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Controllability of multi-agent systems based on agreement protocols

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This paper investigates the controllability of multi-agent systems based on agreement protocols. First, for a group of single-integrator agents, the controllability is studied in a unified framework for both networks with leader-following structure and networks with undirected graph. Some new necessary/sufficient conditions for the controllability of networks of single-integrator agents are established. Second, we prove that, under the same topology and same prescribed leaders, a network of high-order dynamic agents is completely controllable if and only if so is a network of single-integrator agents. Third, how the selection of leaders and the coupling weights of graphs affect the controllability is analyzed. Finally, some numerical simulations are presented to demonstrate the effectiveness of the theoretical results.

multi-agent systems, controllability, structural controllability, agreement protocols, high-order dynamic agents, graphs

1 Introduction

In recent years, studies on multi-agent dynamic systems have received much attention in various research fields[1−24]. Distributed control and coordination of multi-agent systems have made great progress due to the rapid developments of computer science and sensing & communication technologies[4−20]. Applications of these researches pertain to cooperative control of unmanned air vehicles, distributed estimation over sensor networks, swarm-based computing, etc. Research directions in distributed control and coordination of multi-agent systems include flocking motion of multiple autonomous agents[4−6], formation control of multiple mobile robots[7−10], rendezvous problem[11,12], agreement/consensus problem[13−20], and so on.

As a kind of coordination behavior, the agreement of multi-agent systems, that is, the corresponding states of all the agents converge to a common desired quantity by implementing appropriate agreement protocols, has attracted considerable research efforts[13−20]. On the other hand, the controllability is a fundamental and important issue for controlled systems. The controllability plays a basic and fundamental role in numerous research, such as pole assignment, structure decomposition, optimal control and robust control. However, exploring the controllability of multi-agent

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systems is a challenging task. This is because the behavior of networks of dynamic agents is affected by many factors, such as the dynamics of agents, the information flows among agents, and the distributed control laws of the networks involving agreement/consensus protocols.

This paper studies the controllability for both a network of single-integrator agents and a network of high-order dynamic agents based on agreement protocols. The interactions among agents are modeled by graphs. For a given multi-agent system, all the agents are divided into two roles: leader and follower. An agent is regarded as a leader if the agent is actuated by some exogenous control inputs besides the interactions coming from its neighboring agents; an agent is recognized as a follower if the movement of the agent is only dominated by the interactions of its neighbors. We assume that the states of the leaders can be steered to arbitrary values by the exogenous control inputs. The controllability of multi-agent systems reflects whether the states of the leaders are able to drive those of the followers to arbitrary states in finite time or not, or say, reflects the ability of the leaders controlling the followers. For a group of single-integrator agents, the controllability is studied in a unified framework for networks with leader-following structure[21−24] and networks with undirected graph[25,26]. Some new necessary/sufficient conditions for the controllability of networks of single-integrator agents are established. For networks of high-order dynamic agents, we investigate the controllability under two kinds of agreement protocols. It is proved that the controllability of networks of high-order dynamic agents is equivalent to that of networks of single-integrator agents under the same topology and same prescribed leaders. In addition, we analyze that the selection of leaders and the coupling weights of graphs have important influence on the controllability of networks. Based on graphical characterization, a necessary condition for the controllability of networks is established, and a relation between the controllability of networks and the structural controllability of linear systems is revealed.

The remainder of the paper is organized as follows. Section 2 presents some mathematical preliminaries of graph theory. Section 3 studies the controllability of networks of single-integrator agents. Section 4 deals with the controllability for networks of high-order dynamic agents. Section 5 analyzes the effects of the selection of leaders and the coupling weights of graphs on the controllability. Section 6 contains some numerical examples and the last section makes the conclusions.

Notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^m$ denotes the real vector space of real m-vectors. Let I_m be an identity matrix with order $m \times m$. $\mathbf{0}$ denotes a zero matrix with appropriate order. $e_i \in \mathbb{R}^m$ is the ith standard basis vector in $\mathbb{R}^m$. $\underline{m} = \{1, \ldots, m\}$ is an index set. $\otimes$ denotes the Kronecker product of matrices.

2 Mathematical preliminaries

In this section, we present some concepts and basic results on graph theory, which are very useful for the development of the paper.

A digraph (undirected graph) $\mathcal{G}$ consists of a vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, and an arc (edge) set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. An arc (edge) of $\mathcal{G}$, denoted by $e_{ij} = (v_i, v_j)$, is an ordered (unordered) pair of distinct vertices of $\mathcal{V}$. The first vertex and the second vertex are called the tail and the head of e_{ij}, respectively. If $e_{ij} = (v_i, v_j)$ is an arc, then we say that v_i and v_j are adjacent or v_i is a neighbor of v_j. In this paper, we assume that there are no self-loops, i.e., $e_{ii} \notin \mathcal{E}$. Denote the neighbors of vertex v_i by $\mathcal{N}_i = \{v_j : e_{ji} = (v_j, v_i) \in \mathcal{E}\}$. A path from v_i to v_j means that there is a sequence of distinct arcs in $\mathcal{E}$, $(v_i, v_1), (v_1, v_2), \ldots, (v_r, v_j)$; if $v_i = v_j$ we say the sequence of arcs to be a cyclic path. A directed tree is a digraph, where each vertex has exactly one tail except for one special vertex without any tail. The special vertex is called the root of the tree. We say a graph has a spanning tree if there exists a subset of the arcs $\mathcal{E}' \subset \mathcal{E}$ such that the graph $(\mathcal{V}, \mathcal{E}')$ is a directed tree. A directed graph is said to be strongly connected, if there exists a path between any two distinct vertices of the graph. For undirected graph, the strongly connected property is usually called connected. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be two graphs. We call $\mathcal{G}'$ a subgraph of $\mathcal{G}$, denoted by $\mathcal{G}' \subset \mathcal{G}$, if

