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Global practical tracking by output-feedback for nonlinear systems with unknown growth rate

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Abstract This paper is devoted to the global practical tracking by output-feedback for a class of uncertain nonlinear systems with only the tracking error measurable. Different from the closely related works, the systems have unmeasured states dependent growth with unknown constant rate, and the reference signal, as well as its first order derivative, has unknown bound. Mainly because of these, the tracking problem can hardly be solved by straightforwardly extending the existing results. In the paper, motivated by the related stabilization results, and flexibly using the ideas of universal control and dead zone, an adaptive output-feedback controller is designed to make the tracking error prescribed arbitrarily small after a finite time while keeping all the states of the closed-loop system bounded. A numerical example demonstrates the effectiveness of the theoretical results.

Keywords uncertain nonlinear systems, global practical tracking, adaptive control, output-feedback, dynamic high-gain

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1 Introduction

Output tracking control is an important subject in control theory and has been extensively investigated over the last decades [1–16]. Specifically, three types of output tracking are mostly encountered: asymptotic output tracking (see e.g. [1–3]), practical output tracking (or λ-tracking) (see e.g. [5, 8, 10–13]), and output tracking with prescribed transient behavior (see e.g. [14–16]). The first type generally requires the more restrictions on the systems and the reference signal, such as good models, to establish the desired asymptotic behavior of the tracking error. This would make many familiar systems inapplicable since the unmodeled dynamics and uncertainties inevitably exist in the practical control plants. As the degenerated cases, the second and third types are devoted to the specified sufficiently small range achieved for the tracking error, which is adequate for many practical applications. Besides, the latter two types need less conditions than the first type and, particularly, allowing the existence of many kinds of uncertainties and unknowns. Mainly because of these, the latter two types have attracted many attentions during the past three decades and is still an active area of research.

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Up to now, many classes of nonlinear systems have been considered to achieve the practical output tracking [4–9,11–13]. As the recent development, when all the states available for feedback, a series of research results have been obtained (see e.g. [6–9]), and in particular, [6] is devoted to a more general class of strict-feedback uncertain systems, and [7–9] are concerned with high-order uncertain nonlinear systems. When only partial states or output available for feedback, some representative results have been obtained in [11–13]. Specifically, [11] addressed adaptive λ-tracking of nonlinear systems with unknown control coefficient and the growth of polynomial-of-output multiplying an unknown constant, and proposed the backstepping design procedure. Different from [11], [12] further studied the systems with the dominating nonlinearities linearly composed of unmeasurable states with a factor of bounded function and presented a much simpler controller than [11]. Ref. [13], with less information on the reference signal than that in [11, 12], addressed the practical tracking for nonlinear systems with higher-order unmeasurable states dependent growth. It is worth mentioning that both in [11, 12], a dead zone is employed in the updating law to effectively restrain the bursting phenomenon, and moreover to directly establish the desired tracking objective.

This paper is devoted to the global practical tracking for a class of nonlinear systems which have unmeasured states dependent growth with unknown constant rate, and for the reference signal whose derivative and itself have bounds but unknown. To the best of authors' knowledge, the problem has remained open up to now and cannot be straightforwardly solved by the existing methods. This paper requires less information on the reference signal than [11–13]. In fact, in [11, 12], the reference signal is precisely known, and in [13], the bounds of the reference signal and its derivative are required to be known. In addition, the nonlinearities of the systems under investigation are more general than the dominating ones of [12], since the lower-order growing unmeasurable states are permitted in the former, whereas they are excluded in the latter, and also different from [11, 13] for in [11], only output dependent growth is admitted, and in [13], the growth rate is a known constant. Although with similar but somewhat stronger constraints for the systems, the asymptotic stabilization problem has been studied in [17, 18], our adequate investigations suggest that the tracking problem is extremely different from the stabilization one and is more difficult to solve. Mainly motivated by [12, 13, 18], by combining the idea of universal control with dynamic high-gain observer, the aforementioned practical tracking problem has been successfully solved.

The main contributions of the paper are composed of two parts. First, a dynamic-high-gain observer is introduced to reconstruct the system states. The novel updating law of high-gain, which is obtained by using the idea of dead zone, can effectively deal with the uncertainties in the system nonlinearities and reference signal. Based on the dynamic high-gain and the observer, an adaptive output-feedback controller is explicitly designed to make the tracking error prescribed arbitrarily small after a finite time while keeping all the states of the resulting closed-loop system bounded. The proposed output-feedback controller is simple in structure and can easily be implemented in practice. Second, different from the common patterns in the performance analysis, a new one is proposed to successfully overcome the significant technical difficulties caused by the novel updating law for the high-gain. This pattern also offers a deeper understanding on the theoretical results and is no doubt helpful to tackling practical tracking problems of more general nonlinear systems. The remainder of the paper is organized as follows. Section 2 presents the system model and the control objective. Section 3 develops the corresponding globally practical tracking control design scheme and summarizes the main results. In section 4, a numerical example is provided to illustrate the correctness of the theoretical results. Some concluding remarks are given in section 5. The paper ends with an appendix which is an important and necessary part since it collects detailed proofs of a crucial lemma and two fundamental propositions.

Notations. The following notations will be used throughout this paper. $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{R}^+$ denotes the set of all nonnegative real numbers. $\mathbb{R}^n$ denotes the real n-dimensional space. For a given vector or matrix X, X^{T} denotes its transpose; for any $x \in \mathbb{R}^n$, $\|x\|_1$ denotes the 1-norm, i.e., $\|x\|_1 = |x_1| + \cdots + |x_n|$; $\|x\|$ denotes the Euclidean (or 2-) norm of vector x, and for the matrix P, we use $\|P\|$ to denote its norm induced by the 2-norm of the corresponding vector; for any $x \in \mathbb{R}^n$, there always holds $\|x\|_1 \leqslant \sqrt{n}\|x\|$.

2 System model and control objective

2.1 System model

Consider the tracking control problem for a class of uncertain nonlinear systems in the following form:

$$\begin{cases} \dot{\eta}_i = \eta_{i+1} + \psi_i(t,\,\eta), \quad i = 1, \ldots, n-1, \\ \dot{\eta}_n = u + \psi_n(t,\,\eta), \\ y = \eta_1 - y_r, \end{cases} \tag{1}$$

where $\eta = [\eta_1, \ldots, \eta_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the system state vector with the initial value $\eta_0 = \eta(0)$; $u \in \mathbb{R}$, $y \in \mathbb{R}$ and $t \mapsto y_r(t)$, $t \in \mathbb{R}^+$ are the control input, system output (tracking error) and reference signal, respectively; and $\psi_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$ are unknown functions but continuous in the first argument and locally Lipschitz in the second one. In what follows, suppose only the system output is measurable.

