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Geometric construction of energy-minimizing Bézier curves

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Abstract Modeling energy-minimizing curves have many applications and are a basic problem of Geometric Modeling. In this paper, we propose the method for geometric design of energy-minimizing Bézier curves. Firstly, the necessary and sufficient condition on the control points for Bézier curves to have minimal internal energy is derived. Based on this condition, we propose the geometric constructions of three kinds of Bézier curves with minimal internal energy including stretch energy, strain energy and jerk energy. Given some control points, the other control points can be determined as the linear combination of the given control points. We compare the three kinds of energy-minimizing Bézier curves via curvature combs and curvature plots, and present the collinear properties of quartic energy-minimizing Bézier curves. We also compare the proposed method with previous methods on efficiency and accuracy. Finally, several applications of the curve generation technique, such as curve interpolation with geometric constraints and modeling of circle-like curves are discussed.

Keywords curve design, minimal energy, geometric construction, interpolation with constraints, circle-like curves

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1 Introduction

Computing energy minimizing curves or variational curves are fundamental problems of Geometric Computing. They have been employed in many areas such as fairing curves, motion design, computer vision and image processing. For example, in automotive industry, the car body often consists of fair curves with minimal curvature energy [1], which will be considered in this paper. After Holladay introduced the cubic spline for function interpolation and integration in 1957, the modeling of energy minimizing curves has been a hot point of Computer Aided Design. However, most of the previous work employed iteration numerical method to obtain energy-minimizing curves, and the geometric construction of energy-minimizing curves has not been treated before. In particular, given some control points, how to construct other control points such that the resulting Bézier curve has minimal energy, is also a novel topic. In this paper, we will solve the above two problems and present some applications of the curve modeling technique.

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1.1 Related work

There is a huge body of literature on the energy-minimizing curves in CAD. Some work on this topic has been surveyed in [2]. Ferguson [3] introduces the parametric cubic spline curve by applying cubic spline function interpolation to vector valued data. Nielson [4] introduces the ν-spline as a C^1 cubic polynomial alternative to splines in tension. Hagen [5] introduces the τ-spline as a generalization of the ν-spline, which are both energy-minimizing splines. Nielson [6] presents a rational spline whose segments are planar with zero curvature at end points. Splines that minimize the jerk energy are proposed by Merier and Nowachi [7]. Higashi et al. [8] present construction of cubic Bézier curves with smoothly varying curvature specified by position, tangent and curvature. Rando [9] proposes the calculation of Bézier curve minimizing the variation of the radius of curvature. The condition for cubic Bézier curves with positive curvature is proposed by Roulier [10]. Hagen and Bonneau [11] propose a variational approach for designing the weights of a rational curve to achieve a smooth curve with minimal energy integral. Jou and Han [12, 13] investigate the minimal energy splines with various constraints. The construction of curves of minimal energy with prescribed length is discussed in [14]. Nielson [15] introduces the minimum norm network (MNN) by minimizing a linear approximation of strain energy and proposes an interactive design system based on MNN. Moreton and Séquin [16, 17] present a nonlinear algorithm for curves and networks with minimal variation of curvature. The problem of interpolating a sequence of points in the plane by a nonlinear spline curve of minimal energy with prescribed tangents in the endpoints is addressed in [18, 19]. The algorithm for fairing cubic spline curves by minimizing the strain energy is proposed by Zhang et al. [1]. Wang et al. [20] study the B-spline interproximation with different energy forms and parametrization techniques. Brunnett [21] proposes two variational models of fair curves for motion planning. The geometric Hermite curves with minimal strain energy are investigated by Yong and Cheng [22]. Wesselink and Veltkamp [23, 24] propose the construction of constrained variational curves using external energy operators. Based on the data dependent energy representation, Greiner et al. [25–27] propose an improved modeling method of variational curves and surfaces.

From this short review, most of the previous work uses iteration numerical method to generate energy-minimizing curves or curve networks, and there is less work on the geometric construction of energy-minimizing curves.

1.2 Contributions and overview

This paper gives the following contributions:

• By considering the energy as a function of control points, we derive the necessary and sufficient condition on the control points for Bézier curves to have minimal energy.

• Based on the condition, the geometric construction of energy-minimizing curves is proposed. Given the specified control points, the other control points can be determined as the linear combination of the given control points.

• The applications in curve interpolation and modeling of circle-like curves are discussed.

The rest of this paper is arranged as follows. After reviewing the three kinds of internal energy in section 2, the necessary and sufficient condition on the control points to minimize the internal energy of Bézier curves is derived in section 3. In this section, the geometric constructions of three kinds of energy-minimizing Bézier curves are also discussed. In section 4, we compare the three kinds of energy-minimizing Bézier curves, give a comparison with previous methods and present their applications in curve interpolation and modeling of circle-like curves. Finally, conclusions and future work are presented in section 5.

2 Internal energy of curves

The well known examples of internal energy functions are the stretch energy, the strain energy and the jerk energy, which typically depend on geometric information of the planar curve such as the length, curvature and variation of curvature.

The stretch energy measures the length of a curve. It is given by

$$E_{\text{stretch}}(\boldsymbol{p}) = \int_a^b \parallel \boldsymbol{p}'(t) \parallel dt, \quad t \in [a, b].$$

But it is too complicated for most practical applications. The following approximation is often used [23]:

$$\widetilde{E}_{\text{stretch}}(\boldsymbol{p}) = \int_a^b \parallel \boldsymbol{p}'(t) \parallel^2 dt. \tag{1}$$

The strain (or bend) energy measures how much the curve $\boldsymbol{p}(t)$ is bent. It is defined as

$$E_{\text{strain}}(\boldsymbol{p}) = \int_a^b k^2(t) \parallel \boldsymbol{p}'(t) \parallel dt, \tag{2}$$

where $k(t)$ is the curvature of $\boldsymbol{p}(t)$. Assuming that the curve has a nearly arc length parametrization, the following approximation is widely used [1, 23, 25]:

$$\widetilde{E}_{\text{strain}}(\boldsymbol{p}) = \int_a^b \parallel \boldsymbol{p}^{(2)}(t) \parallel^2 dt. \tag{3}$$

It should be mentioned that cubic spline curves has minimal strain energy (3) as presented in [1]. In this paper, we focus on Bézier curves and try to get the geometric construction of unknown control points from given control points such that the resulted Bézier curve has minimal energy.

