{\mathrm{H}}(\widetilde{A}_{\frac{i-1}{n}}, (Z_n(\widetilde{A}))_{\frac{i-1}{n}}) + d_{\mathrm{H}}((Z_n(\widetilde{A}))_{\frac{i-1}{n}}, (Z_n(\widetilde{A}))_\lambda) \\ &\leqslant 2d_{\mathrm{H}}(\widetilde{A}_{\frac{i-1}{n}}, \widetilde{A}_{\frac{i}{n}}) = 2\left[\left(\varphi_1\left(\frac{i}{n}\right) - \varphi_1\left(\frac{i-1}{n}\right)\right) \vee \left(\varphi_2\left(\frac{i-1}{n}\right) - \varphi_2\left(\frac{i}{n}\right)\right)\right] \\ &= \frac{2}{n} \cdot \left(6 - \frac{2i}{n} + \frac{1}{n}\right) \leqslant \frac{2}{n} \cdot \left(6 - \frac{1}{n}\right) < \frac{12}{n}. \end{aligned}$$

Thus, $D(\widetilde{A}, Z_n(\widetilde{A})) = \vee_{\lambda \in (0,1]} d_{\mathrm{H}}(\widetilde{A}_\lambda, (Z_n(\widetilde{A}))_\lambda) \leqslant \frac{12}{n}$. For example, given an error $\varepsilon = 0.1 > 0$, if we approximate this distance with $D(\widetilde{A}, Z_n(\widetilde{A})) \leqslant \frac{12}{n} < 0.1$, then we only need choose $n > 120$. By varying the error and making use of the $n-$polygonal fuzzy number $Z_n(\widetilde{A})$ to approximate to $\widetilde{A}$, we can obtain rough estimates of n (see Table 1).

4 *K*-quasi-additive integrals and *K*-integral norms

In 1998, the K-quasi-additive integral was suggested in [15] by introducing an induced operator, convergence and auto-continuity have been studied in [15–18]. In this section, we shall state the relevant definitions and give the concept of the K-integral norm.

Definition 4.1. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave function that is strictly monotonically increasing. If K satisfies $K(0) = 0, K(1) = 1$ and differentiable on $\mathbb{R}^+$, then K is said to be an induced operator on $\mathbb{R}^+$.

Obviously, its inverse operator K^{-1} exists and is strictly increasing. For example, for any $x \in \mathbb{R}^+$, then $K(x) = x,\ K(x) = \sqrt{x}$ and $K(x) = \log_2(x + 1)$ are clearly induced operators.

Definition 4.2 [15]**.** Let K be an induced operator, for arbitrary $a, b \in \mathbb{R}^+$, define their $K-$quasi-sum $\oplus$ and $K-$quasi-product $\otimes$ as follows $a \oplus b = K^{-1}(K(a) + K(b)); a \otimes b = K^{-1}(K(a)K(b))$.

Theorem 4.1. For any $a, b \in \mathbb{R}^+$, then the following statements hold

(1) $a + b \leqslant a \oplus b$ and $a + b \leqslant a \oplus b$ iff $K(a + b) \leqslant K(a) + K(b)$;

(2) $K(a \oplus b) = K(a) + K(b), \quad K(a \otimes b) = K(a) \cdot K(b)$;

(3) $K^{-1}(a + b) = K^{-1}(a) \oplus K^{-1}(b), \quad K^{-1}(a \cdot b) = K^{-1}(a) \otimes K^{-1}(b)$.

Proof. We only prove (1), the others can be verified directly. In fact, without loss of generality, assume $0 < a < b$, for quasi-sum $\oplus$, there certainly exists an induced operator K. Furthermore, by the Lagrange theorem of mean value, it follows that there exists $\exists\xi_1 \in (0, a)$ and $\exists\xi_2 \in (b, a + b)$ such that

$$K(a) = K(a) - K(0) = K'(\xi_1)a, \quad K(a + b) - K(b) = K'(\xi_2)a.$$

Table 1 Estimation of error

Error	ε=0.1	ε=0.07	ε=0.04	ε=0.02	ε=0.008	ε=0.002
Estimation value $n >$	120	172	300	600	1500	6000

As K is a differentiable concave function iff $K'(x)$ is a decreasing function, it follows that $\xi_1 < a < b < \xi_2 \Rightarrow K'(\xi_2) \leqslant K'(\xi_1)$. Consequently, we obtain $K(a+b) \leqslant K(a) + K(b)$ and therefore $a+b = K^{-1}(K(a+b)) \leqslant K^{-1}(K(a)+K(b)) = a \oplus b$.

Definition 4.3 [14, 15]**.** Let $(X, \Re)$ be an arbitrary measurable space, K be an induced operator, $\hat{\mu} : \Re \to [0, +\infty]$ a set function satisfying the following conditions (1)–(4).

(1) $\hat{\mu}(\emptyset) = 0$;

(2) If $A, B \in \Re$ and $A \cap B = \emptyset$, then $\hat{\mu}(A \cup B) = \hat{\mu}(A) \oplus \hat{\mu}(B)$;

(3) If $A_n \subset \Re$ and $A_n \uparrow A$, then $\hat{\mu}(A_n) \uparrow \hat{\mu}(A)$;

(4) If $A_n \subset \Re$, $A_n \downarrow A$, and there exists $n_0 \in \mathbb{N}$ such that $\hat{\mu}(A_{n_0}) < +\infty$, then $\hat{\mu}(A_n) \downarrow \hat{\mu}(A)$.

Then $\hat{\mu}$ is called a K-quasi-additive measure, and the corresponding triple $(X, \Re, \hat{\mu})$ is said to be a space of K-quasi-additive measure.

Definition 4.4 [17]**.** Let $(X, \Re, \hat{\mu})$ be a space K-quasi-additive measure, K be an induced operator, f a nonnegative measurable function, $A \in \Re$ and $T = \{A_1, A_2, A_3, \ldots, A_n\}$ an arbitrary finite measurable partition of A. Putting $\int_A^{(K)} f\mathrm{d}\hat{\mu} = \sup_T S_K(f, T, A)$ and $S_K(f, T, A) = \oplus \sum_{i=1}^{n} (\inf_{x \in A_i \cap A} f(x) \otimes \hat{\mu}(A_i \cap A))$, then $\int_A^{(K)} f\mathrm{d}\hat{\mu}$ is called a K-quasi-additive integral of f with respect to $\hat{\mu}$ on A. In particular, f is called $\hat{\mu}$-integrable whenever integral is finite, $\int_A^{(K)} f\mathrm{d}\hat{\mu} < +\infty$.

Lemma 5 [17](Integral transformation theorem)**.** Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measures, K an induced operator, and f a nonnegative measurable function on $(X, \Re)$, for all $A \in \Re$, putting $\mu\ (\cdot) = K(\hat{\mu}(\cdot)), A \in \Re$, then μ is a Lebesgue measure, and $\int_A^{(K)} f\mathrm{d}\hat{\mu} = K^{-1}(\int_A K \circ f\mathrm{d}\hat{\mu})$.