To obtain the desired objective of the paper, the following assumptions are imposed on the system (1) and the reference signal y_r.

Assumption 1. There exists an unknown positive constant θ such that $\forall t \in \mathbb{R}^+$, $\forall \eta \in \mathbb{R}^n$,

$$|\psi_i(t,\eta)| \leqslant \theta(|\eta_1| + \cdots + |\eta_i|) + \theta, \quad i = 1, \ldots, n.$$

Assumption 2. The reference signal y_r is continuously differentiable, and moreover, there is an unknown constant $M \geqslant 0$ such that $\sup_{t\geqslant 0}(|y_r(t)| + |\dot{y}_r(t)|) \leqslant M$.

Remark 1. From Assumptions 1 and 2, it is not hard to know that system (1) is substantially different from those in [11–13]. Unlike [11], the nonlinearities of system (1) heavily depend on the unmeasurable states (implied by Assumption 1), and the reference signal is not available for feedback. Different from [12], in the paper, the relative degree of system (1) is not necessarily known a priori, only tracking error (i.e., system output) is measurable (the rationality has been discussed in [13]) and only reference signal and its derivative have bounds but unknown (implied by Assumption 2), and moreover, by Assumption 1, the lower-order growing unmeasurable states are permitted in system (1). Although [13] considered the nonlinear systems with higher-order growing unmeasurable states, the growth rate is known to be constant, so are the bounds of the reference signal and its derivative.

2.2 Control objective

Rigorously speaking, the objective of this paper is to search for an adaptive output-feedback controller for system (1) under Assumptions 1 and 2 as follows:

$$\dot{\chi} = \alpha_\lambda(\chi, y), \qquad u = \beta_\lambda(\chi, y), \tag{2}$$

which guarantees that (i) the solutions of the resulting closed-loop system are well defined and globally bounded on $[0, +\infty)$, (ii) for any initial condition, there is a finite time $T_\lambda > 0$ such that $\sup_{t\geqslant T_\lambda} |y(t)| = \sup_{t\geqslant T_\lambda} |\eta_1(t) - y_r(t)| \leqslant \lambda$, where χ is the state vector of observer and updating law with the appropriate dimension and the initial value $\chi_0 = \chi(0)$. λ is an arbitrary prescribed positive constant to represent the tracking accuracy; functions $\alpha_\lambda(\cdot)$ and $\beta_\lambda(\cdot)$ are vector-valued continuous and scalar continuously differentiable, respectively.

The control just formulated is called global practical tracking control with accuracy λ $[7-9, 13]$, and sometimes it is also referred to as λ-tracking control $[4, 5, 10-12]$.

3 Global practical tracking control via output-feedback

The section is to design a tracking controller for system (1) under Assumptions 1 and 2. As described in section 2, we will explicitly design an adaptive output-feedback controller, which is composed of a full-order dynamic high-gain observer and an adaptive controller.

We first introduce the coordinates $x_1 = y$, $x_i = \eta_i$, $i = 2, \ldots, n$. Then

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(t, x, y_r, \dot{y}_r), \\ \dot{x}_i = x_{i+1} + \phi_i(t, x, y_r), \quad i = 2, \dots, n-1, \\ \dot{x}_n = u + \phi_n(t, x, y_r), \end{cases} \tag{3}$$

where $\phi_1 = \psi_1(t, x_1 + y_r, x_2, \dots, x_n) - \dot{y}_r$, $\phi_i = \psi_i(t, x_1 + y_r, x_2, \dots, x_n), i = 2, \dots, n$.

By Assumptions 1 and 2, it is easy to show that, for $i = 1, \dots, n$,

$$|\phi_i| \leqslant \theta(|x_1 + y_r| + |x_2| + \cdots + |x_i|) + |\dot{y}_r| + \theta \leqslant \theta(|x_1| + |x_2| + \cdots + |x_i|) + \theta_1, \tag{4}$$

where $\theta_1 = \theta(M+1) + M > 0$ is an unknown constant.

Then, we select suitable design parameters $a_i > 0$, $k_i > 0, i = 1, \dots, n$, such that the matrices A and B are Hurwitz, and there exist $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ satisfying

$$\begin{cases} A^{\mathrm{T}}P + PA \leqslant -I \quad \text{and} \quad DP + PD \geqslant 0, \\ B^{\mathrm{T}}Q + QB \leqslant -2I \quad \text{and} \quad DQ + QD \geqslant 0, \end{cases} \tag{5}$$

where $D = \operatorname{diag}\{1, 2, \dots, n\}$,

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad B = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

In view of Lemma 1 in [19], one easily knows that the above mentioned choice of design parameters can always be achieved.

Thus, for any pre-given $\lambda > 0$, we construct the following adaptive output-feedback controller for system (3):

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1} + L^i a_i (y - \hat{x}_1), \quad i = 1, \dots, n-1, \\ \dot{\hat{x}}_n = u + L^n a_n (y - \hat{x}_1), \\ \dot{L} = \max\left\{ \dfrac{2(y - \hat{x}_1)^2 + 2\hat{x}_1^2 - \frac{\lambda^2}{2}}{L^2},\ 0 \right\}, \quad L(0) = 1, \end{cases} \tag{6}$$

and

$$u = -(L^n k_1 \hat{x}_1 + L^{n-1} k_2 \hat{x}_2 + \cdots + L k_n \hat{x}_n), \tag{7}$$

where $\hat{x} = [\hat{x}_1, \dots, \hat{x}_n]^{\mathrm{T}}$ is the observer state vector with the initial value $\hat{x}_0 = \hat{x}(0)$.

Remark 2. From (6) or Proposition 1 below, one can easily seen $L \geqslant 1$. Therefore, the above designed controller consists of a full-order high-gain observer, a novel gain updating law and an adaptive high-gain controller, and obviously is covered by the general form (2). It should be pointed out that the novel updating law of the high-gain is introduced to effectively deal with the uncertainties in the system nonlinearities and reference signal, and highlights the main contribution in the paper. Although the expression of the gain updating law is somewhat complicated, it is essential to establish the boundedness of all the signals in the closed-loop system (composed by (1), (6) and (7)) and achieve the global practical tracking, as will be shown in the proof of Theorem 1.

The following proposition describes the basic properties of the dynamic high-gain L given above.

Proposition 1. The gain L determined by (6) is monotone nondecreasing on its existence interval, and its dynamics are locally Lipschitz in $(y, \hat{x}_1, L)$.