$\mathcal{V}' \subset \mathcal{V}$, $\mathcal{E}' \subset \mathcal{E}$. In addition, if two vertices of $\mathcal{V}'$ are adjacent in $\mathcal{G}'$ if and only if they are adjacent in $\mathcal{G}$, then we say that $\mathcal{G}'$ is an induced subgraph of $\mathcal{G}$. An induced subgraph of $\mathcal{G}$ that is maximal and strongly connected, is said to be a strong component of the graph. For an undirected graph, it is called a connected component. An undirected graph is called complete if every pair of distinct vertices are adjacent.

Next, let $\mathcal{A} = [a_{ij}]$ be a nonnegative matrix with rows and columns indexed by the vertices of $\mathcal{G}$ (a nonnegative matrix means that all the entries of the matrix are nonnegative). A weighted graph is a graph $\mathcal{G} = (\mathcal{V},\ \mathcal{E})$ with a nonnegative matrix $\mathcal{A}$, denoted by $\mathcal{G}(\mathcal{A})$, such that $(v_i,\ v_j) \in \mathcal{E}$ if and only if $a_{ji} > 0$. Here $\mathcal{A}$ is called a weighted adjacency matrix of $\mathcal{G}$, and a_{ji} is called a weight of the arc (v_i, v_j). Particularly, if $\mathcal{A}$ is a 01-matrix (which means that the entries of $\mathcal{A}$ are 0 or 1), then we say $\mathcal{G}(\mathcal{A})$ is an un-weighted graph, or say, $\mathcal{G}(\mathcal{A})$ depicts the topological structure of $\mathcal{G}$. Throughout this paper, we always use $\mathcal{G}$ and $\mathcal{G}(\mathcal{A})$ to represent an un-weighted graph and a weighted graph with the weighted adjacency matrix $\mathcal{A}$, respectively. If $\mathcal{G}(\mathcal{A})$ is an undirected graph then $\mathcal{A}$ is symmetric, i.e., $\mathcal{A}^{\mathrm{T}} = \mathcal{A}$.

The Laplacian matrix $\mathcal{L}(\mathcal{G}(\mathcal{A})) = [l_{ij}] \in \mathbb{R}^{N\times N}$ of a graph $\mathcal{G}(\mathcal{A})$, abbreviated as $\mathcal{L}$, is defined as

$$l_{ij} = \begin{cases} \sum_{v_j\in\mathcal{N}_i} a_{ij}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

It is obvious that the sum of all entries in any row of $\mathcal{L}$ is zero.

3 Networks of single-integrator dynamic agents

In this section, we consider a multi-agent system composed of $N + n_l$ agents, which are labeled 1 through $N + n_l$. The dynamics of each agent is described by

$$\dot{x}_i = u_i, \quad i \in \underline{N + n_l}, \tag{1}$$

where $x_i \in \mathbb{R}^d$ is the state of agent i, and $u_i \in \mathbb{R}^d$ is the control input. In the context of agreement for multi-agent systems, the control input is called an agreement protocol. The interactions or communication links among agents are realized in their control inputs. We employ a graph $\mathcal{G} = (\mathcal{V},\ \mathcal{E})$ to model the interaction relations among agents. Each vertex v_i in $\mathcal{V}$ represents an agent i of the multi-agent system, and each arc e_{ij} in $\mathcal{E}$ means that there is a communication link or an information flow from agent i to agent j. If for any $e_{ij} \in \mathcal{E}$, $e_{ji} \in \mathcal{E}$ as well, then the communication is said to be bidirectional, namely, when agent i can receive information from agent j, agent j can receive information from agent i as well; otherwise, the communication is said to be unidirectional. Interactions among agents are realized through the following typical linear agreement protocol:

$$u_i = \sum_{j\in\mathcal{N}_i} (x_j - x_i), \quad i \in \underline{N + n_l}, \tag{2}$$

which is widely studied in refs. [13–15] from the perspective of the convergence to agreement.

For a given multi-agent system under an agreement protocol, we refer to $\mathcal{G}_x = (\mathcal{G},\ x)$ as a network with value $x \in \mathbb{R}^{d(N+n_l)}$ and graph/topology $\mathcal{G}$, where x is the state collection of all the agents and $\mathcal{G}$ captures the communication links among agents. The controllability problem for networks based on agreement protocols is called the controlled agreement problem for networks. For simplicity, we assume that the dimension of agents $d = 1$. All the results in the present paper are valid for any dimension d, just rewriting the expressions based on Kronecker product of matrices.