Note 4. From Lemma 5, we know that a K-quasi-additive integral reduces to a Lebesgue integral whenever $K(x) = x$. Thus, this kind of integral is a generalization of Lebesgue integrals. In addition, the corresponding quasi-sum and quasi-product reduces to the ordinary sum and product, respectively. In fact, Lemma 5 changes K-quasi-additive integrals into Lebesgue integrals. Hence, some of their properties are very easily to obtain (see [14–18]).

Definition 4.5. Let $F : \mathbb{R}^d \to F_0(\mathbb{R})$ be a fuzzy-valued function, K an induced operator, and $n \in \mathbb{N}$, if there exists a nonnegative $\hat{\mu}$-integrable function $\omega(x)$ such that for any $y \in (Z_n(F(x)))_\lambda$ implies $|\ y\ | \leqslant \omega(x)$ for all $\lambda \in (0, 1]$ and $x \in \mathbb{R}^d$, then F is said to be $\hat{\mu}$-integrable bounded on $\mathbb{R}^d$.

Denote $L^1(\hat{\mu}) = \{F : \mathbb{R}^d \to F_0(\mathbb{R}) \mid F$ is a $\hat{\mu}-$integrable bounded fuzzy-valued function on $\mathbb{R}^d\}$. Obviously, for any $F \in L^1(\hat{\mu})$, $\|\ Z_n(F(x))\ \| = D\ (Z_n(F(x)), Z_n(\{0\}))$ is Lebesgue integrable, and there exists a $\hat{\mu}-$integrable function $\omega(x)$ such that $\|\ Z_n(F(x))\ \| \leqslant \omega(x)$ with $\int_A^{(K)} \omega(x)\mathrm{d}\hat{\mu} < +\infty$.

Definition 4.6. Let $(X, \Re, \hat{\mu})$ be a space of K-quasi-additive measures, and K an induced operator, for given $n \in \mathbb{N}$, for any $F_1, F_2 \in L^1(\hat{\mu})$ and $A \in \Re$, define $H(F_1, F_2) = \int_A^{(K)} D\ (Z_n(F_1(x)), Z_n(F_2(x)))\mathrm{d}\hat{\mu}$. Then H is called a K-integral norm. Clearly, according to Lemma 5, H can be expressed as

$$H(F_1, F_2) = K^{-1}\left(\int_A K(D\ (Z_n(F_1(x)), Z_n(F_2(x))))\mathrm{d}\mu\right).$$

Theorem 4.2. For arbitrary $F_1, F_2 \in L^1(\hat{\mu})$, then $H(F_1, F_2) < +\infty$.

Proof. Actually, because $F_1, F_2 \in L^1(\hat{\mu})$, then there exists $\hat{\mu}$-integrable bounded functions $\omega_1(x)$ and $\omega_2(x)$, respectively, such that $\|\ Z_n(F_1(x))\ \| \leqslant \omega_1(x)$ and $\|\ Z_n(F_2(x))\ \| \leqslant \omega_2(x)$. As D is a metric, then in light of Theorem 4.1(1), for every $x \in \mathbb{R}^d$, we can deduce that

$$\begin{aligned} D\ (Z_n(F_1(x)), Z_n(F_2(x))) &\leqslant D\ (Z_n(F_1(x)), Z_n(\{0\})) + D\ (Z_n(\{0\}), Z_n(F_2(x))) \\ &= \|\ Z_n(F_1(x))\ \| + \|\ Z_n(F_2(x))\ \| \leqslant \omega_1(x) \oplus \omega_2(x). \end{aligned}$$

Combining Lemma 5, Theorem 4.1(2), and the fact that K^{-1} is strictly increasing, it shows that

$$\begin{aligned} H(F_1,F_2) &= K^{-1}\Big(\int_A K(D\ (Z_n(F_1(x)),Z_n(F_2(x))))\mathrm{d}\mu\Big)\\ &\leqslant K^{-1}\Big(\int_A K(\omega_1(x)\oplus\omega_2(x))\mathrm{d}\mu\Big)\\ &= K^{-1}\Big(\int_A K(\omega_1(x))\mathrm{d}\mu\ +\int_A K(\omega_2(x))\mathrm{d}\mu\Big)\\ &= \int_A^{(K)}\omega_1(x)\mathrm{d}\hat{\mu}\oplus\int_A^{(K)}\omega_2(x)\mathrm{d}\hat{\mu}<+\infty. \end{aligned}$$

Theorem 4.3. For arbitrary $F_1,F_2,F_3\in L^1(\hat{\mu})$, then integral norm H satisfies three points inequality with respect to the quasi-sum $\oplus$.

Proof. Since D is a metric on $Z_n(F_0(\mathbb{R}))$, by Theorem 4.1(1), we have for all $x\in\mathbb{R}^d$

$$\begin{aligned} K(D\ (Z_n(F_1(x)),Z_n(F_3(x)))) &\leqslant K(D\ (Z_n(F_1(x)),Z_n(F_2(x)))+D\ (Z_n(F_2(x)),Z_n(F_3(x))))\\ &\leqslant K(D\ (Z_n(F_1(x)),Z_n(F_2(x))))+K(D\ (Z_n(F_2(x)),Z_n(F_3(x)))). \end{aligned}$$

Hence, in accordance with Lemma 5, Theorem 4.1(3), and that K^{-1} is a monotonic increasing function, for every $A\in\Re$, we derive

$$\begin{aligned} H(F_1,F_3) &= K^{-1}\Big(\int_A K(D\ (Z_n(F_1(x)),Z_n(F_3(x))))\mathrm{d}\mu\Big)\\ &\leqslant K^{-1}\Big(\int_A K(D\ (Z_n(F_1(x)),Z_n(F_2(x))))\mathrm{d}\mu+\int_A K(D\ (Z_n(F_2(x)),Z_n(F_3(x))))\mathrm{d}\mu\Big)\\ &= K^{-1}\Big(\int_A K(D\ (Z_n(F_1(x)),Z_n(F_2(x))))\mathrm{d}\mu\Big)\oplus K^{-1}\Big(\int_A K(D\ (Z_n(F_2(x)),Z_n(F_3(x))))\mathrm{d}\mu\Big)\\ &= H(F_1,F_2)\oplus H(F_2,F_3). \end{aligned}$$

Theorem 4.4. The $(L^1(\hat{\mu}),H)$ constitutes a metric space with respect to quasi-addition $\oplus$.

Proof. By Definition 4.6, H satisfies nonnegativity and symmetry; thus by synthesizing Theorem 4.2 and Theorem 4.3, the statement can be proved.

5 Separability of $(L^1(\hat{\mu}),H)$

In the above section, we have outlined the concept of the K-integral norm by introducing K-quasi-additive integrals, and determined that integrable system $(L^1(\hat{\mu}),H)$ constitutes a metric space by means of the integral norm. In this section, we shall go on proving that $(L^1(\hat{\mu}),H)$ constitutes a completely separable metric space. To overcome the shortcoming in [7], we will adopt the method of polygonal fuzzy numbers to develop the space of general fuzzy numbers. This eventuates because n-polygonal fuzzy numbers, handled via the Z_n map constitutes a completely separable metric space. The algorithm is easy to comprehend and the method is simple and clear.