The curvature variation energy is defined as follows [16]:

$$E_{cv}(\boldsymbol{p}) = \int_0^l [\boldsymbol{k}'(s)]^2 ds, \tag{4}$$

where s is the arc parameter, l is the arc length of $\boldsymbol{p}(t)$, and $\boldsymbol{k}(s)$ is the curvature of $\boldsymbol{p}(t)$. The above nonlinear energy function can be approximated by jerk energy, which is introduced by Meier and Nowacki [7]. It is defined in terms of the physical quantity called jerk, which has the following form

$$E_{\text{jerk}}(\boldsymbol{p}) = \int_a^b \parallel \boldsymbol{p}^{(3)}(t) \parallel^2 dt. \tag{5}$$

From (1), (3), (5), the above three kinds of energy can be unified as the L^2 norm of mth derivative of $\boldsymbol{p}(t)$,

$$E_m(\boldsymbol{p}) = \int_a^b \parallel \boldsymbol{p}^{(m)}(t) \parallel^2 dt, \qquad m = 1, 2, 3. \tag{6}$$

In this paper, we assume that $a = 0, b = 1$ for Bézier curves.

3 Geometric construction of energy-minimizing Bézier curves

In this section, we will study the geometric construction of Bézier curves with minimal internal energies discussed in the above section.

Let $\mathcal{C} = \{\boldsymbol{P}_i\}_{i=0}^n$ be the control polygon of a Bézier curve and let

$$\boldsymbol{p}(t) = \sum_{i=0}^n B_i^n(t)\boldsymbol{P}_i, \quad t \in [0, 1]$$

be its associated Bézier curve. Let $\overline{\mathcal{C}} = \{\boldsymbol{P}_i\}_{i\in G}$ be the set of the given control points and $\underline{\mathcal{C}} = \{\boldsymbol{P}_i\}_{i\in D}, 0 \notin D, n \notin D$, be the set of the control points to be determined from the energy-minimizing condition. G and D denote the subscript sets of the control points in the sets $\overline{\mathcal{C}}$ and $\underline{\mathcal{C}}$ respectively, that is, $G \bigcup D = \{0, 1, 2, \ldots, n-1, n\}$. The main result in this paper will translate the condition a control polygon $\mathcal{C}$

minimizes the energy function into a system of linear equations in terms of the control points in $\underline{\mathcal{C}}$. The energy function reaches its extremal when its gradient with respect to the coordinates of control points in $\underline{\mathcal{C}}$ vanishes.

Theorem 3.1. The control points in $\underline{\mathcal{C}} = \{\boldsymbol{P}_i\}_{i\in D}$ minimize the energy function (4) with given control points in $\overline{\mathcal{C}} = \{\boldsymbol{P}_i\}_{i\in G}$ if and only if they satisfy

$$\sum_{j=0}^{n-m} N_{i,j}^{n,m} \Delta^m \boldsymbol{P}_j = \boldsymbol{0}, \qquad i \in D, i \neq 0, i \neq n, \tag{7}$$

where

$$N_{i,j}^{n,m} = \binom{n-m}{j} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\binom{n-m}{i-l}}{\binom{2n-2m}{i+j-l}}, \Delta^m \boldsymbol{P}_j = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \boldsymbol{P}_{j+k}.$$

Proof. The mth derivative of $\boldsymbol{p}(t)$ can be written as (see [21])

$$\boldsymbol{p}^{(m)}(t) = \frac{n!}{(n-m)!} \sum_{i=0}^{n} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} B_{i-l}^{n-m}(t) \boldsymbol{P}_i, = \frac{n!}{(n-m)!} \sum_{j=0}^{n-m} B_j^{n-m}(t) \Delta^m \boldsymbol{P}_j,$$

where $\Delta^m \boldsymbol{P}_j = \sum\limits_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \boldsymbol{P}_{j+k}$.

In the following, the gradients of the energy functional with respect to the coordinates of a control point $\boldsymbol{P}_i = (x_i, y_i) \in \underline{\mathcal{C}}$ will be computed. Then, we get

$$\begin{aligned}
\frac{\partial E_m(\boldsymbol{p})}{\partial x_i} &= 2 \int_0^1 \left\langle \frac{\partial \boldsymbol{p}^{(m)}(t)}{\partial x_i}, \boldsymbol{p}^{(m)}(t) \right\rangle dt \\
&= 2 \int_0^1 \left\langle \frac{n!}{(n-m)!} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} B_{i-l}^{n-m}(t) \boldsymbol{q}^x, \boldsymbol{p}^{(m)}(t) \right\rangle dt \\
&= 2 \int_0^1 \frac{n!}{(n-m)!} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} B_{i-l}^{n-m}(t) \left\langle \boldsymbol{q}^x, \frac{n!}{(n-m)!} \sum_{j=0}^{n-m} B_j^{n-m}(t) \Delta^m \boldsymbol{P}_j \right\rangle dt,
\end{aligned}$$

where $\boldsymbol{q}^x = (1, 0)$.

From the following identity of Bernstein polynomials:

$$B_i^n(t) B_j^m(t) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{m+n}{i+j}} B_{i+j}^{m+n}(t), \qquad \int_0^1 B_i^n(t) dt = \frac{1}{n+1},$$

we have

$$\frac{\partial E_m(\boldsymbol{p})}{\partial x_i} = K_{m,n} \sum_{j=0}^{n-m} \binom{n-m}{j} \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} \frac{\binom{n-m}{i-l}}{\binom{2n-2m}{i+j-l}} (\Delta^m \boldsymbol{P}_j)_x = K_{m,n} \sum_{j=0}^{n-m} N_{i,j}^{n,m} (\Delta^m \boldsymbol{P}_j)_x,$$

where $K_{m,n} = \dfrac{2(n!)^2}{[(n-m)!]^2(2n-2m+1)}$, $N_{i,j}^{n,m}$ is defined in (7), and $(\Delta^m \boldsymbol{P}_j)_x$ denotes the x-component of the vector $\Delta^m \boldsymbol{P}_j$.