In what follows, we first recall some typical topological structures of networks in refs. [21–26], and the modeling methods which transform the agreement dynamics of a network into a controlled linear system. Furthermore, we establish some new necessary/sufficient conditions for the controllability of the associated controlled linear system. The results of this section extend and improve the existing results to a certain extent.

We start with the partition of agents into leaders and followers. For a given multi-agent system, an agent is called a leader if the agent is actuated by some exogenous control inputs besides the interactions coming from its neighboring agents; otherwise, the agent is called a follower.

3.1 Topological structures

In the literature, the controlled agreement problems were mainly discussed for two kinds of networks: networks with leader-following structure[21−24] and networks with undirected graph[25,26], which are restated as follows.

3.1.1. Networks with leader-following structure[21−24]1). The communication links among the leaders and the followers are unidirectional, that is, there only exist information flows from the leaders to the followers; the communication links among the followers are bidirectional. The dynamics of the followers abides by (2), while the dynamics of the leaders selects control inputs indifferently and freely. In Figure 1, if we delete the links represented by dashed lines, the graph with vertices $1, 2, 3, 4$ and 5 is an example of such a network.

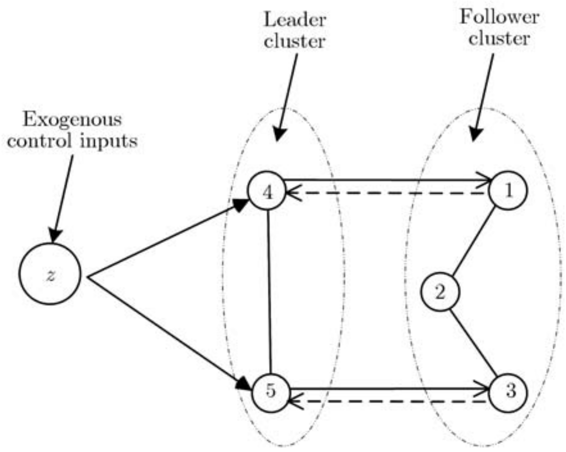

Figure 1 Schematic diagram for topology structures of networks.

Suppose there are N followers and n_l leaders over a network with leader-following structure. Unless it is explicitly specified, we will assume that the followers have small labels and the leaders have large ones, that is, we will label the followers from 1 to N and the leaders from $N+1$ to $N+n_l$. For the network, the associated Laplacian matrix can be written as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathcal{L}_l \end{bmatrix}, \tag{3}$$

where $\mathcal{L}_f$ corresponds to the indices of the followers, and $\mathcal{L}_l$ corresponds to the indices of the leaders.

Assume the n_l leaders are governed by the exogenous control input $z \in \mathbb{R}^{n_l}$ which can steer the states of the leaders to arbitrary values. Based on the partition of agents, we can write the agreement dynamics (1)–(2) as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ \mathbf{0} & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ z \end{bmatrix},$$

where x is the stacked vector of the followers' states and y is the stacked vector of the leaders' states. Then the dynamics of the followers can be viewed as the controlled linear time-invariant system

$$\dot{x} = -\mathcal{L}_f x - l_{fl} y \tag{4}$$

with the control input being the leaders' states y. We call the controlled linear time-invariant system above a controlled agreement system of the network. The following definition presents the concept of a network being completely controllable.

Definition 1. For a given network $\mathcal{G}_x$, we say the network is completely controllable under some prescribed leaders, if its associated controlled agreement system is completely controllable (see ref. [27]).

3.1.2 Networks with undirected graph[25,26]. The communication links of the whole network are bidirectional. In order to transform the agreement dynamics (1)–(2) into a controlled agreement system, some agents are appointed the leaders. The movements of these leaders are dominated by some exogenous control inputs, besides the state information obtained from their neighbors. In Figure 1, if we view the two pairs of arcs in opposite directions, connecting $4, 1$ and $5, 3$, as two edges, then the graph is an example of such a network.

Suppose there are N followers and n_l leaders over a network with undirected graph, then the associated Laplacian matrix is in the form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix}, \tag{5}$$

where $\mathcal{L}_f$ and $\mathcal{L}_l$ have the same meanings as in (3). Then from the dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ z \end{bmatrix},$$

1) Note that we use the terminology "leader-following structure" just to emphasize that there only exist information flows from the leaders to the followers.

we can derive the controlled agreement system of the network is taken in the form (4) as well.

Remark 1. By observing the properties of these two kinds of networks, we find that a network with leader-following structure has the same controlled agreement system as a network with undirected graph, under the same partition of leaders and followers and the same block matrices $\mathcal{L}_f$ and l_{fl} in their respective Laplacian matrices. Thus it is natural to think that there may exist some connections between the controllability of these two kinds of networks.

Based on the observation above, we next establish a unified framework for the controllability of these two kinds of networks. To this end, we provide the following definition on the underlying undirected graph of a graph.

Definition 2. For a given graph $\mathcal{G} = (\mathcal{V},\ \mathcal{E})$, we say an undirected graph to be the underlying undirected graph of $\mathcal{G}$, denoted by $\mathcal{G}^u = (\mathcal{V}^u,\ \mathcal{E}^u)$, if $\mathcal{V}^u = \mathcal{V}$ and $\mathcal{E}^u$ is an edge set of unordered pairs of distinct vertices of $\mathcal{V}^u$, where an edge $e^u_{ij} \in \mathcal{E}^u$ if $e_{ij} \in \mathcal{E}$ or $e_{ji} \in \mathcal{E}$.