Definition 5.1. Let $(\mathbb{R}^d,\Re,\hat{\mu})$ be a space of K-quasi-additive measure, $\Omega\subset\mathbb{R}^d$, mapping $S:\Omega\to F_0(\mathbb{R}),\{E_i\mid i=1,2,\ldots,m\}$ be a finite partition of Ω, i.e., $\bigcup_{i=1}^m E_i=\Omega$ and $E_i\cap E_j=\emptyset(i\neq j)$ where $E_i\in\Re, i=1,2,\ldots,m$. For any $x=(x_1,x_2,\ldots,x_d)\in\Omega$, if there exists a set of fuzzy numbers $\widetilde{A}_1,\widetilde{A}_2,\ldots,\widetilde{A}_m\in F_0(\mathbb{R})$ with $S(x)=\sum_{i=1}^m\widetilde{A}_i\cdot\chi_{E_i}(x)$, where $\chi_{E_i}(x)$ is a characteristic function, then S is called a fuzzy-valued simple function defined on Ω.

Let $S_n(\Omega)$ denote the family of all fuzzy-valued simple functions on Ω. Obviously, from Theorem 3.1 and Definition 3.2, we obtain $Z_n(S(x))=\sum_{i=1}^m Z_n(\widetilde{A}_i)\cdot\chi_{E_i}(x)$ for every $x\in\Omega$.

Definition 5.2. For given $n \in \mathbb{N}$, let $Q : \Omega \to Z_n(F_0(\mathbb{R}))$ be a polygonal fuzzy-valued function, $x_0 \in \Omega$, for arbitrary $\varepsilon > 0$, if there exists a $\delta > 0$ such that $D\ (Q(x), Q(x_0)) < \varepsilon$ whenever $\eta(x, x_0) < \delta$, then Q is said to be continuous at point x_0, where η is a metric in $\Omega \subset \mathbb{R}^d$.

In addition, for every $x = (x_1, x_2, \dots, x_d) \in \Omega$, the polygonal fuzzy-valued function Q can be denoted as $Q(x) = (f_0^1(x), f_1^1(x), \dots, f_n^1(x), f_n^2(x), \dots, f_1^2(x), f_0^2(x)) \in Z_n(F_0(\mathbb{R}))$.

Definition 5.3. Let $F : \Omega \to F_0(\mathbb{R})$, for given $n \in \mathbb{N}, x_0 \in \Omega$, if polygonal fuzzy valued function $Z_n(F(\cdot))$ is continuous at point x_0, then F is said to be continuous at point x_0, if F is continuous at an arbitrary point on Ω, then F is said to be continuous on Ω.

Taking Lemma 4 into account, F is continuous on Ω iff $Z_n(F(\cdot))$ is continuous on Ω and iff each $f_j^q(x)$ is continuous on Ω for $q = 1, 2; j = 0, 1, 2, \dots, n$.

Next, we will verify that the class of fuzzy-valued simple functions $S_n(\Omega)$ is dense on the space $L^1(\hat{\mu})$ of integrable bounded functions; that is, $S_n(\Omega)$ possesses a universal approximation with respect to $L^1(\hat{\mu})$ in the sense of K-integral norms.

Theorem 5.1. Let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a space of a K-quasi-additive measure, $\hat{\mu}(\Omega) < +\infty$ with $\Omega \subset \mathbb{R}^d, F : \Omega \to F_0(\mathbb{R})$ be $\hat{\mu}$-integrable, K an induced operator, $n \in \mathbb{N}$, then $S_n(\Omega)$ can approximate F to arbitrary accuracy with respect to K-integral norms.

Proof. For given $n \in \mathbb{N}$ and any $\varepsilon > 0$, we need only prove that there exists $S_0 \in S_n(\Omega)$ such that $H(F, S_0) < \varepsilon$ for every $F \in L^1(\hat{\mu})$.

From Lemma 3, we know that the completely metric space $(Z_n(F_0(\mathbb{R})), D)$ is separable. Without loss of generality, suppose $\{\widetilde{A}_i^z \mid i \in \mathbb{N}\}$ is a countably-dense subset of $Z_n(F_0(\mathbb{R}))$, where each n-polygonal fuzzy number $\widetilde{A}_i^z \in Z_n(F_0(\mathbb{R}))$, $i = 1, 2, \dots$. Thus, for every $\varepsilon > 0$, there exists $i \in \mathbb{N}$ and $\widetilde{X} \in F_0(\mathbb{R})$ such that $D(Z_n(\widetilde{X}), \widetilde{A}_i^z) < \varepsilon$. Let

$$
\begin{aligned}
&E_1 = \{x \in \Omega \mid D(Z_n(F(x)), \widetilde{A}_1^z < \varepsilon\},\\
&E_2 = \{x \in \Omega \mid D(Z_n(F(x)), \widetilde{A}_1^z) \geqslant \varepsilon,\ D(Z_n(F(x)), \widetilde{A}_2^z) < \varepsilon\},\\
&\quad \dots\\
&E_k = \{x \in \Omega \mid D(Z_n(F(x)), \widetilde{A}_i^z) \geqslant \varepsilon\ (i = 1, 2, \cdots, k-1),\ D(Z_n(F(x)), \widetilde{A}_k^z) < \varepsilon\},\\
&\quad \dots
\end{aligned}
$$

Clearly, these sets fulfill $E_i \cap E_j = \emptyset (i \neq j)$, and $\bigcup_{k=1}^{\infty} E_k = \Omega$, where every E_k is measurable. In fact, $\bigcup_{k=1}^{\infty} E_k \subset \Omega$ is obvious. On the contrary, assume for any $x \in \Omega$, then (1) if $x \in E_1$, then $\Omega \subset \bigcup_{k=1}^{\infty} E_k$ holds; else (2) if $x \notin E_1$, then in regard to the sequence $\{E_i\}$ of the sets, we know that there exists an $i_0 \in \mathbb{N}$ such that $D\ (Z_n(F(x)), \widetilde{A}_{i_0}^z) < \varepsilon$.

As for the i_0-th term, if $D(Z_n(F(x)), \widetilde{A}_i^z) \geqslant \varepsilon$ for every $i \in \{1, 2, \dots, i_0-1\}$, by means of the definition of the sequence $\{E_i\}$ of sets, it follows that $x \in E_{i_0} \subset \bigcup_{k=1}^{\infty} E_k$; otherwise, if there exists $i_k \in \{1, 2, \dots, i_0-1\}$ such that $D\ (Z_n(F(x)), \widetilde{A}_{i_k}^z) < \varepsilon$, then may be this i_k does not sole. Let i_{k_0} be the smallest of all i_k with $D(Z_n(F(x)), \widetilde{A}_i^z) \geqslant \varepsilon,\ i = 1, 2, \dots, i_0 - 1$, thus, $x \in E_{i_{k_0}} \subset \bigcup_{k=1}^{\infty} E_k$. Therefore, $\Omega = \bigcup_{k=1}^{\infty} E_k$.