Proof. Observe that by (6), $\dot{L} \geqslant 0$ and therefore L is monotone nondecreasing and for any time t in its existence interval, $L(t) \geqslant L(0) = 1$. As a result, $(2(y - \hat{x}_1)^2 + 2\hat{x}_1^2 - \frac{\lambda^2}{2})/L^2$ is smooth with respect to $(y, \hat{x}_1, L)$. Then, it can be proven that $\dot{L}$ is locally Lipschitz in $(y, \hat{x}_1, L)$. In fact, for any two points $(y', \hat{x}_1', L')$ and $(y'', \hat{x}_1'', L'')$ in certain neighborhood, one easily gets

$$
\begin{aligned}
|\dot{L}' - \dot{L}''| &\leqslant \left| \frac{2(y' - \hat{x}_1')^2 + 2(\hat{x}_1')^2 - \frac{\lambda^2}{2}}{(L')^2} - \frac{2(y'' - \hat{x}_1'')^2 + 2(\hat{x}_1'')^2 - \frac{\lambda^2}{2}}{(L'')^2} \right| \\
&\leqslant N(|y' - y''| + |\hat{x}_1' - \hat{x}_1''| + |L' - L''|),
\end{aligned}
$$

where N is a proper positive constant related with the neighborhood.

By Proposition 1, it is easy to verify that the right-hand side of the resulting closed-loop system is continuous and locally Lipschitz in (η, $\hat{x}$, L) in an open neighborhood of the initial condition, and hence the closed-loop system has a unique solution on a small interval $[0,\, t_s)$ (see Theorem 3.1, page 18 of [20]). Let $[0,\, t_f)$ be its maximal interval on which a unique solution exists, where $0 < t_f \leqslant +\infty$ (see Theorem 2.1, page 17 of [20]). If one can prove $t_f = +\infty$, then all the closed-loop system states are well defined on $[0,\, +\infty)$.

Before addressing the main results of the paper, we first present two fundamental propositions (whose detailed proofs are given in Appendix A). Specifically, Proposition 2 characterizes the dynamic behavior of the closed-loop system via a Lyapunov candidate function. It is worth emphasizing that the degree of L is required to be -2 in the second term on the right-hand side of (8). This point is very important for guaranteeing the boundedness of L, as will be shown in the proof of Lemma 1 (see in Appendix A3). Proposition 3 reveals the intrinsic relationship between the high-gain L and the other variables and shows that it suffices to prove the boundedness of L for that of the closed-loop system.

Proposition 2. Define $e_i = x_i - \hat{x}_i$, $\varepsilon_i = \frac{e_i}{L^i}$, $z_i = \frac{\hat{x}_i}{L^i}$, $i = 1, \ldots, n$, and denote $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_n]^{\mathrm{T}}$, $z = [z_1, \ldots, z_n]^{\mathrm{T}}$. Then, there exist positive constants γ and Θ, such that $V(\varepsilon, z) = \gamma V_1(\varepsilon) + V_2(z) := \gamma\varepsilon^{\mathrm{T}} P \varepsilon + z^{\mathrm{T}} Q z$ satisfies the following inequality on $[0,\, t_f)$:

$$
\dot{V} \leqslant -(L - \Theta)(\|\varepsilon\|^2 + \|z\|^2) + \frac{\Theta}{L^2}, \tag{8}
$$

where P and Q are symmetric positive definite matrices satisfying (5).

Proposition 3. For the resulting closed-loop system, if L is bounded on $[0,\, t_f)$, then z and ε are bounded on $[0,\, t_f)$ as well.

Now, we are ready to address the main results in this paper, which are summarized in the following theorem.

Theorem 1. Consider system (1) under Assumptions 1 and 2. If the design parameters a_i, k_i, $i = 1, \ldots, n$ are suitably chosen such that Proposition 2 holds, then based on the dynamic high-gain observer (6), the output-feedback controller (7) guarantees that, for any initial condition, all the states of the resulting closed-loop system are well defined and bounded on $[0,\, +\infty)$, and furthermore, the global practical tracking (with tolerance λ) can be achieved, i.e., for any given $\lambda > 0$, there exists a finite time T_λ such that $|y(t)| \leqslant \lambda$, $\forall t \geqslant T_\lambda$.

Proof. To successfully prove the boundedness of the closed-loop solutions, it is necessary to first show that $t_f = +\infty$, and thus the solution of the closed-loop system exists and is unique on $[0,\, +\infty)$. Then, by a contradiction argument, we prove the boundedness of L. From this and Proposition 3, we conclude the boundedness of the solution of the closed-loop system. In fact, the first claim of the theorem is directly obtained from the following lemma whose proof is provided in Appendix A for the sake of compactness.

Lemma 1. For the closed-loop system given in Theorem 1 with the proper design parameters, $t_f = +\infty$ and all the states are bounded on $[0,\, +\infty)$.

Next, let us show the second claim in Theorem 1, i.e., for any given $\lambda > 0$, there exists a finite time T_λ, such that $|y(t)| \leqslant \lambda$, $\forall t \geqslant T_\lambda$. It is easy to see that L is continuously differentiable and $\lim_{t\to+\infty} L(t)$ exists by virtue of the boundedness and the nondecreasing property of L on $[0,\, +\infty)$. Using (3), (6) and the boundedness of y, $\hat{x}_1$, L on $[0, +\infty)$, we conclude that $\dot{y}$ (i.e., $\dot{x}_1$), $\dot{\hat{x}}_1$ and $\dot{L}$ are bounded on $[0, +\infty)$, and hence $(2(y - \hat{x}_1)^2 + 2\hat{x}_1^2 - \frac{\lambda^2}{2})/L^2$ in $\dot{L}$ has a bounded derivative on $[0, +\infty)$. Denoting by N_1 this upper bound, one can prove that $\dot{L}$ is uniformly continuous on $[0, +\infty)$. In fact, $\forall \epsilon > 0$, taking $\delta = \frac{\epsilon}{2N_1}$, for any $t_1, t_2 \in [0, +\infty)$, if only $|t_1 - t_2| < \delta$, then

$$
\begin{aligned}
|\dot{L}(t_1)-\dot{L}(t_2)| &\leqslant \left|\frac{2(y(t_1)-\hat{x}_1(t_1))^2+2\hat{x}_1^2(t_1)-\frac{\lambda^2}{2}}{L^2(t_1)}-\frac{2(y(t_2)-\hat{x}_1(t_2))^2+2\hat{x}_1^2(t_2)-\frac{\lambda^2}{2}}{L^2(t_2)}\right| \\
&\leqslant N_1|t_1-t_2|<\epsilon.
\end{aligned}
$$

Thus, by Barbălat's Lemma[1)], we have $\lim_{t\to+\infty}\dot{L}(t)=0$. That is, for any initial condition $(\eta(0),\,\hat{x}(0))$, there exists a finite time $T_\lambda>0$ such that for all $t>T_\lambda$, $(2(y(t)-\hat{x}_1(t))^2+2\hat{x}_1^2(t)-\frac{\lambda^2}{2})/L^2(t)\leqslant\dot{L}(t)\leqslant\frac{\lambda^2}{2L^2(t)}$ holds, which implies $|y(t)|=|\eta_1(t)-y_r(t)|\leqslant\lambda,\forall t>T_\lambda$. The proof of the theorem is complete.