Given a network $\mathcal{G}_x$ with some prescribed leaders, denote the associated Laplacian matrix of $\mathcal{G}$ as $\mathcal{L} = [\begin{smallmatrix} \mathcal{L}_f & l_{fl} \\ l_{lf} & \mathcal{L}_l \end{smallmatrix}]$. Let $\mathfrak{S}$ be a collection of graphs, where $\mathcal{G} \in \mathfrak{S}$ means that the Laplacian matrix associated to its underlying undirected graph $\mathcal{G}^u$ has the form

$$\mathcal{L}^u = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \widetilde{\mathcal{L}}_l \end{bmatrix}, \tag{6}$$

where $\mathcal{L}_f$ and $\widetilde{\mathcal{L}}_l$ are symmetric matrices.

Remark 2. It is evident that networks with leader-following structure studied in refs. [21–24] and networks with undirected graph studied in refs. [25, 26] belong to the collection $\mathfrak{S}$. Thus $\mathfrak{S}$ brings these two kinds of networks into a unified framework. In addition, for a given network $\mathcal{G}_x$ with $\mathcal{G} \in \mathfrak{S}$, the controlled agreement system of the network $\mathcal{G}^u_x$ is the same as that of $\mathcal{G}_x$. Consequently, the controllability of $\mathcal{G}_x$ is equivalent to that of $\mathcal{G}^u_x$.

Under the unified framework, we derive some new necessary/sufficient conditions for the controllability of networks with $\mathcal{G} \in \mathfrak{S}$ in the following subsection.

3.2 Controllability criteria

Assumption 1[2]. For a given graph $\mathcal{G}$, let $\mathcal{G}_f$ and $\mathcal{G}_l$ be the induced subgraphs on the followers and the leaders, respectively; $\mathcal{G}_f$ and $\mathcal{G}_l$ are called the follower subgraph and the leader subgraph, respectively. We assume that the leader subgraph $\mathcal{G}_l$ is linked to all the connected components of the follower subgraph $\mathcal{G}_f$. In other words, for each of the connected components of $\mathcal{G}_f$, there exists at least one leader in $\mathcal{G}_l$ and one follower in the connected component, such that there is a path from the leader to the follower.

Proposition 1. For a given network $\mathcal{G}_x$ with the dynamics (1)–(2) and $\mathcal{G} \in \mathfrak{S}$, suppose there are N followers and $n_l \geqslant 1$ leaders, and the underlying undirected graph is $\mathcal{G}^u$ with Laplacian matrix $\mathcal{L}^u$. If Assumption 1 is satisfied, then the corresponding controlled agreement system (4) is completely controllable if and only if there are no common eigenvalues of $\mathcal{L}^u$ and $\mathcal{L}_f$.

Proof. The proof is similar to the proof of Lemma 2.2 in ref. [26], and hence is omitted.

Proposition 1 presents a necessary and sufficient condition for the controllability of $\mathcal{G}_x$ with $n_l \geqslant 1$ leaders based on the eigenvalues of $\mathcal{L}_f$. We next establish a necessary condition for the controllability of the network characterized by the eigenvalues and the eigenvectors of $\mathcal{L}_f$, and provide a concise proof theoretically.

Before establishing Theorem 1, we recall some properties of symmetric matrices. It is well known that any symmetric matrix has real eigenvalues and is unitarily diagonalizable. Moreover, the left eigenvector and the right eigenvector corresponding to an eigenvalue of a symmetric matrix are transpose mutually. Relations between the spectrum of a symmetric matrix and that of its principle sub-matrix are stated as follows.

Lemma 1 (Theorem 9.1.1 in ref. [28]). Suppose S is a real symmetric $n \times n$ matrix and T is

2) Note that this assumption is a necessary condition for the controllability of controlled linear systems, which had been proved in ref. [24], and it indicates that the state of each follower has direct or indirect connection with the control inputs (or the states of the leaders). We will extend the necessary condition to the case of networks with general graph in section 5.

a principal sub-matrix of S with order $m \times m$. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$ be the respective eigenvalues of S and T. Then for $i = 1, \ldots, m$,

$$\lambda_{n-m+i} \leqslant \mu_i \leqslant \lambda_i.$$

Theorem 1. For a given network $\mathcal{G}_x$ with the dynamics (1)–(2) and $\mathcal{G} \in \mathfrak{S}$, suppose there are N followers and $n_l \geqslant 1$ leaders, and Assumption 1 is satisfied. If the associated controlled agreement system (4) is completely controllable, then 1) there exists no eigenvalue of $\mathcal{L}_f$ with multiplicity more than n_l; 2) if there exists an eigenvalue of $\mathcal{L}_f$ with multiplicity $k \leqslant n_l$, then the product matrix Ml_{fl} has full row rank, where $M \in \mathbb{R}^{k\times N}$ is composed of the k linearly independent left eigenvectors corresponding to the eigenvalue.