Furthermore, let $\mu\ (\cdot) = K(\hat{\mu}\ (\cdot))$, by Lemma 5, we can see that μ is a Lebesgue measure that satisfies $\mu\ (E) = K(\hat{\mu}\ (E)) \leqslant K(\hat{\mu}\ (\Omega)) < +\infty$ for all $E \in \Re$. Taking advantage of the countable additivity of the Lebesgue measure μ with regard to sequence $\{E_k\}$ of measurable sets, we find that $\sum_{k=1}^{\infty} \mu\ (E_k) = \mu\ (\bigcup_{k=1}^{\infty} E_k) = \mu\ (\Omega) = K(\hat{\mu}\ (\Omega)) < +\infty$. Hence, the series of positive terms $\sum_{k=1}^{\infty} \mu\ (E_k)$ is convergent, and thus, for arbitrary $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ and whenever $n \geqslant N$ such that

$$
\mu \left(\bigcup_{k=N+1}^{\infty} E_k \right) = \sum_{k=N+1}^{\infty} \mu\ (E_k) = \left| \sum_{k=1}^{\infty} \mu\ (E_k) - \sum_{k=1}^{N} \mu\ (E_k) \right| < \varepsilon.
$$

Setting $E_0 = \bigcup_{k=N+1}^{\infty} E_k$, then the above can be rewritten as $\mu\ (E_0) < \varepsilon$.

Synthesizing the above discussion, $\Omega = (\bigcup_{k=1}^{N} E_k) \cup E_0$ and $\{E_1, E_2, \dots, E_N, E_0\}$ constitutes a new finite measurable partition on Ω. Now, we select the front $n-$polygonal fuzzy numbers $\{\widetilde{A}_k^z \mid k = 1, 2, \dots, N\}$; of course, $\widetilde{A}_k^z \in F_0(\mathbb{R}),\ k = 1, 2, \dots, N$. Let $S_0(x) = \sum_{k=0}^{N} \widetilde{A}_k^z \cdot \chi_{E_k}(x)$ for all $x \in \Omega$.

Whenever $k = 0$, replenish $\widetilde{A}_0^z = (0, 0, \ldots, 0, 0, \ldots, 0, 0)$, then S_0 is a fuzzy-valued simple function on Ω, i.e., $S_0 \in S_n(\Omega)$. From Definition 3.2, we obtain $Z_n(\widetilde{A}_k^z \cdot \chi_{E_k}(x)) = Z_n(\widetilde{A}_k^z) \cdot Z_n(\chi_{E_k}(x)) = \widetilde{A}_k^z \cdot \chi_{E_k}(x)$. Consequently, $Z_n(S_0(x)) = S_0(x)$. In addition, the distance function $D(Z_n(F(x)), S_0(x))$ is bounded and Lebesgue integrable on Ω, as also is $K(D(Z_n(F(x)), S_0(x)))$. According to the absolute continuity of the Lebesgue integrals, taking $\delta = \varepsilon > 0$, whenever $\mu(E_0) < \varepsilon = \delta$, we derive from

$$\int_{E_0} K(D(Z_n(F(x)), S_0(x)))\mathrm{d}\mu < \varepsilon. \tag{1}$$

In light of the sequence $\{E_k\}$, for all $x \in E_k$, we can infer

$$D(Z_n(F(x)), S_0(x)) = D(Z_n(F(x)), \widetilde{A}_k^z) < \varepsilon, \ k = 1, 2, \ldots, N. \tag{2}$$

By use of Lemma 5, combining the monotonicity of K^{-1}, (1) and (2), we have

$$\begin{aligned} H(F, S_0) &= K^{-1}\left(\int_{\bigcup_{k=1}^N E_k} K(D(Z_n(F(x)), S_0(x)))\mathrm{d}\mu + \int_{E_0} K(D(Z_n(F(x)), S_0(x)))\mathrm{d}\mu \right) \\ &\leqslant K^{-1}\left(\sum_{k=1}^{N} \int_{E_k} K(\varepsilon)\mathrm{d}\mu + \varepsilon \right) = K^{-1}\left(K(\varepsilon) \cdot \mu\left(\bigcup_{k=1}^{N} E_k \right) + \varepsilon \right) \\ &\leqslant K^{-1}(\mu(\Omega) \cdot K(\varepsilon) + \varepsilon). \end{aligned}$$

Because K and K^{-1} are strictly increasing, $\mu(\Omega)$ is finite, thus, for all $\varepsilon > 0$, it follows that $\mu\ (\Omega) \cdot K(\varepsilon) + \varepsilon$ can be made arbitrary small, and consequently, the expression $K^{-1}(\mu\ (\Omega) \cdot K(\varepsilon) + \varepsilon)$ still can be arbitrary small. Hence, $S_n(\Omega)$ can approximate F with respect to the K-integral norm to arbitrary accuracy.

Theorem 5.2. Let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, $\Omega \subset \mathbb{R}^d$ be a bounded measurable set, K an induced operator, for given $n \in \mathbb{N}$, let $C(\Omega) = \{F : \Omega \to F_0(\mathbb{R}) \mid F \text{ is continuous on } \Omega\}$, then $C(\Omega)$ is dense in $S_n(\Omega)$.

Proof. Select a bounded set $B \subset \Omega$, and construct the function and sequence of functions on Ω as follows

$$\rho(x, B) = \inf_{y \in B} \eta(x, y), \quad G_m(x) = \frac{1}{1 + m\rho(x, B)}, \quad \forall x \in \Omega, \ m = 1, 2, \ldots,$$

where η is a metric on $\mathbb{R}^d$, for all $x \in \Omega$, the function $\rho(x, B)$ and $G_m(x)$ can be shown to be uniformly continuous on Ω, and thus continuous on Ω satisfying

$$\lim_{m \to \infty} G_m(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases} = \chi_B\ (x).$$

Now for any $\widetilde{A} \in F_0(\mathbb{R})$, and for a given $n \in \mathbb{N}$, then $Z_n(\widetilde{A}) \in Z_n(F_0(\mathbb{R})) \subset F_0(\mathbb{R})$. Let $S(x) = Z_n(\widetilde{A}) \cdot \chi_B(x)$, $F_m(x) = Z_n(\widetilde{A}) \cdot G_m(x)$, for arbitrary $x \in \Omega$, $m = 1, 2, \ldots$, then $S \in S_n(\Omega)$ and $F_m(x) \in Z_n(F_0(\mathbb{R}))$. By Note 2, we find $Z_n(S(x)) = S(x)$ and $Z_n(F_m(x)) = F_m(x)$.

Next, we are going to prove that the polygonal fuzzy-valued function $F_m(x)$ is continuous on Ω. Actually, for each $x_0 \in \Omega$, since each real function $G_m(x)$ is continuous at point x_0, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mid G_m(x) - G_m(x_0) \mid < \varepsilon$ for all $x \in \Omega$ and whenever $\eta(x, x_0) < \delta$. Applying Theorem 3.2(1) and Note 2, we immediately deduce

$$D\ (F_m(x), F_m(x_0)) \leqslant \| Z_n(\widetilde{A}) \| \cdot \mid G_m(x) - G_m(x_0) \mid < \| Z_n(\widetilde{A}) \| \ \varepsilon.$$

By Definition 5.2, we get that each $F_m(x)$ is continuous at point x_0, furthermore, it is continuous on Ω. Consequently, $F_m \in C(\Omega), m = 1, 2, \ldots$.