Analogously, we get the following gradient:

$$\frac{\partial E_m(\boldsymbol{p})}{\partial y_i} = K_{m,n} \sum_{j=0}^{n-m} N_{i,j}^{n,m} (\Delta^m \boldsymbol{P}_j)_y,$$

where $(\Delta^m \boldsymbol{P}_j)_y$ denotes the y-component of the vector $\Delta^m \boldsymbol{P}_j$.

The condition for the extremals of energy functional is $\frac{\partial E_m(\boldsymbol{p})}{\partial x_i} = \frac{\partial E_m(\boldsymbol{p})}{\partial y_i} = 0$. Thus, eq. (7) is proved.

From Theorem 3.1, we can obtain the unknown control points $\boldsymbol{P}_i \in \underline{\mathcal{C}}$ via solving the system of linear equations (7). In fact, the unknown control points can be formulated as the the linear combination of the given control points.

Remark 3.1. When $\boldsymbol{p}(t)$ can degenerate to a curve of degree $m-1$, that is, $\boldsymbol{p}^{(m)}(t) = 0$, obviously it satisfies condition (7). In fact, in some cases, the resulting energy-minimizing curves are degenerated curves (see subsection 3.3).

Remark 3.2. For the case with high degree, efficient computation of binomial coefficient in (7) is a key issue. The iteration method in [28] can be used for computing binomial coefficient. It is efficient and has numerical stability.

3.1 Bézier curves with minimal stretch energy

When $m = 1$, $E_m(\boldsymbol{p})$ is stretch energy. After simple computation from Theorem 3.1, we obtain the following proposition.

Proposition 3.1. Given the control points $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{i-1}, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_n$, the control point $\boldsymbol{P}_i$ to be determined can be constructed as follows: select $\boldsymbol{S}$ on the straight line $\boldsymbol{P}_{i-1}\boldsymbol{P}_{i+1}$ such that $\frac{\|\boldsymbol{P}_{i-1}\boldsymbol{S}\|}{\|\boldsymbol{S}\boldsymbol{P}_{i+1}\|} = -\frac{N_{i,i}^{n,1}}{N_{i,i-1}^{n,1}}$, and move $\boldsymbol{S}$ to $\boldsymbol{P}_i$ such that $\boldsymbol{S}\boldsymbol{P}_i = -\frac{\sum_{j=0,\, j\neq i-1,i}^{n-1} N_{i,j}^{n,1}(\boldsymbol{P}_{j+1}-\boldsymbol{P}_j)}{N_{i,i-1}^{n,1}-N_{i,i}^{n,1}}$. Then the corresponding Bézier curve of degree n has minimal stretch energy.

In the case $n = 2$, we have

$$\boldsymbol{P}_1 = \frac{1}{2}(\boldsymbol{P}_0 + \boldsymbol{P}_2).$$

In particular, given the two end points, the resulting Bézier curves from Proposition 3.1 are straight lines. In fact, for the other two energies discussed in the following two subsections, we have the same results. Hence, in the remainder of this paper, we only discuss the cases of Bézier curves of higher degree with more than two given control points. For simplicity, we only present the results of case $n = 3, 4$.

In the case $n = 3$, we have

$$\boldsymbol{P}_1 = \frac{1}{4}(3\boldsymbol{P}_0 + \boldsymbol{P}_2) + \frac{1}{2}(\boldsymbol{P}_3 - \boldsymbol{P}_2), \quad \boldsymbol{P}_2 = \frac{1}{4}(3\boldsymbol{P}_3 + \boldsymbol{P}_1) + \frac{1}{2}(\boldsymbol{P}_0 - \boldsymbol{P}_1).$$

Obviously, the construction of $\boldsymbol{P}_2$ is symmetric with the construction of $\boldsymbol{P}_1$.

In the case $n = 4$, we get

$$\boldsymbol{P}_1 = \frac{1}{6}(5\boldsymbol{P}_0 + \boldsymbol{P}_2) + \frac{5}{12}(\boldsymbol{P}_3 - \boldsymbol{P}_2) + \frac{1}{4}(\boldsymbol{P}_4 - \boldsymbol{P}_3), \quad (8)$$

$$\boldsymbol{P}_2 = \frac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{P}_3) + (\boldsymbol{P}_0 - \boldsymbol{P}_1) + (\boldsymbol{P}_4 - \boldsymbol{P}_3), \quad (9)$$

$$\boldsymbol{P}_3 = \frac{1}{6}(5\boldsymbol{P}_4 + \boldsymbol{P}_2) + \frac{5}{12}(\boldsymbol{P}_1 - \boldsymbol{P}_2) + \frac{1}{4}(\boldsymbol{P}_0 - \boldsymbol{P}_1).$$

Figure 1(a) illustrates the geometric construction of $\boldsymbol{P}_2$. In this paper, we assume that the circles are the given control points, and the stars are the unknown control points.

Corollary 3.1. Given the control points $\boldsymbol{P}_0, \boldsymbol{P}_3$ and $\boldsymbol{P}_4$, the unknown control point $\boldsymbol{P}_1$ divides the line segment $\boldsymbol{P}_0\boldsymbol{P}_3$ in the ratio 1 : 2, and the other unknown control point $\boldsymbol{P}_2$ satisfies $\boldsymbol{P}_1\boldsymbol{P}_2 = \boldsymbol{P}_3\boldsymbol{P}_4$. Then the corresponding quartic Bézier curve has minimal stretch energy.