Proof. Assume the underlying undirected graph of $\mathcal{G}$ is $\mathcal{G}^u$ with the Laplacian matrix $\mathcal{L}^u$. According to Proposition 1, the system (4) being completely controllable indicates that there are no common eigenvalues of $\mathcal{L}^u$ and $\mathcal{L}_f$. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_{N+n_l}$ and $\mu_1 \geqslant \cdots \geqslant \mu_N$ be the respective eigenvalues of $\mathcal{L}^u$ and $\mathcal{L}_f$. Then Lemma 1 implies that $\lambda_{n_l+i} \leqslant \mu_i \leqslant \lambda_i$ for $i = 1, \cdots, N$. If 1) does not hold, without loss of generality, we assume $n_l = 2$ and $\mu_1 = \mu_2 = \mu_3$, i.e. μ_1 is an eigenvalue of $\mathcal{L}_f$ with multiplicity three. Thus $\lambda_3 \leqslant \mu_1 \leqslant \lambda_1$, $\lambda_4 \leqslant \mu_2 \leqslant \lambda_2$ and $\lambda_5 \leqslant \mu_3 \leqslant \lambda_3$. It follows that $\lambda_3 = \mu_1$, which contradicts the fact that there are no common eigenvalues of $\mathcal{L}^u$ and $\mathcal{L}_f$. Next, if 2) does not hold, we assume the eigenvalue μ_s of $\mathcal{L}_f$ is with multiplicity $k \leqslant n_l$ and the associated eigenvectors are $u_1, u_2, \ldots, u_k$, such that

$$\begin{bmatrix} u_1^{\mathrm{T}} \\ \vdots \\ u_k^{\mathrm{T}} \end{bmatrix} l_{fl}$$

is row linearly dependent. It follows that there exist some real numbers $a_1, \ldots, a_k$, not all zero, such that $a_1 u_1^{\mathrm{T}} l_{fl} + \cdots + a_k u_k^{\mathrm{T}} l_{lf} = 0$. Let $\beta = a_1 u_1 + \cdots + a_k u_k$. Then $\beta \neq 0$, $\mathcal{L}_f \beta = \mu_s \beta$ and

$$\mathcal{L}^u \begin{bmatrix} \beta \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \widetilde{\mathcal{L}}_l \end{bmatrix} \begin{bmatrix} \beta \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f \beta \\ l_{fl}^T \beta \end{bmatrix} = \mu_s \begin{bmatrix} \beta \\ \mathbf{0} \end{bmatrix},$$

which shows that μ_s is a common eigenvalue of $\mathcal{L}_f$ and $\mathcal{L}^u$. This is also a contradiction. The proof is completed.

Remark 3. In the case of networks with one leader, the result of Theorem 1 is consistent with that of Theorem IV.1 of ref. [21].

Remark 4. Proposition 1 and the proof of Theorem 1 are derived based on Remark 2, i.e., the equivalence in the context of modeling the controlled agreement systems for the network $\mathcal{G}_x \in \mathfrak{S}$ and the network $\mathcal{G}_x^u$. In the viewpoint of networks with leader-following structure, Proposition 1 provides a simple proof of Theorem 1. This improves the results of refs. [21, 22, 24]. In the viewpoint of networks with undirected graph, Proposition 1 expands the applicable ranges for the results of ref. [26]. In other words, Proposition 1 indicates that if the network with undirected graph studied in ref. [26] is replaced by $\mathcal{G}_x^u$, then the results of Theorem 4.5 and Corollary 4.6 in ref. [26] can be used to determine the controllability of the network $\mathcal{G}_x \in \mathfrak{S}$.

4 Networks of high-order dynamic agents

In this section, we consider the controlled agreement problem for a multi-agent system with high-order dynamic agents. The dynamics of each agent is given by the following mth order differential equation:

$$\begin{aligned} &\dot{x}_i^{(1)} = x_i^{(2)}, \ldots, \dot{x}_i^{(m-1)} = x_i^{(m)}, \\ &\dot{x}_i^{(m)} = u_i, \quad i \in \underline{N+n_l}, \end{aligned} \tag{7}$$

where m is a positive integer and denotes the order of the differential equation (7); $x_i^{(1)} \in \mathbb{R}$ is called an information variable of agent i for convenience and $x_i^{(k+1)}$, $k \in \underline{m-1}$ is the kth order derivative of $x_i^{(1)}$; $u_i \in \mathbb{R}$ is the control input. We will study the controlled agreement problem for such a multi-agent system under two agreement protocols: one is with the feedbacks of all the relative state information between neighboring agents (see ref. [29]),

$$\begin{aligned} u_i = -\sum_{j\in\mathcal{N}_i} \sum_{k=0}^{m-1} c_k (x_i^{(k+1)} - x_j^{(k+1)}), \\ i \in \underline{N+n_l}; \end{aligned} \tag{8}$$

the other is with the feedbacks of partial relative state information between neighboring agents (proposed by ref. [30]),

$$u_i=\sum_{k=1}^{m-1}c_kx_i^{(k+1)}-\sum_{j\in\mathcal{N}_i}(x_i^{(1)}-x_j^{(1)}),\qquad(9)$$

for $i\in\underline{N+n_l}$, where $c_0,\ c_1,\ \ldots,\ c_{m-1}$ are nonzero feedback gains.