In addition, for any $x = (x_1, x_2, \ldots, x_d) \in \Omega$, as $\chi_B(x), G_m(x) \in \mathbb{R}$, utilizing Note 2, Note 3, and Theorem 3.2(1), it is straightforward to see that

$$D\ (S(x), F_m(x)) \leqslant \| Z_n(\widetilde{A}) \| \cdot D\ (\chi_B(x), G_m(x)) = \| Z_n(\widetilde{A}) \| \cdot \mid G_m(x) - \chi_B(x) \mid \to 0\ (m \to \infty).$$

Hence, $\lim_{m\to\infty} D\ (S(x), F_m(x)) = 0$, and by continuity of K, implies $\lim_{m\to\infty} K(D\ (S(x), F_m(x))) = K(0) = 0$, applying the continuity of K^{-1} and the dominant convergence theorem of Lebesgue's integral

$$\lim_{m\to\infty} H(S, F_m) = K^{-1}\left(\int_\Omega \lim_{m\to\infty} K(D\ (S(x), F_m(x)))\mathrm{d}\mu\right) = K^{-1}(0) = 0.$$

Therefore, the polygonal fuzzy-valued simple function S may be approximated by polygonal fuzzy-valued function F_m, that is to say that $C(\Omega)$ is dense in $S_n(\Omega)$.

Definition 5.4 [8]. Let $f : [0,1]^d \to F_0(\mathbb{R})$ be a $d-$dimensional fuzzy-valued function, for all $x = (x_1, x_2, \ldots, x_d) \in [0,1]^d$ and $m \in \mathbb{N}$, we introduce the expression $B_m(f; x) = \sum_{i_1,i_2,\ldots,i_d=0}^m Q_{m;\ i_1,i_2,\ldots,i_d}(x)\cdot J(\frac{i_1}{m}, \frac{i_2}{m}, \ldots, \frac{i_d}{m})$. Then $B_m(f; x)$ is called a $d-$dimensional fuzzy-valued Bernstein polynomial of f, where $Q_{m;\ i_1,i_2,\ldots,i_d}(x) = C_m^{i_1} C_m^{i_2} \cdots C_m^{i_d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d} (1-x_1)^{m-i_1}(1-x_2)^{m-i_2} \cdots (1-x_d)^{m-i_d}$ is a real-valued multi-variable polynomial function.

Note 5. Here $\sum_{i_1,i_2,\ldots,i_d=0}^m = \sum_{i_1=0}^m \cdot \sum_{i_2=0}^m \cdots \sum_{i_d=0}^m$, and $\sum_{i_1,i_2,\ldots,i_d=0}^m Q_{m;\ i_1,i_2,\ldots,i_d}(x) = 1$.

Lemma 6 [9]. Let $F : [a,b]^d \to F_0(\mathbb{R})$ be a continuous fuzzy-valued function and D a metric in $F_0(\mathbb{R})$. For arbitrary $\varepsilon > 0$, then there exists a d-dimensional fuzzy-valued Bernstein polynomial $B_m(F; x)$ such that $D(B_m(F; x), F(x)) < \varepsilon$ for any $x \in [a,b]^d$.

Theorem 5.3. Let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a space of K-quasi-additive measure, $\Omega \subset \mathbb{R}^d$ a bounded measurable set, and K an induced operator, for given $n \in \mathbb{N}$, denote $P(\Omega) = \{F : \Omega \to F_0(\mathbb{R}) \mid F(x) = \sum_{i_1,i_2,\ldots,i_d=0}^m \widetilde{A}_{i_1,i_2,\ldots,i_d} \cdot Q_{m;\ i_1,i_2,\ldots,i_d}(x), \widetilde{A}_{i_1,i_2,\ldots,i_d} \in F_0(\mathbb{R})\}$. Then $P(\Omega)$ is dense in $C(\Omega)$.

Proof. Obviously, $P(\Omega)$ is a countable set, since Ω is bounded. Thus, there exists a closed d-dimensional rectangular parallelepiped $[a,b]^d$ such that $\Omega \subset [a,b]^d$.

Indeed, for every $F \in C(\Omega)$, by Lemma 6, we know that for any $\varepsilon > 0$, there exists a $d-$dimensional Bernstein polynomial $B_m(F; x) = \sum_{i_1,i_2,\ldots,i_d=0}^m \widetilde{B}_{i_1,i_2,\ldots,i_d} \cdot Q_{m;\ i_1,i_2,\ldots,i_d}(x)$ such that $D(B_m(F; x), F(x)) < \varepsilon$ for all $x \in \Omega$, where $\widetilde{B}_{i_1,i_2,\ldots,i_d} \in F_0(\mathbb{R})$. Making use of Theorem 3.1, for given $n \in \mathbb{N}$,

$$Z_n(B_m(F; x)) = \sum_{i_1,i_2,\ldots,i_d=0}^m Z_n(\widetilde{B}_{i_1,i_2,\ldots,i_d}) \cdot Q_{m;\ i_1,i_2,\ldots,i_d}(x) \in Z_n(F_0(\mathbb{R})).$$

By means of Lemma 4, for each $x \in \Omega$

$$D\ (Z_n(B_m(F; x)), Z_n(F(x))) \leqslant D\ (B_m(F; x), F(x)) < \varepsilon, \tag{3}$$

In accordance with Lemma 3, suppose $\aleph = \{\widetilde{A}_1^z, \widetilde{A}_2^z, \ldots, \widetilde{A}_k^z, \ldots\}$ is a countably dense subset of $Z_n(F_0(\mathbb{R}))$, then there exists corresponding polygonal fuzzy numbers $\{\widetilde{A'}^z_{i_1,i_2,\ldots,i_d}\} \subset \aleph$ for a cluster of polygonal fuzzy numbers $Z_n(\widetilde{B}_{i_1,i_2,\ldots,i_d})$ such that $D(\widetilde{A'}^z_{i_1,i_2,\ldots,i_d}, Z_n(\widetilde{B}_{i_1,i_2,\ldots,i_d})) < \varepsilon$ for any $i_1, i_2, \ldots, i_d \in \{0, 1, 2, \ldots, m\}$.

Let $P_m(x) = \sum_{i_1,i_2,\ldots,i_d=0}^m \widetilde{A'}^z_{i_1,i_2,\ldots,i_d} \cdot Q_{m;\ i_1,i_2,\ldots,i_d}(x)$, for all $x \in \Omega$, then $P_m \in P(\Omega)$. From Theorem 3.1(2), we can obtain $Z_n(P_m(x)) = P_m(x) \in Z_n(F_0(\mathbb{R})) \subset F_0(\mathbb{R})$. According to Theorem 3.2(2) and Note 5, we immediately derive

$$\begin{aligned} D(P_m(x), Z_n(B_m(F; x))) &\leqslant \sum_{i_1,i_2,\ldots,i_d=0}^m Q_{m;\ i_1,i_2,\ldots,i_d}(x) \cdot D(\widetilde{A'}^z_{i_1,i_2,\ldots,i_d}, Z_n(\widetilde{B}_{i_1,i_2,\ldots,i_d})) \\ &= 1.\ D(\widetilde{A'}^z_{i_1,i_2,\ldots,i_d}, Z_n(\widetilde{B}_{i_1,i_2,\ldots,i_d})) < \varepsilon. \end{aligned} \tag{4}$$