Proof. Combining (8) and (9), we have

$$\boldsymbol{P}_1 = \frac{2}{3}\boldsymbol{P}_0 + \frac{1}{3}\boldsymbol{P}_3,$$

$$\boldsymbol{P}_2 = \frac{1}{3}(2\boldsymbol{P}_0 - 2\boldsymbol{P}_3 + 3\boldsymbol{P}_4) = \frac{2}{3}\boldsymbol{P}_0 + \frac{1}{3}\boldsymbol{P}_3 + (\boldsymbol{P}_4 - \boldsymbol{P}_3).$$

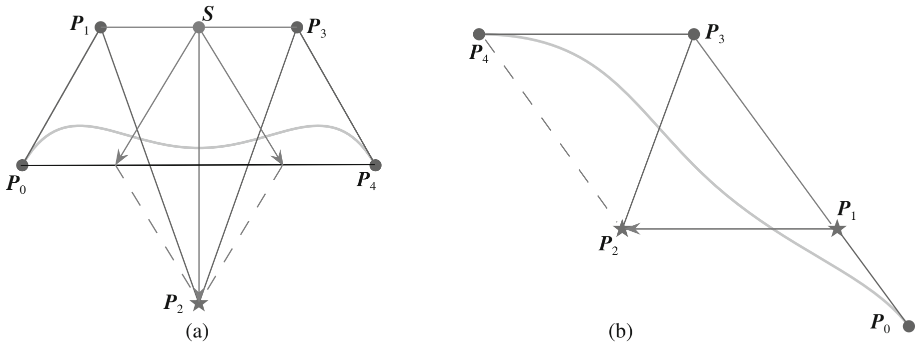


Figure 1 Geometric construction of quadratic Bézier curve with minimal stretch energy. (a) $\boldsymbol{P}_2$ is unknown; (b) $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are unknown.

Thus, the geometric construction is obtained, see Figure 1(b).

Similarly, if the control points $\boldsymbol{P}_0, \boldsymbol{P}_3$ and $\boldsymbol{P}_4$ are given, the control points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ to be determined can be constructed symmetrically.

3.2 Bézier curves with minimal strain energy

In the case $m = 2$, $E_m(\boldsymbol{p})$ in (6) is strain energy. We have the following proposition.

Proposition 3.2. Given the control points $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{i-1}, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_n$, the unknown control point $\boldsymbol{P}_i$ can be constructed as follows: select $\boldsymbol{S}$ on the straight line $\boldsymbol{P}_{i-1}\boldsymbol{P}_{i+1}$ such that $\frac{\|\boldsymbol{P}_{i-1}\boldsymbol{S}\|}{\|\boldsymbol{S}\boldsymbol{P}_{i+1}\|} = \frac{N^{n,2}_{i,i-2}-N^{n,2}_{i,i-1}}{N^{n,2}_{i,i}-N^{n,2}_{i,i-1}}$, and move $\boldsymbol{S}$ to $\boldsymbol{P}_i$ such that $\boldsymbol{S}\boldsymbol{P}_i = \frac{1}{N^{n,2}_{i,i}-2N^{n,2}_{i,i-1}+N^{n,2}_{i,i-2}}[N^{n,2}_{i,i}\ (\boldsymbol{P}_{i+1} - \boldsymbol{P}_{i+2}) + N^{n,2}_{i,i-2}(\boldsymbol{P}_{i-1} - \boldsymbol{P}_{i-2}) - \sum^{n-2}_{\substack{j=0\\ j\neq i-2,i-1,i}} N^{n,2}_{i,j}\Delta^2\boldsymbol{P}_j]$. Then the corresponding Bézier curve of degree n has minimal strain energy.

In the case $n = 3$, we have

$$\boldsymbol{P}_1 = \frac{1}{2}(\boldsymbol{P}_0 + \boldsymbol{P}_2), \quad \boldsymbol{P}_2 = \frac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{P}_3).$$

Hence, the unknown control point $\boldsymbol{P}_1$ is independent of $\boldsymbol{P}_3$; the unknown control point $\boldsymbol{P}_1$ is independent of $\boldsymbol{P}_0$.

In the case $n = 4$, we get

$$\begin{aligned}
\boldsymbol{P}_1 &= \frac{1}{16}(9\boldsymbol{P}_0 + 7\boldsymbol{P}_2) + \frac{3}{16}(\boldsymbol{P}_3 - \boldsymbol{P}_2) + \frac{1}{16}(\boldsymbol{P}_3 - \boldsymbol{P}_4),\\
\boldsymbol{P}_2 &= \frac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{P}_3) + \frac{1}{6}(\boldsymbol{P}_1 - \boldsymbol{P}_0) + \frac{1}{6}(\boldsymbol{P}_3 - \boldsymbol{P}_4),\\
\boldsymbol{P}_3 &= \frac{1}{16}(9\boldsymbol{P}_4 + 7\boldsymbol{P}_2) + \frac{3}{16}(\boldsymbol{P}_1 - \boldsymbol{P}_2) + \frac{1}{16}(\boldsymbol{P}_1 - \boldsymbol{P}_0).
\end{aligned}$$

Figure 2(a) illustrates the geometric construction method of $\boldsymbol{P}_2$. Note that the resulting Bézier curve is degenerated into a cubic polynomial curve, which is consistent with the result presented in [1]; that is, a cubic Hermite curve has the minimum strain energy among all C^1 cubic polynomial spline curves satisfying the same endpoint conditions.

Corollary 3.2 Given the control points $\boldsymbol{P}_0, \boldsymbol{P}_3$ and $\boldsymbol{P}_4$, the unknown control points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can be constructed as follows: select $\boldsymbol{S}$ on the straight line $\boldsymbol{P}_0\boldsymbol{P}_4$ such that $\frac{\|\boldsymbol{P}_0\boldsymbol{S}\|}{\|\boldsymbol{S}\boldsymbol{P}_4\|} = 3$, and move $\boldsymbol{S}$ to $\boldsymbol{P}_2$ such that $\boldsymbol{S}\boldsymbol{P}_2 = \boldsymbol{P}_3\boldsymbol{P}_4$; the other unknown control point $\boldsymbol{P}_1$ is the middle point of line segment $\boldsymbol{P}_0\boldsymbol{P}_2$. Then the corresponding quartic Bézier curve has minimal strain energy (see Figure 2(b)).