For the convenience of expression, we refer a network with the dynamics (7) and (8) to be network I, and a network with the dynamics (7) and (9) to be network II throughout this section.

We first explain why the high-order integrator is employed to describe the dynamics of agents. The idea is inspired by the following four facts. First, any completely controllable continuous-time linear time-invariant (LTI) system, having the state-space equation $\dot{x}=Ax+Bv$, can be equivalently brought into a collection of decoupled and independently controlled chains of integrators, under an appropriate nonsingular linear transformation and a suitable state feedback (see ref. [31]). Second, if we simply use the matrix pair (A,B) to denote the controlled system $\dot{x}=Ax+Bu$, then the set of all completely controllable pairs (A,B) is open and dense in the space composed of all matrix pairs (A,B) (see ref. [32] and the references therein). Third, take a single-input LTI system $\dot{x}=Ax+bu$ for example, where $(A,\ b)$ is completely controllable and can be transformed into the mth order integrator (7). (Note that any completely controllable multi-input LTI system can be transformed into a completely controllable single-input LTI system[33].) If the multi-agent system (7) with the protocol (8) or (9) is completely controllable, then we can derive suitable agreement protocols, under which the multi-agent system with agents modeled as $\dot{x}_i=Ax_i+bv_i$ is completely controllable. (The agreement protocols are given later on.) Finally, the high-order-integrator model of agents is a generalization of the single/double-integrator model, which were widely studied in refs. [13, 14, 18, 20]. Hence it is of physical interest and of theoretical interest to investigate the controllability of multi-agent systems with agents modeled by high-order integrator.

Next, we investigate the controlled agreement problem for networks of high-order dynamic agents with topology modeled by a graph $\mathcal{G}$. Denote $x^{[k]}=[x_1^{(k)}\cdots x_{N+n_l}^{(k)}]^{\mathrm{T}},\ k\in\underline{m}$ as the collection of the states of the system (7). Assume there are N followers and $n_l\geqslant 1$ leaders over the network. Label the followers 1 through N, and the leaders $N+1$ through $N+n_l$. The movements of these leaders are dominated by some exogenous control input $z=[z_1\ \cdots\ z_{n_l}]^{\mathrm{T}}\in\mathbb{R}^{n_l}$, which can drive the states of all the leaders to arbitrary values. Specifically, the dynamics of the leaders takes the form of

$$\dot{x}_{N+i}^{(1)}=x_i^{(2)},\ \ldots,\ \dot{x}_{N+i}^{(m-1)}=x_{N+i}^{(m)},$$
$$\dot{x}_{N+i}^{(m)}=u_i+z_i,\quad i\in\underline{n_l},$$

where u_i is given in (8) or (9). The Laplacian matrix of $\mathcal{G}$ can be written as

$$\mathcal{L}=\begin{bmatrix}\mathcal{L}_f & l_{fl}\\ l_{lf} & \mathcal{L}_l\end{bmatrix},\qquad(10)$$

where $\mathcal{L}_f\in\mathbb{R}^{N\times N}$ and $\mathcal{L}_l\in\mathbb{R}^{n_l\times n_l}$ have the same meanings as those of (3), and l_{lf} indicates the communication links from the followers to the leaders.

4.1 Controllability of network I

According to the partition of leaders and followers, the multi-agent system (7) under the protocol (8) can be rewritten as the following stacked form

$$\begin{bmatrix}\dot{x}^{[1]}\\ \vdots\\ \dot{x}^{[m]}\end{bmatrix}=\varOmega\begin{bmatrix}x^{[1]}\\ \vdots\\ x^{[m]}\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ z\end{bmatrix},\qquad(11)$$

where

$$\varOmega=E_m\otimes I_{N+n_l}-F_m\otimes\begin{bmatrix}\mathcal{L}_f & l_{fl}\\ l_{lf} & \mathcal{L}_l\end{bmatrix},$$
$$E_m=\begin{bmatrix}\mathbf{0} & I_{m-1}\\ 0 & \mathbf{0}\end{bmatrix},$$
$$F_m=\begin{bmatrix}\mathbf{0}\\ \theta^T\end{bmatrix},$$
$$\theta=[c_0\ c_1\ \cdots c_{m-1}]^{\mathrm{T}},$$

and $z\in\mathbb{R}^{n_l}$ is the exogenous control input of the leaders. Consequently, we can interpret the dynamics of the followers as the controlled LTI system below:

$$\dot{\xi}=(E_m\otimes I_N-F_m\otimes\mathcal{L}_f)\xi-(F_m\otimes l_{fl})\varphi,\qquad(12)$$

where $\xi = [x_1^{(1)} \ \cdots \ x_N^{(1)} \ \cdots \ x_1^{(m)} \ \cdots \ x_N^{(m)}]^{\mathrm{T}}$ is the stacked vector of all the followers' states, $\varphi = [x_{N+1}^{(1)} \ \cdots \ x_{N+n_l}^{(1)} \ \cdots \ x_{N+1}^{(m)} \ \cdots \ x_{N+n_l}^{(m)}]^{\mathrm{T}}$ is the stacked vector of all the leaders' states. Note that the control input of the system in (12) is all the states of the leaders.