Furthermore, by applying Theorem 4.3 and Lemma 5, and combining (3) and (4), we can infer

$$\begin{aligned} H(P_m, F) &\leqslant H(P_m, B_m(F)) \oplus H(B_m(F), F) \\ &= K^{-1}\left(\int_\Omega K(D\ (P_m(x), Z_n(B_m(F; x))))\mathrm{d}\mu\right) \end{aligned}$$

$$\oplus K^{-1}\bigg(\int_{\Omega} K(D\ (Z_n(B_m(F;x)), Z_n(F(x))))\mathrm{d}\mu\bigg)$$
$$\leqslant K^{-1}\bigg(\int_{\Omega} K(\varepsilon)\mathrm{d}\mu\bigg) \oplus K^{-1}\bigg(\int_{\Omega} K(\varepsilon)\mathrm{d}\mu\bigg) = K^{-1}\Big(2\mu\ (\Omega)\cdot K(\varepsilon)\Big).$$

Evidently, for arbitrary $\varepsilon > 0$, expression $K^{-1}(2\mu\ (\Omega)\cdot K(\varepsilon))$ will still arbitrary small. Therefore, for every continuous fuzzy-valued operator F in $C(\Omega)$ can be approximated by the operator P_m of a fuzzy-valued Bernstein polynomial in $P(\Omega)$. This means that $P(\Omega)$ is dense in $C(\Omega)$.

Theorem 5.4. let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a space of $K-$quasi-additive fuzzy measure, $\Omega \subset \mathbb{R}^d$ a bounded measurable set, and K an induced operator. Then $(L^1(\hat{\mu}), H)$ is a completely-separable metric space.

Proof. That $(L^1(\hat{\mu}), H)$ constitutes a metric space has been demonstrated in the proof of Theorem 4.3. In addition, repeating the arguments for the completeness with respect to the integrable space in functional analysis, we may prove the completeness of $(L^1(\hat{\mu}), H)$. Thus, we need only demonstrate the separability of $(L^1(\hat{\mu}), H)$.

Applying Theorem 5.3, Theorem 5.2, and Theorem 5.1, we immediately know that $P(\Omega)$ is also dense in $L^1(\hat{\mu})$, that is to say that $P(\Omega)$ is a dense subset in $L^1(\hat{\mu})$. Hence, $(L^1(\hat{\mu}), H)$ is a completely-separable metric space.

6 Universal approximation of polygonal fuzzy neural networks

A polygonal fuzzy number is solely determined by a finite number of points on a straight line $\mathbb{R}$, which can be used to approximate to a class of bounded fuzzy numbers up to arbitrary accuracy. Thus, it not only is a generalization of trigonometric fuzzy numbers or a ladder fuzzy numbers, but also can give an approximation of general bounded fuzzy numbers. In simplifying the extension principle (Definition 3.2 and Theorem 3.1), polygonal fuzzy numbers both assure the closeness of their four arithmetic operations, and maintain similar properties to ladder fuzzy numbers. At the same, the space of polygonal fuzzy numbers and Euclidean space have analogous properties.

The polygonal fuzzy neural networks introduced in this paper are a class of network systems in which connection weights as well as threshold values take values that are polygonal fuzzy numbers, and their inner operations are based on the simplified extension principle. Indeed, the structure of the following polygonal fuzzy neural networks can be described as an operational system combining both addition and multiplication with respect to polygonal fuzzy numbers. In other words, polygonal fuzzy neural networks finish fuzzy information processing by a finite number of points that determine the polygonal fuzzy numbers. Consequently, for a polygonal fuzzy neural network, its approximately-expressible capability is readily solved by means of the linear operational properties of polygonal fuzzy numbers. In this section, we shall discuss the universal approximation of four-layer regular polygonal fuzzy neural networks with respect to the class of $\hat{\mu}$-integrable bounded fuzzy-valued functions in the sense of K-integral norms.

For the rest of this paper, we will always let u_{ikj}, v_{kj}, and $\widetilde{w}_k$ be a connected weight between the i-th input neuron and the j-th neuron in the first hidden layer, the j-th neuron in the first hidden layer and the k-th neuron in the second hidden layer as well as the k-th neuron in the second hidden layer and output neuron, respectively, where $u_{ikj}, v_{kj} \in \mathbb{R}$, and $\widetilde{w}_k \in F_0(\mathbb{R})$. Let the neurons in the input layer and the second hidden layer as well as the output layer be linear, and the activation function σ in the first hidden layer be a bounded continuous function on $\mathbb{R}$, where $u_k(j) = (u_{1kj}, u_{2kj}, \ldots, u_{dkj}) \in \mathbb{R}^d$. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, and given $n \in \mathbb{N}$, then a four-layer regular polygonal fuzzy neural network is denoted as

$$\Im_0[\sigma] = \Bigg\{\ G_{pq} : \mathbb{R}^d \to Z_n(F_0(\mathbb{R}))\ |\ \ G_{pq}(x) = \sum_{k=1}^{q} \widetilde{W}_k \cdot \bigg(\sum_{j=1}^{p} v_{kj}\cdot\sigma(\langle u_k(j), x\rangle + \theta_{kj}\ )\bigg),$$
$$p, q \in \mathbb{N}, \widetilde{W}_k \in Z_n(F_0(\mathbb{R})), v_{kj}, \theta_{kj} \in \mathbb{R}, u_k(j) \in \mathbb{R}^d\ \Bigg\}.$$

where $\theta_{kj} \in \mathbb{R}$ is a threshold value for the j-th neuron in the first hidden layer. Let p and q be the number of neurons in the first hidden layer and second layer, respectively.

Actually, every element in $\Im_0[\sigma]$, where $\sigma : \mathbb{R} \to \mathbb{R}^+$ is a four-layer regular polygonal fuzzy neural network consisting of two hidden layer. As an activation function of the first hidden layer, it is bounded with $\lim_{x\to-\infty} \sigma(x) = 0$, $\lim_{x\to+\infty} \sigma(x) = 1$.

Definition 6.1 [6]. Let $\Lambda = \{ F : \mathbb{R}^d \to F_0(\mathbb{R}) \mid F$ is a fuzzy valued function$\}$, $\Gamma \subset \Lambda$, for arbitrary $\varepsilon > 0$, for any $F \in \Gamma$ and a compact set $U \subset \mathbb{R}^d$, if there exists $p, q \in \mathbb{N}$, connection weight $\widetilde{W}_k \in F_0(\mathbb{R}), v_{kj} \in \mathbb{R}, u_k(j) \in \mathbb{R}^d$ and threshold value $\theta_{kj} \in \mathbb{R}$ for $i = 1, 2, \ldots, d; j = 1, 2, \ldots, p; k = 1, 2, \ldots, q$ such that $D\ (Z_n(F(x)), G_{pq}(x)) < \varepsilon$ for all $x = (x_1, x_2, \ldots, x_d) \in U$, then we say that the four-layer regular polygonal fuzzy neural network $\Im_0[\sigma]$ possesses a universal approximation to Γ, or it is called a universal approximator of Γ.

Definition 6.2 [6]. Let $\sigma : \mathbb{R} \to \mathbb{R}^+$ be an activation function and $f : \mathbb{R}^d \to \mathbb{R}$ be continuous function, for arbitrary $\varepsilon > 0$ and for each compact set $U \subset \mathbb{R}^d$, if there exists m hidden neurons, connection weight $v_j \in \mathbb{R}, W_j = (w_{1j}, w_{2j}, \ldots, w_{dj}) \in \mathbb{R}^d$, and threshold value $\theta_j \in \mathbb{R}$ such that $| \sum_{j=1}^m v_j \cdot \sigma(\langle W_j, x\rangle + \theta_j) - f(x) | < \varepsilon$ for all $x = (x_1, x_2, \ldots, x_d) \in U$, then σ is called a Tauber-Wiener function.