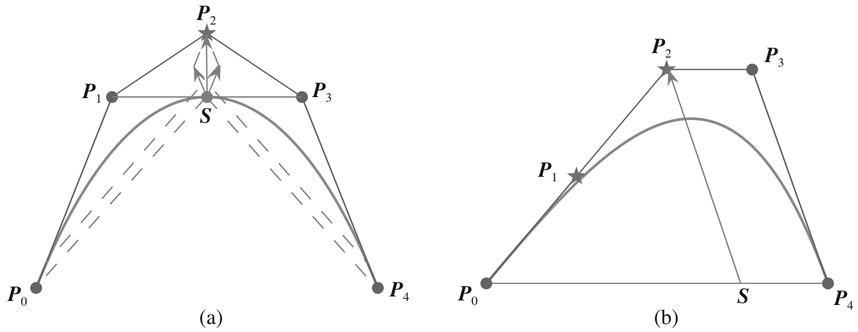


Figure 2 Geometric construction of quartic Bézier curve with minimal strain energy. (a) $\boldsymbol{P}_2$ is unknown; (b) $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are unknown.

3.3 Bézier curves with minimal jerk energy

In the case $m = 3$, $E_m(\boldsymbol{p})$ in (6) is jerk energy. We have the following proposition.

Proposition 3.3. Given the control points $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{i-1}, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_n$, the unknown control point $\boldsymbol{P}_i$ can be constructed as follows: select $\boldsymbol{S}$ on the straight line $\boldsymbol{P}_{i-1}\boldsymbol{P}_{i+1}$ such that $\frac{\|\boldsymbol{P}_{i-1}\boldsymbol{S}\|}{\|\boldsymbol{S}\boldsymbol{P}_{i+1}\|} = \frac{3N_{i,i-1}^{n,3} - N_{i,i}^{n,3}}{N_{i,i-3}^{n,3} - 3N_{i,i-2}^{n,3}}$, move $\boldsymbol{S}$ to $\boldsymbol{P}_i$ such that $\boldsymbol{S}\boldsymbol{P}_i = \frac{1}{N_{i,i-3}^{n,3} - 3N_{i,i-2}^{n,3} + 3N_{i,i-1}^{n,3} - N_{i,i}^{n,3}} [N_{i,i}^{n,3}(\boldsymbol{P}_{i+2} - \boldsymbol{P}_{i+3}) + 2N_{i,i}^{n,3}(\boldsymbol{P}_{i+2} - \boldsymbol{P}_{i+1}) + N_{i,i-1}^{n,3}(\boldsymbol{P}_{i-1} - \boldsymbol{P}_{i+2}) + N_{i,i-2}^{n,3}(\boldsymbol{P}_{i-2} - \boldsymbol{P}_{i+1}) + 2N_{i,i-3}^{n,3}(\boldsymbol{P}_{i-1} - \boldsymbol{P}_{i-2}) + N_{i,i-3}^{n,3}(\boldsymbol{P}_{i-3} - \boldsymbol{P}_{i-2}) - \sum_{\substack{j=0 \\ j \neq i-3, i-2, i-1, i}}^{n-3} N_{i,j}^{n,3} \Delta^3 \boldsymbol{P}_j]$. Then the corresponding Bézier curve of degree n has minimal jerk energy.

In the case $n = 3$, for $i = 1$ and $i = 2$, the same result is obtained:

$$\Delta^3 \boldsymbol{P}_0 = 0. \tag{10}$$

For $i = 1$, we get

$$\boldsymbol{P}_1 = \frac{1}{3}(2\boldsymbol{P}_2 + \boldsymbol{P}_0) + \frac{1}{3}(\boldsymbol{P}_2 - \boldsymbol{P}_3).$$

By (10), the resulting curves can degenerate into quadratic Bézier curves. This is consistent with Remark 3.1.

When $n = 4$, we have

$$\begin{aligned} \boldsymbol{P}_1 &= \frac{1}{7}(5\boldsymbol{P}_2 + 2\boldsymbol{P}_0) + \frac{1}{14}(\boldsymbol{P}_0 - \boldsymbol{P}_3) + \frac{1}{7}(\boldsymbol{P}_2 - \boldsymbol{P}_3) + \frac{1}{14}(\boldsymbol{P}_3 - \boldsymbol{P}_4), \\ \boldsymbol{P}_2 &= \frac{1}{2}(\boldsymbol{P}_1 + \boldsymbol{P}_3) + \frac{1}{6}(\boldsymbol{P}_1 - \boldsymbol{P}_0) + \frac{1}{6}(\boldsymbol{P}_3 - \boldsymbol{P}_4), \\ \boldsymbol{P}_3 &= \frac{1}{7}(5\boldsymbol{P}_2 + 2\boldsymbol{P}_4) + \frac{1}{14}(\boldsymbol{P}_4 - \boldsymbol{P}_1) + \frac{1}{7}(\boldsymbol{P}_2 - \boldsymbol{P}_1) + \frac{1}{14}(\boldsymbol{P}_1 - \boldsymbol{P}_0). \end{aligned} \tag{11}$$

Figure 3 (a) shows the geometric construction of $\boldsymbol{P}_3$.

Corollary 3.3. Given the control points $\boldsymbol{P}_0, \boldsymbol{P}_3$ and $\boldsymbol{P}_4$, the unknown control points $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can be constructed as follows: let $\boldsymbol{S}$ be the middle point of straight line $\boldsymbol{P}_0\boldsymbol{P}_4$, and move $\boldsymbol{S}$ to $\boldsymbol{P}_1$ such that $\boldsymbol{S}\boldsymbol{P}_1 = \boldsymbol{P}_4\boldsymbol{P}_3$; then move $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$ such that $\boldsymbol{P}_1\boldsymbol{P}_2 = \frac{1}{3}\boldsymbol{P}_0\boldsymbol{P}_3$. Then the corresponding quartic Bézier curve has minimal jerk energy (see Figure 3(b)).