The terminologies and notations which appear in this section have the same meanings as those in section 3.

Theorem 2. For a given network I with a graph $\mathcal{G}$, suppose there are N followers and n_l leaders. Then the controlled agreement system (12) is completely controllable if and only if the controlled agreement system (4), i.e., $(-\mathcal{L}_f, \ -l_{fl})$, is completely controllable.

Proof. Denote $A = E_m \otimes I_N - F_m \otimes \mathcal{L}_f$ and $B = -F_m \otimes l_{fl}$. We start the proof by observing the relations between the spectrum of A and that of $\mathcal{L}_f$. Let λ be an eigenvalue of A and $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}]$ with $\beta_k \in \mathbb{R}^N, \ k \in \underline{m}$ be the associated left eigenvector. Then we have

$$\begin{cases} -c_0\beta_m^{\mathrm{T}}\mathcal{L}_f = \lambda\beta_1^{\mathrm{T}}, \\ \beta_1^{\mathrm{T}} - c_1\beta_m^{\mathrm{T}}\mathcal{L}_f = \lambda\beta_2^{\mathrm{T}}, \\ \qquad \vdots \\ \beta_{m-1}^{\mathrm{T}} - c_{m-1}\beta_m^{\mathrm{T}}\mathcal{L}_f = \lambda\beta_m^{\mathrm{T}}. \end{cases}$$

Consequently,

$$\begin{cases} \beta_1^{\mathrm{T}} = \lambda^{m-1}\beta_m^{\mathrm{T}} + c_{m-1}\lambda^{m-2}\beta_m^{\mathrm{T}}\mathcal{L}_f + \cdots \\ \qquad +c_1\beta_m^{\mathrm{T}}\mathcal{L}_f, \\ \qquad \vdots \\ \beta_{m-2}^{\mathrm{T}} = \lambda^2\beta_m^{\mathrm{T}} + c_{m-1}\lambda\beta_m^{\mathrm{T}}\mathcal{L}_f + c_{m-2}\beta_m^{\mathrm{T}}\mathcal{L}_f, \\ \beta_{m-1}^{\mathrm{T}} = \lambda\beta_m^{\mathrm{T}} + c_{m-1}\beta_m^{\mathrm{T}}\mathcal{L}_f, \end{cases} \tag{13}$$

and

$$-(c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0)\beta_m^{\mathrm{T}}\mathcal{L}_f = \lambda^m\beta_m^{\mathrm{T}}. \tag{14}$$

We say that $f(\lambda) := c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0 \neq 0$. Otherwise, (14) means that $\lambda = 0$ or $\beta_m^{\mathrm{T}} = 0$. If $\lambda = 0$, then $f(\lambda) = 0$ results in $c_0 = 0$, which contradicts the fact that c_0 is nonzero number. If $\beta_m^{\mathrm{T}} = 0$, then (13) leads to $\beta_1^{\mathrm{T}} = \cdots = \beta_{m-1}^{\mathrm{T}} = \mathbf{0}$, which contradicts $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}]$ being a left eigenvector of matrix A. As a result, (14) implies that $-\frac{\lambda^m}{c_{m-1}\lambda^{m-1}+\cdots+c_1\lambda+c_0}$ is an eigenvalue of $\mathcal{L}_f$, denoted by μ, and β_m^{T} is the corresponding left eigenvector. For now, it follows that for any eigenvalue λ of matrix $E_m \otimes I_N - F_m \otimes \mathcal{L}_f$ with a left eigenvector $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}]$, $\mu = -\frac{\lambda^m}{c_{m-1}\lambda^{m-1}+\cdots+c_1\lambda+c_0}$ is an eigenvalue of $\mathcal{L}_f$ with the corresponding left eigenvector β_m^{T}. Conversely, for any given eigenvalue μ of $\mathcal{L}_f$ with a corresponding left eigenvector β^{T}, we can obtain that the roots of the polynomial $s^m + \mu c_{m-1}s^{m-1} + \cdots + \mu c_0 = 0$ with respect to s are the eigenvalues of $E_m \otimes I_N - F_m \otimes \mathcal{L}_f$ and $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_{m-1}^{\mathrm{T}} \ \beta^{\mathrm{T}}]$ is their corresponding eigenvectors with

$$\begin{cases} \beta_1 = (s^{m-1} + \mu c_{m-1}s^{m-2} + \cdots + \mu c_1)\beta, \\ \quad \vdots \\ \beta_{m-2} = (s^2 + \mu c_{m-1}s + \mu c_{m-2})\beta, \\ \beta_{m-1} = (s + \mu c_{m-1})\beta. \end{cases}$$

Next, we prove the conclusion of the theorem.