Theorem 6.1. Let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a finite space of k-quasi-additive measure, σ a Tauber-Wiener function, K an induced operator, for given $n \in \mathbb{N}$, then $\Im_0[\sigma]$ possesses a universal approximation for $S_n(\Omega)$ with respect to K-integral norm H.

Proof. For arbitrary $\varepsilon > 0$ and $S \in S_n(\Omega)$ with $\Omega \subset \mathbb{R}^d$ a compact set, we need only prove that there exists $G_{m\lambda} \in \Im_0[\sigma]$ such that $H(G_{m\lambda}, S) < \varepsilon$.

Practically, from Definition 5.1, choose any fuzzy valued simple function $S \in S_n(\Omega)$ for given $n \in \mathbb{N}$, let $S(x) = \sum_{i=1}^m \widetilde{A}_i \cdot \chi_{E_i}(x)$ for all $x = (x_1, x_2, \ldots, x_d) \in \Omega$, where $\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_m \in F_0(\mathbb{R})$ and with $\bigcup_{i=1}^m E_i = \Omega, E_i \cap E_j = \emptyset\ (i \neq j)$. Without loss of generality, we can assume that from the norms $\| Z_n(\widetilde{A}_1) \|, \| Z_n(\widetilde{A}_2) \|, \ldots, \| Z_n(\widetilde{A}_m) \|$, at least exist one $\| Z_n(\widetilde{A}_i) \|$ satisfies $\| Z_n(\widetilde{A}_i) \| \neq 0$ where $1 \leqslant i \leqslant m$.

Since each characteristic function $\chi_{E_i}(x)$ is a nonnegative measurable on $\Omega, i \in \{1, 2, \ldots, m\}$, then, in light of the Lusin theorem, for arbitrary $\varepsilon > 0$, it follows that there exists a closed subset $\Delta_i \subset \Omega$ such that $\chi_{E_i}(x)$ is continuous on Δ_i with $\mu(\Omega - \Delta_i) < \frac{\varepsilon}{m}$, where $\mu(\cdot) = K(\hat{\mu}(\cdot)), \mu$ is a Lebesgue measure, every $\chi_{E_i}(x)$ takes value 1 or 0 on Δ_i.

Moreover, as $\Omega \subset \mathbb{R}^d$ is a compact set iff Ω is a bounded closed set, each $\Delta_i (i = 1, 2, \ldots, m)$ is a compact set, and $\mu\ (\Omega) < +\infty$. Because σ is a Tauber-Wiener function, for every continuous function $\chi_{E_i}(x)$(it always is 1 or 0) on Δ_i, by Definition 6.2, for arbitrary $\varepsilon > 0$, we know that there exists quantity $\lambda_i \in \mathbb{N}$ of the neurons in the hidden layer, connected weights $u'_i(1), u'_i(2), \ldots, u'_i(\lambda_i) \in \mathbb{R}^d$, $v'_{i1}, v'_{i2}, \ldots, v'_{i\lambda_i} \in \mathbb{R}$, and threshold values $\theta'_{i1}, \theta'_{i2}, \ldots, \theta'_{i\lambda_i} \in \mathbb{R}$ such that

$$\left| \sum_{j=1}^{\lambda_i} v'_{ij} \cdot \sigma(\langle u'_i(j), x\rangle + \theta'_{ij}\ ) - \chi_{E_i}(x) \right| < \frac{\varepsilon}{m}, \tag{5}$$

for arbitrary $x = (x_1, x_2, \ldots, x_d) \in \Delta_i,\ i = 1, 2, \ldots, m$. Now, for all $x \in \Omega$, we let

$$G'(x) = \sum_{i=1}^m Z_n(\widetilde{A}_i) \cdot \left(\sum_{j=1}^{\lambda_i} v'_{ij} \cdot \sigma(< u'_i(j), x > +\theta'_{ij}\ ) \right),$$

and write $\max_{1\leqslant i\leqslant m} \| Z_n(\widetilde{A}_i) \| = a$, where a is regarded as a given constant. In accordance with Theorem 3.2 and (5), it is straightforward to see that

$$D(G'(x), Z_n(S(x))) \leqslant \sum_{i=1}^m \| Z_n(\widetilde{A}_i) \| \cdot | \sum_{j=1}^{\lambda_i} v'_{ij} \cdot \sigma(< u'_i(j), x > +\theta'_{ij}\ ) - \chi_{E_i}(x) | < ma \cdot \frac{\varepsilon}{m} = a\varepsilon.$$

Therefore, we obtain $\int_{\Omega} K(D(G'(x), Z_n(S(x))))\mathrm{d}\mu \leqslant \int_{\Omega} K(a\varepsilon)\mathrm{d}\mu = \mu\ (\Omega) \cdot K(a\varepsilon) < +\infty$. This means that function $K(D(G'(x), Z_n(S(x))))$ is Lebesgue integrable on Ω. Putting $\Delta = \bigcap_{i=1}^{m} \Delta_i$, then $\Delta \subset \Delta_i \subset \Omega$, and Δ is still a compact set. Applying the sub-countable additivity of the Lebesgue measure μ, we immediately can infer $\mu(\Omega - \Delta) = \mu\left(\bigcup_{i=1}^{m}(\Omega - \Delta_i)\right) \leqslant \sum_{i=1}^{m} \mu(\Omega - \Delta_i) \leqslant \sum_{i=1}^{m} \frac{\varepsilon}{m} = \varepsilon$.

Obviously, $\Delta \neq \emptyset$, as if not then this would imply $\mu(\Omega) < \varepsilon$ from the above formula contradicting the fact that $\mu(\Omega)$ is finite. Taking advantage of the absolute continuity of the Lebesgue integral, for arbitrary $\varepsilon > 0$, select $\delta = \varepsilon > 0$, whenever $\mu\ (\Omega - \Delta) < \varepsilon$, it is not hard to see that

$$\int_{\Omega-\Delta} K(D\ (G'(x), Z_n(S(x))))\mathrm{d}\mu\ < \varepsilon. \tag{6}$$

Next, we shall construct the transformation of a system. Assume $\lambda = \sum_{i=1}^{m} \lambda_i,\ \beta_i = \sum_{k=1}^{i-1} \lambda_k$, and take $\beta_1 = 0,\ i = 2, 3, \ldots, m$. Let

$$v_{ij} = \begin{cases} v'_{i(j-\beta_i)}, & \beta_i < j \leqslant \beta_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \quad \theta_{ij} = \begin{cases} \theta'_{i(j-\beta_i)}, & \beta_i < j \leqslant \beta_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$
$$u_i(j) = \begin{cases} u_i(j-\beta_i), & \beta_i < j \leqslant \beta_{i+1}, \\ 0, & \text{otherwise}. \end{cases}$$

Then for $m, \lambda \in \mathbb{N}$, and for arbitrary $x \in \Delta = \bigcap_{i=1}^{m} \Delta_i \subset \Delta_i$, according to the transformation of above system, the following formula holds

$$\sum_{j=1}^{\lambda} v_{ij} \cdot \sigma(\langle u_i(j), x\rangle +\ \theta_{ij}\ ) = \sum_{j=1}^{\lambda_i} v'_{ij} \cdot \sigma(\langle u'_i(j), x\rangle + \theta'_{ij}). \tag{7}$$

Let $G_{m\lambda}(x) = \sum_{i=1}^{m} Z_n(\widetilde{A}_i) \cdot \sum_{j=1}^{\lambda} v_{ij} \cdot \sigma(< u_i(j), x > +\theta_{ij})$ for arbitrary $x = (x_1, x_2, \ldots, x_d) \in \Omega$, then we can infer that $G_{m\lambda} \in \Im_0[\sigma]$ with $Z_n(G_{m\lambda}(x)) = G_{m\lambda}(x)$. Of course, by the absolute continuity of the Lebesgue integral, we have $\int_{\Omega-\Delta} K(D\ (G_{m\lambda}(x), Z_n(S(x))))\mathrm{d}\mu\ < \varepsilon$.