4 A numerical example

In this section, we give a numerical example to illustrate the correctness and effectiveness of the theoretical results by considering the following second-order nonlinear system:

$$
\begin{cases}
\dot{\eta}_1=\eta_2,\\
\dot{\eta}_2=u-\theta\mathrm{sign}(\eta_2)\big(1+|\eta_2|\big),\\
y=\eta_1-y_r,
\end{cases}
$$

where $\mathrm{sign}(\cdot)$ denotes the signal function, that is $\mathrm{sign}(\eta_2)=1(=-1)$ when $\eta_2>0$ (<0) and $\mathrm{sign}(\eta_2)=0$ when $\eta_2=0$; y_r is the signal to be tracked. Suppose $\theta=0.5$, $y_r=\sin(t)$.

Then a direct application of our proposed method yields an adaptive output-feedback controller as follows:

$$
\begin{cases}
\dot{\hat{x}}_1=\hat{x}_2+La_1(y-\hat{x}_1),\\
\dot{\hat{x}}_2=u+L^2a_2(y-\hat{x}_1),\\
\dot{L}=\max\left\{\dfrac{2(y-\hat{x}_1)^2+2\hat{x}_1^2-\frac{\lambda^2}{2}}{L^2},\ 0\right\}\quad\text{with}\quad L(0)=1,
\end{cases}
$$

and $u=-(L^2k_1\hat{x}_1+Lk_2\hat{x}_2)$, where $a_1=1$, $a_2=10$, $k_1=12$, $k_2=1$. For the chosen a_1, a_2 and k_1, k_2, by solving the matrix inequalities (5), we have

$$
P=\begin{bmatrix}5.5000 & -0.5000\\ -0.5000 & 0.6000\end{bmatrix},\qquad Q=\begin{bmatrix}13.0833 & 0.0833\\ 0.0833 & 1.0833\end{bmatrix},
$$

and therefore, in virtue of the proof of Proposition 2 in Appendix A1, one can easily obtain suitable positive constants γ and Θ. This shows that the chosen a_1, a_2 and k_1, k_2 are appropriate design parameters.

Let the tracking accuracy be $\lambda=0.1$, and initial conditions be $\eta_1(0)=0.5,\eta_2(0)=1$, $\hat{x}_1(0)=2$, $\hat{x}_2(0)=-3$, we obtain the following Figures 1–5 by numerical simulation. From these figures, all the signals in the closed-loop system are bounded. From Figure 1, it can be see after about seven seconds, the tracking error satisfies $|y|=|\eta_1-y_r|\leqslant 0.1$, which means that the prescribed tracking performance is achieved.

5 Concluding remarks

In this paper, the global practical tracking problem has been investigated for a class of uncertain nonlinear systems under weaker conditions. By flexibly using the the idea of dead zone and the techniques in universal control theory, an adaptive output-feedback controller has been successfully constructed to achieve the prescribed tracking performance. It is worth pointing out that in order to solve the global

1) Barbălat's Lemma: Suppose that $\omega:[0,+\infty)\to\mathbb{R}$ is a continuously differentiable function, and $\lim_{t\to+\infty}\omega(t)$ exists and is finite. If $\dot{\omega}(t),t\in[0,+\infty)$ is uniformly continuous, then $\lim_{t\to+\infty}\dot{\omega}(t)=0$. For more basic and alterative forms of Barbălat's Lemma, we refer the reader to [21]

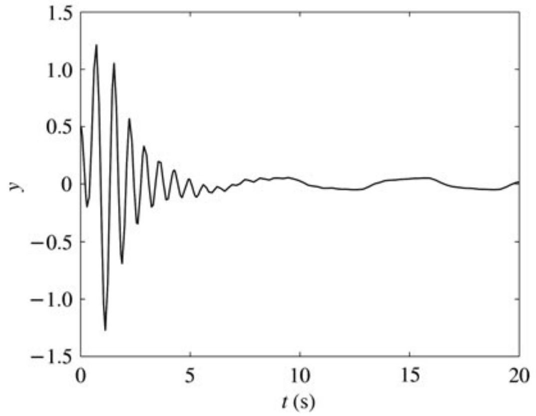

Figure 1   The trajectory of tracking error y.

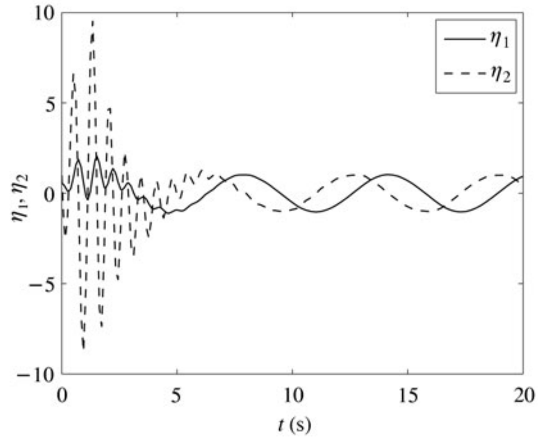

Figure 2   The trajectories of system states η_1 and η_2.

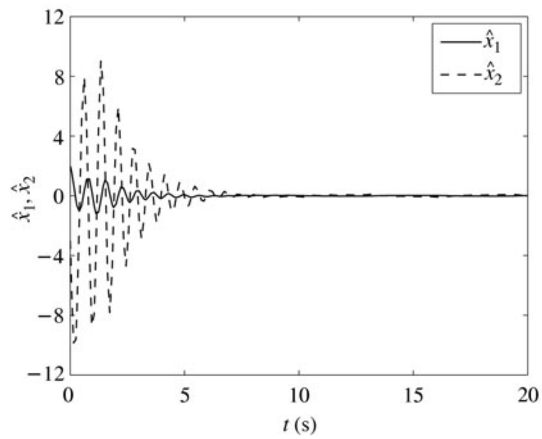

Figure 3   The trajectories of observer states $\hat{x}_1$ and $\hat{x}_2$.