4 Comparison and application

In this section, we will compare the three kinds of Bézier curves with different minimal energy, give a comparison with previous methods, and present several applications of the curve generation method.

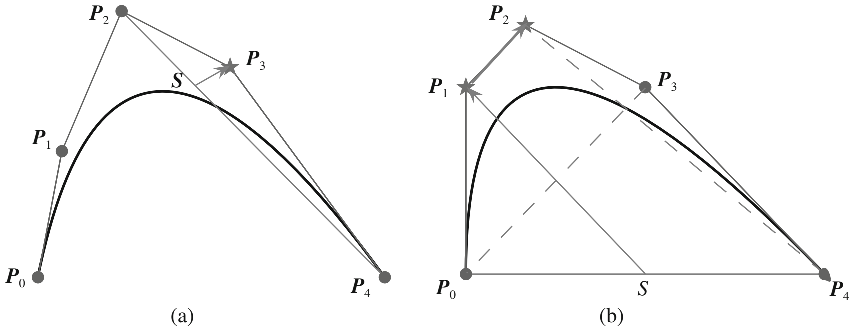


Figure 3 Geometric construction of quartic Bézier curve with minimal jerk energy. (a) $\boldsymbol{P}_3$ is unknown; (b) $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are unknown.

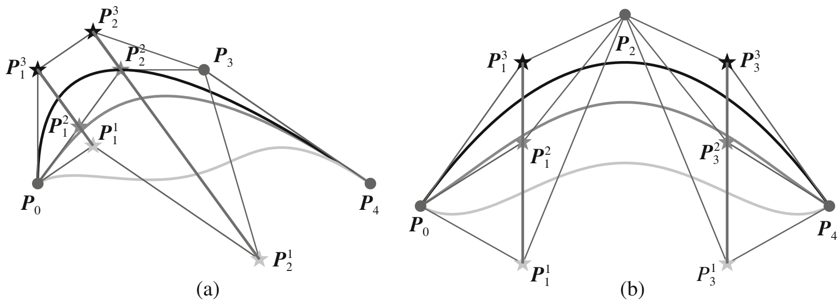


Figure 4 Comparison of three kinds of energy-minimizing quadratic Bézier curves, the curve with $\boldsymbol{P}_1^1$ and $\boldsymbol{P}_2^1$ is the stretch energy-minimizing curve, the curve with $\boldsymbol{P}_1^2$ and $\boldsymbol{P}_2^2$ is the strain energy-minimizing curve, the curve with $\boldsymbol{P}_1^3$ and $\boldsymbol{P}_2^3$ is the jerk energy-minimizing curve. (a) $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are unknown; (b) $\boldsymbol{P}_1$ and $\boldsymbol{P}_3$ are unknown.

4.1 Comparison of Bézier curves with different minimal energy

The three kinds of energy discussed in this paper have different geometric interpretation. The stretch energy is related to the length of the curve, bend energy is related to the curvature, and jerk energy is an approximation to the variation of curvature. We present the three kinds of energy-minimizing curves with the same given control points in Figure 4. In fact, for quartic Bézier curves, we have the following proposition.

Proposition 4.1. For $n=4$, given the control points $\boldsymbol{P}_0, \boldsymbol{P}_3$ and $\boldsymbol{P}_4$, $\boldsymbol{P}_1^m$ and $\boldsymbol{P}_2^m$ with respect to different internal energy are collinear, and $\frac{\|\boldsymbol{P}_1^1\boldsymbol{P}_1^2\|}{\|\boldsymbol{P}_1^2\boldsymbol{P}_1^3\|}=\frac{1}{3}$, $\frac{\|\boldsymbol{P}_2^1\boldsymbol{P}_2^2\|}{\|\boldsymbol{P}_2^2\boldsymbol{P}_2^3\|}=5$.

Proposition 4.2. For $n=4$, given the control points $\boldsymbol{P}_0, \boldsymbol{P}_2$ and $\boldsymbol{P}_4$, $\boldsymbol{P}_1^m$ and $\boldsymbol{P}_3^m$ with respect to different internal energy are collinear, and $\frac{\|\boldsymbol{P}_1^1\boldsymbol{P}_1^2\|}{\|\boldsymbol{P}_1^2\boldsymbol{P}_1^3\|}=\frac{\|\boldsymbol{P}_3^1\boldsymbol{P}_3^2\|}{\|\boldsymbol{P}_3^2\boldsymbol{P}_3^3\|}=\frac{38}{25}$.

The above two propositions can be proved after simple computation. The collinear property is useful for curve design: when two kinds of energy-minimizing quartic Bézier curves are obtained, the remaining one can be constructed from Proposition 4.1 or Proposition 4.2.

Figure 5 analyzes the shapes of the energy-minimizing curves by curvature combs and curvature plots. The stretch-energy-minimizing curve has a spiky curvature plot with several peaks, and large curvature at the endpoints $\boldsymbol{P}_0$; the bend-energy-minimizing curve has a considerably smoother curvature plot, and zero curvature at $\boldsymbol{P}_0$; the jerk-energy-minimizing curve has a very smooth curvature plot, and constant curvature at $\boldsymbol{P}_0$.