Making use of Theorem 3.2(2), for arbitrary $x = (x_1, x_2, \ldots, x_d) \in \Delta = \bigcap_{i=1}^{m} \Delta_i$, it follows that

$$\begin{aligned} D\ (G_{m\lambda}(x), Z_n(S(x))) &\leqslant \sum_{i=1}^{m} \parallel Z_n(\widetilde{A}_i) \parallel \cdot \left| \sum_{j=1}^{\lambda} v_{ij} \cdot \sigma(< u_i(j), x > + \ \theta_{ij}\ ) - \chi_{E_i}(x) \right| \\ &= \sum_{i=1}^{m} \parallel Z_n(\widetilde{A}_i) \parallel \cdot \left| \sum_{j=1}^{\lambda_i} v'_{ij} \cdot \sigma(< u'_i(j), x > + \ \theta'_{ij}\ ) - \chi_{E_i}(x) \right| \\ &\leqslant ma \cdot \frac{\varepsilon}{m} = a\varepsilon. \end{aligned}$$

By Lemma 5 and eq. (6), and combining the monotonicity of K^{-1}, we derive

$$\begin{aligned} H(G_{m\lambda}, S) &= K^{-1}\left(\int_{\Delta\cup(\Omega-\Delta)} K(D\ (Z_n(G_{m\lambda}(x)), Z_n(S(x))))\mathrm{d}\mu \right) \\ &= K^{-1}\left(\int_{\Delta} K(D\ (Z_n(G_{m\lambda}(x)), Z_n(S(x))))\mathrm{d}\mu + \int_{\Omega-\Delta} K(D\ (Z_n(G_{m\lambda}(x)), Z_n(S(x))))\mathrm{d}\mu \right) \\ &\leqslant K^{-1}\left(\int_{\Delta} K(a\varepsilon)\mathrm{d}\mu + \varepsilon \right) = K^{-1}(K(a\varepsilon) \cdot \mu\ (\Delta) + \varepsilon). \end{aligned}$$

In fact, for any $\forall \varepsilon > 0$, as K is strictly increasing, hence $\mu(\Delta) \leqslant \mu(\Omega) < +\infty$. Consequently, the expression $K(a\varepsilon) \cdot \mu(\Delta) + \varepsilon$ is an infinitesimal quantity. Furthermore, since K^{-1} is strictly increasing, which implies that the expression $K^{-1}(K(a\varepsilon) \cdot \mu(\Delta) + \varepsilon)$ remains arbitrary small. Synthesizing the above discussion, we argue that the four-layer regular fuzzy neural network $\Im_0[\sigma]$ possesses a universal approximation for the class of fuzzy-valued simple functions with respect to the K-integral norm H.

Theorem 6.2. Let $(\mathbb{R}^d, \Re, \hat{\mu})$ be a finite space of K-quasi-additive measure, σ a Tauber-Wiener function, and K an induced operator, for given $n \in \mathbb{N}$ and arbitrary $F \in L^1(\hat{\mu})$, then $\Im_0[\sigma]$ can approximate F to arbitrary accuracy with respect to K-integral norm H.

Proof. In accordance with Theorem 5.1, we know that a fuzzy-valued simple function can approximate F to arbitrary accuracy with respect to K-integral norm. This means that there exists $S_0 \in S_n(\Omega)$ for any $\varepsilon > 0$ and $F \in L^1(\hat{\mu})$, such that $H(S_0, F) < \varepsilon$.

Using Theorem 6.1, the polygonal fuzzy neural network $\Im_0[\sigma]$ possesses a universal approximation with respect to $S_n(\Omega)$, i.e., for the above-given $S_0 \in S_n(\Omega)$, there exists a polygonal fuzzy-valued function $G_{m\lambda} \in \Im_0[\sigma]$ such that $H(G_{m\lambda}, S_0) < \varepsilon$.

Applying Theorem 5.4 and using the three-point inequality for the integral norm H, it is straightforward to see that

$$H(G_{m\lambda}, F) \leqslant H(G_{m\lambda}, S_0) \oplus H(S_0, F) < \varepsilon \oplus \varepsilon = K^{-1}(2K(\varepsilon)).$$

Actually, for all $\varepsilon > 0$, as K^{-1} and K are strictly increasing, we argue that the expression $K^{-1}(2K(\varepsilon))$ remains arbitrary small. Therefore, for every $F \in L^1(\hat{\mu})$, the polygonal fuzzy neural network $\Im_0[\sigma]$ can approximate F to arbitrary accuracy with respect to K-integral norm H.

7 Conclusions

It is well known that operations between general fuzzy numbers are not simply linear, but depends on Zadeh's complex extension principle. Thus, studies of the applications of fuzzy numbers are very difficult, even operations for the most simplest trigonometric or ladder fuzzy numbers do not possess closeness. The big question is: how can one realize these nonlinear operations between fuzzy numbers? Solving this problem has important significance in constructing a suitable fuzzy neural network that approximates a given nonlinear function, and in studying learning algorithm, fuzzy inference, and fuzzy information processing. In this context, The polygonal fuzzy number was presented in [7]. It overcomes the above shortcomings by simplifying the extension principle, and consequently, such numbers were able to replace traditional ones.

Moreover, a polygonal fuzzy neural network has the following merits: 1) it extends the scope over which fuzzy valued functions had been approximated in the past, that is to say, extends it to $\hat{\mu}$-integrable bounded fuzzy-valued functions; 2) it is similar to handling trigonometric fuzzy number information, and their learning algorithms can be easily designed; 3) compared with traditional fuzzy neural networks, its input-output capability is more stronger; and 4) its approximation capability has been improved. In fact, the K-quasi-additive integral is a generalization of a traditional the Lebesgue integral. In addition, a polygonal fuzzy neural network is far superior to traditional neural networks, through use of this kind of integral to define the K-integral norm and the approximation afforded by adopting polygonal fuzzy neural networks. All of these aspects undoubtedly generalize [11, 12]. Consequently, we shall be continuing and developing Liu Puyin's work [6–10]. Systems involving the class of integrable functions are pervasive in research work, therefore, continued study of the approximation capability of fuzzy neural networks to various fuzzy integrable functions will have important significance in theory as well as applications.

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