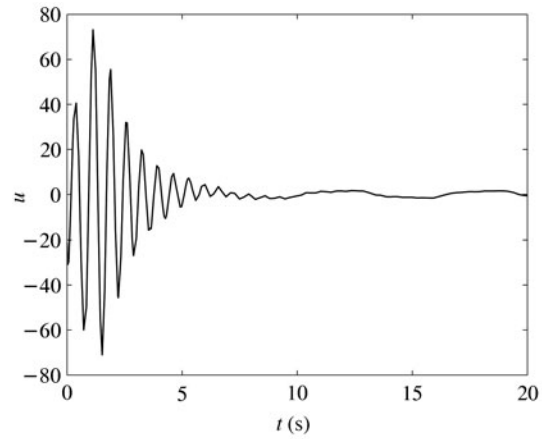

Figure 4   The trajectory of control input u.

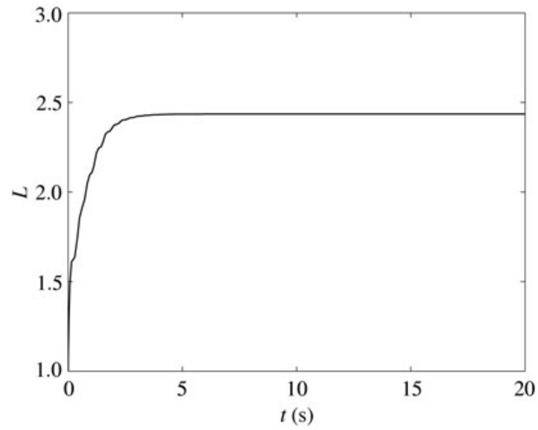

Figure 5   The trajectory of high-gain L.

practical tracking control problem of this paper, we introduce the novel updating law of the high-gain and propose the new pattern of performance analysis process. Based on the main ideas and results of the paper, the future work will be devoted to the practical tracking of other classes of uncertain nonlinear systems, such as the nonlinear systems with unknown control coefficients.

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Appendix A

The appendix provides the rigorous proofs of fundamental Propositions 2 and 3 and crucial Lemma 1, which are collected here for the sake of compactness.

A1 The proof of Proposition 2

Keeping in mind the definitions of e_i's, ε_i's and z_i's, and in terms of (3) and (6), we have the following dynamics

$$\begin{cases} \dot{e}_1 = e_2 - La_1e_1 + \phi_1(t, x, y_r, \dot{y}_r), \\ \dot{e}_i = e_{i+1} - L^ia_ie_1 + \phi_i(t, x, y_r), \quad i = 2, \dots, n-1, \\ \dot{e}_n = -L^na_ne_1 + \phi_n(t, x, y_r), \end{cases} \tag{A1}$$

and

$$\begin{cases} \dot{\varepsilon} = LA\varepsilon + \Phi(t, x, y_r, \dot{y}_r, L) - \frac{\dot{L}}{L} D\varepsilon, \\ \dot{z} = LBz + La\varepsilon_1 - \frac{\dot{L}}{L} Dz, \end{cases} \tag{A2}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}}$, $\Phi = [\frac{1}{L}\phi_1(t, x, y_r, \dot{y}_r), \frac{1}{L^2}\phi_2(t, x, y_r), \ldots, \frac{1}{L^n}\phi_n(t, x, y_r)]^{\mathrm{T}}$; $a = [a_1, \ldots, a_n]^{\mathrm{T}}$; and A, B, D have been defined in (5).

For system (A2), choose the Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$

$$V(\varepsilon, z) \; = \; \gamma V_1(\varepsilon) + V_2(z), \tag{A3}$$

where $\gamma = \|Qa\|^2 + 1$ and $V_i : \mathbb{R}^n \to \mathbb{R}^+$, $i = 1, 2$ defined by $V_1(\varepsilon) = \varepsilon^{\mathrm{T}} P\varepsilon, V_2(z) = z^{\mathrm{T}} Qz$.

Then, on the interval $[0,\, t_f)$, the derivative of V along the solutions of (A2) and using matrix inequality (5) satisfies

$$\dot{V} \leqslant -L\gamma\|\varepsilon\|^2 - \gamma\frac{\dot{L}}{L}\varepsilon^{\mathrm{T}}(DP + PD)\varepsilon - 2L\|z\|^2 - \frac{\dot{L}}{L}z^{\mathrm{T}}(DQ + QD)z + 2\gamma\varepsilon^{\mathrm{T}}P\Phi + 2L\varepsilon_1 z^{\mathrm{T}}Qa.$$

With the help of relationship (5) and the fact that $\dot{L} \geqslant 0$, $L \geqslant 1$ on $[0, t_f)$, it is easy to obtain that

$$\dot{V} \leqslant -L\gamma\|\varepsilon\|^2 - 2L\|z\|^2 + 2\gamma\varepsilon^{\mathrm{T}}P\Phi + 2L\varepsilon_1 z^{\mathrm{T}}Qa. \tag{A4}$$

For the last term of the above inequality, by the method of completing square, one has $2L\varepsilon_1 z^{\mathrm{T}}Qa \leqslant L\|z\|^2 + L\|Qa\|^2\|\varepsilon\|^2$.

In addition, observe that

$$\left|\frac{\phi_i}{L^i}\right| \leqslant \frac{\theta}{L^i}(|\hat{x}_1| + \cdots + |\hat{x}_i| + |e_1| + \cdots + |e_i|) + \frac{\theta_1}{L^i} \leqslant \theta\sqrt{n}(\|z\| + \|\varepsilon\|) + \frac{\theta_1}{L^i}.$$

Then, for the third term on the right-hand side of (A4), one has

$$\begin{aligned} |2\gamma\varepsilon^{\mathrm{T}}P\Phi| &\leqslant 4\theta\gamma n\|P\|\|\varepsilon\| \cdot (\|z\| + \|\varepsilon\|) + 4\gamma\sqrt{n}\|P\|\|\varepsilon\| \cdot \frac{\theta_1}{L} \\ &\leqslant 6\theta\gamma n\|P\|(\|\varepsilon\|^2 + \|z\|^2) + \frac{4\theta_1^2\gamma\|P\|}{\theta L^2}. \end{aligned}$$

Thus, from (A4), it follows that on $[0,\, t_f)$,

$$\dot{V} \leqslant -(L - 6\theta\gamma n\|P\|)(\|\varepsilon\|^2 + \|z\|^2) + \frac{4\theta_1^2\gamma\|P\|}{\theta L^2}. \tag{A5}$$

Let $\Theta = \max\{6\theta\gamma n\|P\|, \frac{4\theta_1^2\gamma\|P\|}{\theta}\}$. Then (A5) becomes (8), and this completes the proof of Proposition 2.