4.2 Comparison with previous method

In this subsection, from the view of accuracy and efficiency, we will compare the proposed method with some previous work, such as the nonlinear method for minimal variation of curvature proposed by Moreton

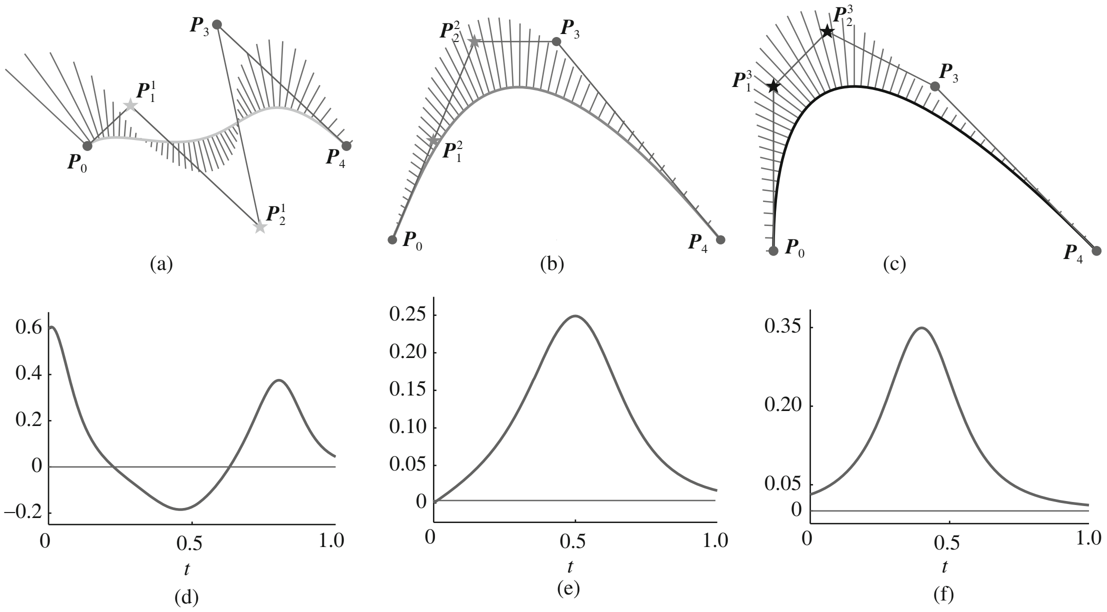


Figure 5 Shape analysis of the three kinds of energy-minimizing curves presented in Figure 4(a). (a)(d) Curvature comb and curvature plot of stretch energy-minimizing Bézier curves; (b)(e) curvature comb and curvature plot of strain energy-minimizing Bézier curves; (c)(f) curvature comb and curvature plot of jerk energy-minimizing Bézier curves.

Figure 6 Comparison with previous method. (a) Comparison with the method in [12] and the minimization result of exact curvature energy; (b) comparison with the method in [16] and the minimization result of exact curvature variation energy.

and Séquin [16], and the constraint optimization method for minimal curvature integral presented by Jou and Han [12].

Figure 6(a) compares the proposed method with the method in [12] and the minimization result of exact curvature energy in (2); Figure 6(b) presents the comparison with the method in [16] and the minimization result of exact curvature variation energy in (4). The medium control points are obtained by the proposed method, and the biggest control points are constructed by the method to be compared, and the smallest control points are obtained from the minimization of exact energy function by iteration method. Table 1 presents the corresponding energy values of curves in Figure 6. Table 1 shows that our method can efficiently generate better approximation to the minimization of exact energy function.

4.3 Applications

In this subsection, we will apply the curve generation technique in curve interpolation with constraints and modeling of circle-like curves.

Curve interpolation with constraints. In our method, the given control points often express the position, tangent and curvature information at the end points. Hence, they can be used for curve interpolation with geometric constraints. In traditional energy-minimizing curve interpolation problem,

Table 1 Comparison on both accuracy and efficiency

Term	Proposed method	Comparison method	Exact energy
Energy value in Figure 6(a)	23.486	25.372	22.585
Computational time in Figure 6(a)	0.22s	0.57s	0.93s
Energy value in Figure 6(b)	38.684	42.256	37.134
Computational time in Figure 6(b)	0.36s	0.79s	1.23s

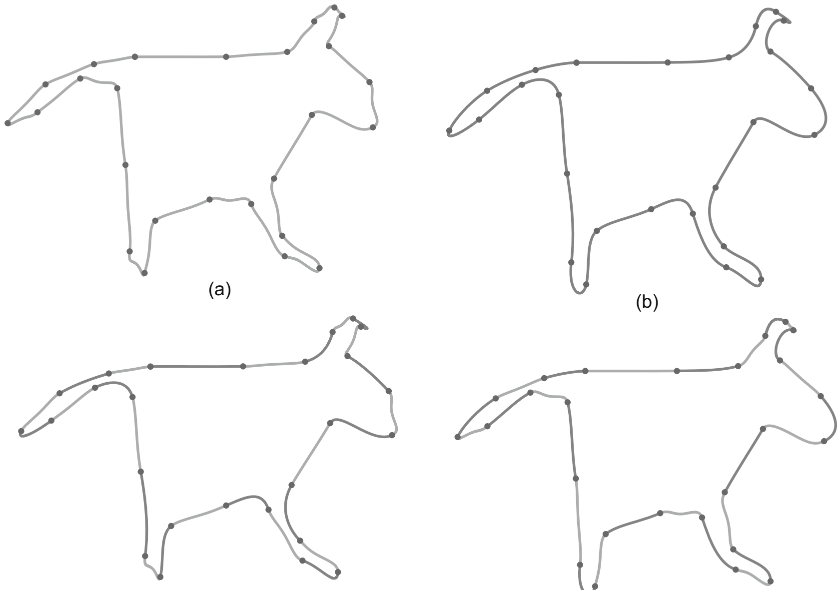


Figure 7 Curve interpolation with the same constraints. (a) Interpolating curves consisting of stretch-energy-minimizing segments; (b) interpolating curves consisting of strain-energy-minimizing segments; (c)(d) interpolating curves consisting of stretch-energy-minimizing segments and strain-energy-minimizing segments.