A2 The proof of Proposition 3

Note that $L(t_f) = \sup_{0 \leqslant t < t_f} L(t)$ since L is monotone nondecreasing, continuous, and bounded on $[0, t_f)$. Let's first show that z is bounded on $[0,\, t_f)$. Consider the function $V_2(z) = z^T Qz$ for the z-dynamic system of (A2). By a simple calculation, on the interval $[0, t_f)$, one has

$$\begin{aligned} \dot{V}_2 \leqslant & -L\|z\|^2 + \|Qa\|^2 L\varepsilon_1^2 \\ \leqslant & -\|z\|^2 + \|Qa\|^2 L\left(2\varepsilon_1^2 + 2z_1^2 - \frac{\lambda^2}{2L^2}\right) + \|Qa\|^2\lambda^2 \\ \leqslant & -\mu V_2 + \|Qa\|^2 L(t_f)\dot{L} + \|Qa\|^2\lambda^2, \end{aligned} \tag{A6}$$

where $\mu = \frac{1}{\lambda_{\max}(Q)}$. From (A6), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}V_2(z(t))) \leqslant \|Qa\|^2 L(t_f)\mathrm{e}^{\mu t}\dot{L}(t) + \|Qa\|^2\lambda^2\mathrm{e}^{\mu t}, \quad \forall t \in [0,\, t_f).$$

Integrating both sides of the above inequality yields

$$\begin{aligned} \mathrm{e}^{\mu t}V_2(z(t)) &\leqslant V_2(z(0)) + \|Qa\|^2 L(t_f)\int_0^t \mathrm{e}^{\mu t}\mathrm{d}L(t) + \|Qa\|^2\lambda^2\int_0^t \mathrm{e}^{\mu t}\mathrm{d}t \\ &\leqslant V_2(z(0)) + \|Qa\|^2 L^2(t_f)\mathrm{e}^{\mu t} + \|Qa\|^2\lambda^2\lambda_{\max}(Q)\mathrm{e}^{\mu t}, \quad \forall t \in [0,\, t_f), \end{aligned}$$

from which it follows that $\forall t \in [0,\, t_f)$,

$$\|z(t)\|^2 \leqslant \frac{1}{\lambda_{\min}(Q)}(z^{\mathrm{T}}(0)Qz(0) + \|Qa\|^2 L^2(t_f) + \|Qa\|^2\lambda^2\lambda_{\max}(Q)).$$

This indicates that z is bounded on $[0,\, t_f)$.

We next turn to prove the boundedness of ε on $[0,\, t_f)$. For this purpose, we introduce the following change of coordinates: $\xi_i = \frac{e_i}{(L^*)^i}, i = 1, \ldots, n$, where L^* is a constant satisfying

$$L^* \geqslant \max\{L(t_f),\, 6\theta n\|P\| + 3\}. \tag{A7}$$

Then, under the new change of coordinates, the error dynamics (A1) is transformed into

$$\begin{aligned}
\dot{\xi}_1 &= L^*\xi_2 - L^*a_1\xi_1 + L^*a_1\xi_1 - La_1\xi_1 + \frac{\phi_1(t,x,y_r,\dot{y}_r)}{L^*},\\
\dot{\xi}_i &= L^*\xi_{i+1} - L^*a_i\xi_1 + L^*a_i\xi_1 - L\left(\frac{L}{L^*}\right)^{i-1}a_i\xi_1 + \frac{\phi_i(t,x,y_r)}{(L^*)^i}, \quad i = 2, \ldots, n-1,\\
\dot{\xi}_n &= -L^*a_n\xi_1 + L^*a_n\xi_1 - L\left(\frac{L}{L^*}\right)^{n-1}a_n\xi_1 + \frac{\phi_n(t,x,y_r)}{(L^*)^n},
\end{aligned}$$

which can also be rewritten in the following compact form

$$\dot{\xi} = L^*A\xi + L^*a\xi_1 - L\Gamma a\xi_1 + \Phi^*(t,x,y_r,\dot{y}_r), \tag{A8}$$

where $\Gamma = \mathrm{diag}\{1, \frac{L}{L^*}, \ldots, (\frac{L}{L^*})^{n-1}\}$,

$$\Phi^* = \left[\frac{\phi_1(t,x,y_r,\dot{y}_r)}{L^*}, \frac{\phi_2(t,x,y_r)}{(L^*)^2}, \ldots, \frac{\phi_n(t,x,y_r)}{(L^*)^n}\right]^{\mathrm{T}}.$$

Along the solutions of (A8), differentiating the quadratic function $V_3(\xi) = \xi^{\mathrm{T}}P\xi$,

$$\dot{V}_3 \leqslant -L^*\|\xi\|^2 + 2\xi_1L^*a^{\mathrm{T}}P\xi - 2\xi_1La^{\mathrm{T}}\Gamma P\xi + 2\Phi^{*\mathrm{T}}P\xi$$

holds on $[0, t_f)$. By the method of completing the square, for the second and third terms on the right-hand side of the above inequality, one gets

$$\left|2\xi_1L^*a^{\mathrm{T}}P\xi\right| \leqslant L^{*2}\|a^{\mathrm{T}}P\|^2\xi_1^2 + \|\xi\|^2, \quad |2\xi_1La^{\mathrm{T}}\Gamma P\xi| \leqslant L^2\|a^{\mathrm{T}}\Gamma P\|^2\xi_1^2 + \|\xi\|^2. \tag{A9}$$

Moreover, with the definitions of ε_i and ξ_i, using (4) and (A7), one obtains

$$\left|\frac{\phi_i}{(L^*)^i}\right| \leqslant \theta\sqrt{n}(\|z\| + \|\xi\|) + \frac{\theta_1}{L^i}, \quad |2\Phi^{*\mathrm{T}}P\xi| \leqslant 6\theta n\|P\|(\|\xi\|^2 + \|z\|^2) + 4\|P\|\theta_1^2/\theta,$$

which, together with (A7) and (A9), means that on $[0, t_f)$,

$$\begin{aligned}
\dot{V}_3 &\leqslant -(L^* - 6\theta n\|P\| - 2)\|\xi\|^2 + 6\theta n\|P\|\|z\|^2 + (L^{*2}\|a^{\mathrm{T}}P\|^2 + L^2\|a^{\mathrm{T}}\Gamma P\|^2)\xi_1^2 + 4\|P\|\theta_1^2/\theta\\
&\leqslant -\|\xi\|^2 + \theta_2\|z\|^2 + \theta_2\varepsilon_1^2 + \theta_2\\
&\leqslant -\|\xi\|^2 + \theta_2\left(\sup_{0\leqslant t<t_f}\|z(t)\|\right)^2 + \theta_2(2\varepsilon_1^2 + 2z_1^2 - \frac{\lambda^2}{2L^2}) + \theta_2\lambda^2 + \theta_2\\
&\leqslant -\frac{1}{\lambda_{\max}(P)}V_3 + \theta_2\dot{L} + \theta_2\left(\sup_{0\leqslant t<t_f}\|z(t)\|\right)^2 + \theta_2\lambda^2 + \theta_2,
\end{aligned}$$

where $\theta_2 = \max\{6\theta n\|P\|,\, L^{*2}\|a^{\mathrm{T}}P\|^2 + L^2(t_f)\|a^{\mathrm{T}}\Gamma P\|^2,\, 4\|P\|\theta_1^2/\theta\}$. Moreover, following similar arguments to the proof of the boundedness of z, we easily have

$$\|\xi(t)\|^2 \leqslant \frac{1}{\lambda_{\min}(P)}\left(\theta_2\left(\left(\sup_{0\leqslant t<t_f}\|z(t)\|\right)^2 + \lambda^2 + 1\right)\lambda_{\max}(P) + \xi^{\mathrm{T}}(0)P\xi(0) + \theta_2L(t_f)\right), \quad \forall t \in [0,\, t_f).$$

This, together with the definitions of ε_i and ξ_i and the boundedness of z (just proved), implies that ε is bounded on $[0,\, t_f)$.

A3 The proof of Lemma 1

In virtue of (8) and flexibly applying contradiction argument, we now prove the lemma. Noticing the definitions of x_i's, e_i's, ε_i's and z_i's, the boundedness of all closed-loop signals is implied by that of $(\varepsilon,\, z,\, L)$, both on $[0,\, t_f)$. The proof of the lemma consists of two parts. The first part shows that $t_f = +\infty$, or the solution of the closed-loop system exists on $[0,\, +\infty)$. Then, boundedness of all closed-loop signals on $[0,\, +\infty)$ is proven in the second part.

Part I $t_f = +\infty$.

This can be done by a contradiction argument. Suppose that t_f is finite. Then there must exist at least one of the following two cases: (1) L is bounded on $[0,\, t_f)$; (2) L is unbounded on $[0,\, t_f)$.

In case(1), by Proposition 3, we know that ε and z are bounded on $[0,\, t_f)$. However, since the maximal existence interval $[0,\, t_f)$ is finite (by supposition) and L is bounded on $[0,\, t_f)$, at least one of ε and z is unbounded on $[0,\, t_f)$, which leads to a contradiction. So, case(1) does not occur.

In case(2), by Proposition 1, L is a monotone nondecreasing function. Thus, there exists a finite time $t^* \in (0,\, t_f)$, such that $L(t) \geqslant \Theta + 1, \forall t \in [t^*,\, t_f)$, under which, on the finite interval $[t^*,\, t_f)$, (8) becomes $\dot{V} \leqslant -(\|\varepsilon\|^2 + \|z\|^2) + \Theta \leqslant -\mu_1 V + \Theta$, where $\mu_1 = \frac{1}{\max\{\gamma\lambda_{\max}(P),\, \lambda_{\max}(Q)\}}$. From the above inequality, it is easy to see that ε and z are bounded on $[t^*, t_f)$. Integrating $\dot{L} = \max\{2\varepsilon_1^2 + 2z_1^2 - \frac{\lambda^2}{2L^2},\, 0\}$ on the finite interval $[t^*,\, t_f)$ yields

$$+\infty \;=\; L(t_f) - L(t^*) \;=\; \int_{t^*}^{t_f} \dot{L}(t)\mathrm{d}t \;\leqslant\; \int_{t^*}^{t_f} (2\varepsilon_1^2(t) + 2z_1^2(t))\mathrm{d}t \;<\; +\infty,$$

which is a contradiction and hence shows that case (2) is also unlikely to occur.

In both cases, this contradicts the assumption on finiteness of t_f. Therefore, $t_f = +\infty$.

Part II Boundedness of all closed-loop signals over the interval $[0, +\infty)$.

As mentioned in the above discussion, it suffices to prove the boundedness of $(\varepsilon,\, z,\, L)$ over the interval $[0, +\infty)$ for that of all closed-loop signals. By Proposition 3, the key is to prove the boundedness of L on the interval $[0, +\infty)$.

Suppose that L is unbound on $[0,\, +\infty)$, i.e., $\lim_{t\to+\infty} L(t) = +\infty$. Since L is a continuous and monotone nondecreasing function, there exists a finite time $T_\sigma \in (0,\, +\infty)$, such that

$$L(t) \;\geqslant\; \Theta + \frac{1}{\sigma\mu_1}, \quad \forall t \in [T_\sigma,\, +\infty), \tag{A10}$$

where $\sigma \leqslant 1$ is an arbitrary positive constant. Furthermore, on $[T_\sigma,\, +\infty)$, (8) becomes

$$\dot{V} \leqslant -\frac{1}{\sigma\mu_1}(\|\varepsilon\|^2 + \|z\|^2) + \frac{\Theta}{L^2} \leqslant -\frac{1}{\sigma}V + \frac{\Theta}{L^2}.$$

Multiplying both sides of the above inequality by $\mathrm{e}^{\frac{1}{\sigma}t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\frac{1}{\sigma}t}V(t)) \leqslant \frac{\Theta}{L^2(t)}\mathrm{e}^{\frac{1}{\sigma}t}, \quad \forall t \in [T_\sigma,\, +\infty). \tag{A11}$$

Integrating (A11) yields

$$V(t) \leqslant \mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}V(T_\sigma) + \frac{\sigma\Theta}{L^2(T_\sigma)}, \quad \forall t \in [T_\sigma,\, +\infty). \tag{A12}$$

This means that V is bounded on $[T_\sigma,\, +\infty)$. From (A10), we conclude that T_σ increases when σ decreases, and so $T_\sigma \geqslant T_1, \forall\sigma \in (0,\, 1]$ holds, and when $\sigma = 1$, denote T_σ by T_1. Furthermore, from (A12), one has $V(T_\sigma) \leqslant V(T_1) + \Theta$ and

$$V(t) \leqslant e^{\frac{1}{\sigma}(T_\sigma - t)}(V(T_1) + \Theta) + \frac{\sigma\Theta}{L^2(T_\sigma)}, \quad \forall t \in [T_\sigma,\, +\infty). \tag{A13}$$

Now, we are in a position to find a finite time such that $2(y - \hat{x}_1)^2 + 2\hat{x}_1^2$must enter the interval $[0,\, \frac{\lambda^2}{2}]$ and cannot escape from there forever. For this purpose, by searching for a sufficiently small analysis parameter $\sigma > 0$, such that after the finite time each term on the right-hand side of (A13) is suitably small and L keeps unchanged.