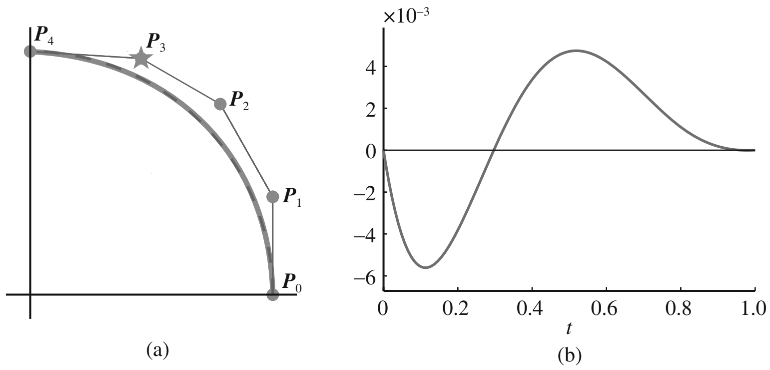
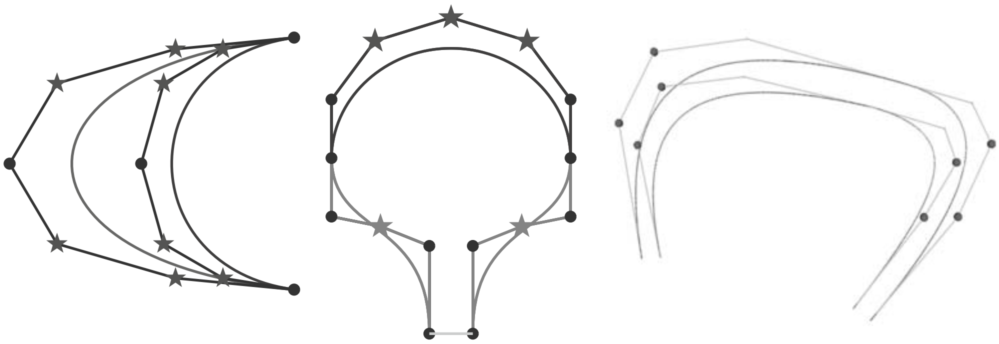


Figure 8 Approximation of circular arcs. (a) The quarter circle (dot line) and the approximation results using the jerk-energy-minimizing quartic Bézier curves (solid line); (b) error function in (a).

the energy of the entire interpolating curve is minimized. However, in some cases, we often require that different curve segments have different minimal energy.

Figure 7 shows different interpolating curves with the same geometric constraints. The curve segments in Figures 7(a) and 7(b) have the same minimizing energy, and the segments in Figures 7(c) and 7(d) have different minimizing energy. The curve segments between two data points (circles) are quartic Bézier curves with unknown control point $\boldsymbol{P}_2$. The given control points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_3$ and $\boldsymbol{P}_4$ of every curve segment guarantee that the whole curve is G^1 continuous.



(a) (b) (c)

Figure 9 Modeling examples by circle-like curves. (a) The moon consisting of two jerk energy-minimizing Bézier curves of degree six; (b) the table tennis racket consisting of three kinds of energy-minimizing curves; (c) highway design by using G^1 jerk energy-minimizing curves, where the circle control points are obtained from the energy-minimizing condition.

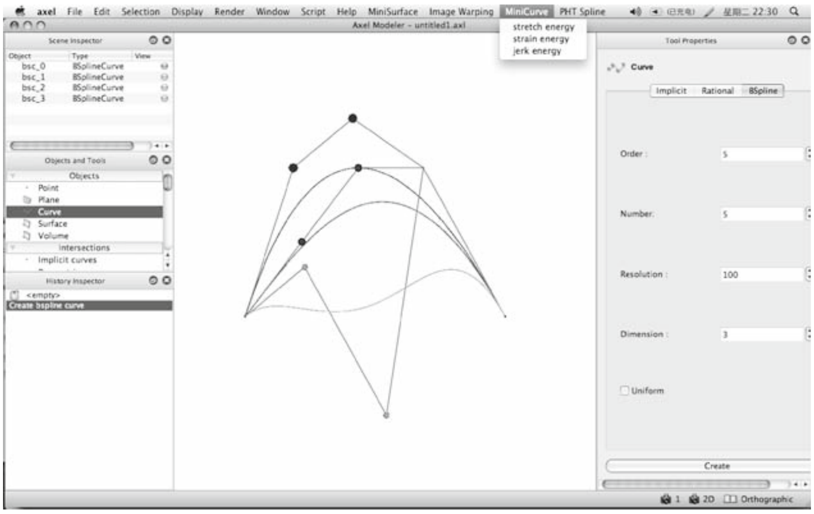


Figure 10 Implementation in AXEL.

Modeling of circle-like curves. Circular shapes is a desirable feature of curve construction in the aircraft and the automobile industries. The jerk-energy-minimizing curve approximately minimizes the variation of curvature, and it forms circle-like curves when constraints allow.

Kim and Ahn [29] proposed an approximation method of circular arcs by quartic Bézier curves. If we specify the control points $\boldsymbol{P}_0, \boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_4$ according to Kim and Ahn's method, then the resulting jerk-energy-minimizing Bézier curve can be seen as an approximation of circular arc (see Figure 8(a)). For the circular arc with angle $\alpha = \frac{\pi}{2}$ and radius 1, the Hausdorff distance between the jerk-energy-minimizing Bézier curve and the quarter circle is 5.61×10^{-3} (see Figure 8(b)). Though the result is not so good as Kim and Ahn's method, it demonstrates that we can use it for modeling of circle-like curves when constraints allow. Three modeling examples are shown in Figure 9.

5 Conclusion and future work

In this paper, the problem of geometric construction of energy-minimizing Bézier curves is addressed. Given some control points, the unknown control points can be obtained by geometric operation such that the resulted Bézier curve has minimal energy. Three kinds of internal energies are considered, including stretch energy, strain energy and jerk energy. Comparison and applications of the energy-minimizing Bézier curves are also presented. The curve modeling technique is simple and has clear geometric meanings, and has been implemented in AXEL as a plugin (see Figure 10).

The geometric construction of other kinds of curves such as C-B-spline curves [30] with minimal energy is a part of future work.

Acknowledgements

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