First, using (6), (A3) and (A13), one has

$$\begin{aligned}\dot{L}(t) = {} & \max\left\{2\varepsilon_1^2(t) + 2z_1^2(t) - \frac{\lambda^2}{2L^2(t)}, 0\right\} \leqslant \max\{\mu_2 V(t), 0\} \\ & \leqslant \mu_2(V(T_1) + (\sigma + 1)\Theta), \quad \forall t \in [T_\sigma,\, +\infty),\end{aligned}$$

where $\mu_2 = \frac{2}{\min\{\gamma\lambda_{\min}(P),\, \lambda_{\min}(Q)\}}$. Integrating the above inequality leads to

$$L(t) \leqslant \mu_2(V(T_1) + (\sigma+1)\Theta)(t - T_\sigma) + L(T_\sigma), \quad \forall t \in [T_\sigma, +\infty). \tag{A14}$$

If $\sigma > 0$ is sufficiently small, then it is easy to verify that when

$$t \geqslant T := T_\sigma + \sigma \ln \frac{16\mu_2 L^2(T_\sigma)(V(T_1)+\Theta)}{\lambda^2 - 32\sigma^2\mu_2^3(V(T_1)+\Theta)(V(T_1)+(\sigma+1)\Theta)^2} \;>\; T_\sigma,$$

one has

$$16\mu_2 \mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(V(T_1)+\Theta)(2\mu_2^2(V(T_1)+(\sigma+1)\Theta)^2\sigma^2\mathrm{e}^{\frac{1}{\sigma}(t-T_\sigma)} + L^2(T_\sigma)) \;\leqslant\; \lambda^2.$$

From this and $\frac{\upsilon^2}{2} < \mathrm{e}^{\upsilon}$, $\forall \upsilon \geqslant 0$, it is not difficult to prove that

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(V(T_1)+\Theta) \leqslant \frac{\lambda^2}{8\mu_2\big(\mu_2(V(T_1)+(\sigma+1)\Theta)(t-T_\sigma)+L(T_\sigma)\big)^2}, \quad \forall t \geqslant T. \tag{A15}$$

Furthermore, from (A14), the first term on the right-side of (A13) satisfies

$$\mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(V(T_1)+\Theta) \leqslant \frac{\lambda^2}{8\mu_2 L^2(t)}, \quad \forall t \geqslant T. \tag{A16}$$

For the second term on the right-side of (A13), using (A14), one gets

$$\begin{aligned}\frac{\sigma\Theta}{L^2(T_\sigma)} &= \frac{\sigma\Theta L^2(T)}{L^2(T_\sigma)L^2(T)} \leqslant \frac{\sigma\Theta\left(\mu_2(V(T_1)+(\sigma+1)\Theta)\sigma\ln\frac{16\mu_2L^2(T_\sigma)(V(T_1)+\Theta)}{\lambda^2-32\sigma^2\mu_2^3(V(T_1)+\Theta)(V(T_1)+(\sigma+1)\Theta)^2} + L(T_\sigma)\right)^2}{L^2(T_\sigma)L^2(T)}\\ &\leqslant \frac{\sigma\Theta\left(\frac{\sigma\mu_2(V(T_1)+(\sigma+1)\Theta)}{L(T_\sigma)}\ln\frac{16\mu_2L^2(T_\sigma)(V(T_1)+\Theta)}{\lambda^2-32\sigma^2\mu_2^3(V(T_1)+\Theta)(V(T_1)+(\sigma+1)\Theta)^2}+1\right)^2}{L^2(T)}.\end{aligned} \tag{A17}$$

Obviously, when σ is sufficiently small, the numerator on the right-side of (A17) is less than any given positive constant.

From the preceding arguments, for any given positive constant λ, by choosing sufficiently small σ, there hold

$$\begin{cases} \mathrm{e}^{\frac{1}{\sigma}(T_\sigma - t)}(V(T_1)+\Theta) \leqslant \dfrac{\lambda^2}{8\mu_2 L^2(t)}, \quad \forall t \geqslant T, \\ \dfrac{\sigma\Theta}{L^2(T_\sigma)} \leqslant \dfrac{\lambda^2}{8\mu_2 L^2(T)}. \end{cases} \tag{A18}$$

From (A13) and the monotone nondecreasing property of L, we have $V(t) \leqslant \frac{\lambda^2}{4\mu_2 L^2(T)}, \forall t \geqslant T$, which together with (A3) leads to

$$2\varepsilon_1^2(t) + 2z_1^2(t) \;\leqslant\; \mu_2 V(t) \;\leqslant\; \frac{\lambda^2}{4L^2(T)} \;<\; \frac{\lambda^2}{2L^2(T)}, \quad \forall t \geqslant T. \tag{A19}$$

Next, based on sufficiently small σ, under the premise of (A15)–(A19), let us establish the contradiction.

It is not difficult to verify that the closed-loop solution is continuous on $[0, +\infty)$. By supposition $L(+\infty) = +\infty$, there must be a finite time $T' > T$, such that $L(T') > L(T)$. Then by (A19) and the expression of $\dot{L}$, there must exist some finite times satisfying $2\varepsilon_1^2 + 2z_1^2 - \frac{\lambda^2}{2L^2} = 0$ on the interval $(T, T']$. In such finite times, denote the first time by T''. Thus, there holds

$$\begin{cases} 2\varepsilon_1^2(T'') + 2z_1^2(T'') = \dfrac{\lambda^2}{2L^2(T'')}, \\ 2\varepsilon_1^2(t) + 2z_1^2(t) < \dfrac{\lambda^2}{2L^2(t)}, \quad \forall t \in [T, T''). \end{cases} \tag{A20}$$

From the second relation of (A20) and the expression (6) of $\dot{L}$, it follows that $\dot{L}(t) \equiv 0$, $\forall t \in [T, T'')$, and then by the continuity of L, we have $L(T'') = L(T)$. From this and (A19), one easily obtains $2\varepsilon_1^2(T'')+2z_1^2(T'') < \frac{\lambda^2}{2L^2(T'')}$, which contradicts with the first relation of (A20) and shows that L is bounded on $[0, +\infty)$.

The boundedness of z and ε can be established immediately by Proposition 3. By the definitions of z and ε, x and $\hat{x}$ are bounded on $[0, +\infty)$, so is η by Assumption 2. Using (7), we conclude that the control u is also bounded on $[0, +\infty)$. This completes the proof of the boundedness of all closed-loop signals.