By contradiction, if the system (12) is uncontrollable, then there exists an eigenvalue λ of $E_m \otimes I_N - F_m \otimes \mathcal{L}_f$ with an associated left eigenvector $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}]$ and $\beta_k \in \mathbb{R}^N, \ k \in \underline{m}$, such that $-[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}](F_m \otimes l_{fl}) = 0$. It follows that $\mu = -\frac{\lambda^m}{c_{m-1}\lambda^{m-1}+\cdots+c_1\lambda+c_0}$ is an eigenvalue of $\mathcal{L}_f$ with β_m^{T} being the corresponding left eigenvector, and $\beta_m^{\mathrm{T}}l_{fl} = \mathbf{0}$. This contradicts that $(-\mathcal{L}_f, \ -l_{fl})$ is completely controllable. Conversely, if the system $(-\mathcal{L}_f, \ -l_{fl})$ is uncontrollable, then we can easily derive a contradiction to the assumption that the system (12) is completely controllable according to the relations between the spectrum of the two system matrices. Hence the details are omitted.

4.2 Controllability of network II

In the case of network II, we have the following dynamics of the whole closed-loop system:

$$\begin{bmatrix} \dot{x}^{[1]} \\ \vdots \\ \dot{x}^{[m]} \end{bmatrix} = \Phi \begin{bmatrix} x^{[1]} \\ \vdots \\ x^{[m]} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ z \end{bmatrix}, \tag{15}$$

where $z \in \mathbb{R}^{n_l}$ is given in (11),

$$\Phi = G_m \otimes I_{N+n_l} - H_m \otimes \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{lf} & \mathcal{L}_l \end{bmatrix},$$

$$G_m = \begin{bmatrix} \mathbf{0} & I_{m-1} \\ 0 & \vartheta^T \end{bmatrix}, \ H_m = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} \end{bmatrix},$$
$$\vartheta = [c_1 \ \cdots c_{m-1}]^{\mathrm{T}}.$$

This results in the dynamics of the followers becoming the controlled LTI system below:

$$\dot{\xi} = (G_m \otimes I_N - H_m \otimes \mathcal{L}_f)\xi - (e_m \otimes l_{fl})\varphi_1, \quad (16)$$

where ξ is given in (12), $\varphi_1 = [x_{N+1}^{(1)} \ \cdots \ x_{N+n_l}^{(1)}]^{\mathrm{T}}$ and $e_m \in \mathbb{R}^m$ is the mth standard basis vector in $\mathbb{R}^m$. Note that the control input of the controlled agreement system (16) is the leaders' information variables instead of all the states of the leaders.

Analogously, we derive the following necessary and sufficient condition for the controllability of network II.

Theorem 3. For a given network II with a graph $\mathcal{G}$, suppose there are N followers and n_l leaders. Then the associated controlled agreement system (16) is completely controllable if and only if the controlled agreement system (4), i.e., $(-\mathcal{L}_f, \ -l_{fl})$, is completely controllable.

Proof. The proof is similar to that of Theorem 2. We only state the relations between the spectrum of $G_m \otimes I_N - H_m \otimes \mathcal{L}_f$ and that of $\mathcal{L}_f$. Let λ be an eigenvalue of $G_m \otimes I_N - H_m \otimes \mathcal{L}_f$ and $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_m^{\mathrm{T}}]$ with $\beta_k \in \mathbb{R}^N, \ k \in \underline{m}$ be the associated left eigenvector, then $\mu = -(\lambda^m - c_{m-1}\lambda^{m-1} - \cdots - c_1\lambda)$ is an eigenvalue of $\mathcal{L}_f$ and β_m^{T} is the associated left eigenvector. Conversely, for a given eigenvalue μ of $\mathcal{L}_f$ with the associated left eigenvector β^{T}, then the roots of the polynomial $s^m - c_{m-1}s^{m-1} - \cdots - c_1 s + \mu = 0$ with respect to s are the eigenvalues of $G_m \otimes I_N - H_m \otimes \mathcal{L}_f$ and their corresponding left eigenvectors are $[\beta_1^{\mathrm{T}} \ \cdots \ \beta_{m-1}^{\mathrm{T}} \ \beta^{\mathrm{T}}]$, where

$$\begin{cases} \beta_1 = (s^{m-1} - c_{m-1}s^{m-2} - \cdots - c_2 s - c_1)\beta, \\ \beta_2 = (s^{m-2} - c_{m-1}s^{m-3} - \cdots - c_2)\beta, \\ \quad \vdots \\ \beta_{m-1} = (s - c_{m-1})\beta. \end{cases}$$

The results above are very interesting as the controllability of networks of high-order dynamic agents is equivalent to that of networks of single-integrator agents under the same topology and the same prescribed leaders. That is to say, the controllability of network I or II is indifferent of the dynamics of agents, and only determined by their topologies. In this sense, for a network I or II with graph $\mathcal{G} \in \mathfrak{S}$, the criteria for the controllability of networks of single-integrator agents established in Proposition 1 and Theorem 1 are suitable for determining the controllability of the network.

4.3 Controllability of general networks

In this part, we first propose two agreement protocols for networks of agents with dynamics modeled by a completely controllable LTI system, and then give some criteria for the controllability of the general networks.

For the simplicity of expression, we consider networks of agents with dynamics modeled by a single-input LTI system

$$\dot{x}_i = Ax_i + bv_i, \ i \in \underline{N+n_l}, \quad (17)$$

where $x_i \in \mathbb{R}^m$ is the state of agent i, $v_i \in \mathbb{R}$ is the control input to be designed and $(A, \ b)$ is completely controllable. Suppose the characteristic polynomial of A is $s^m - a_m s^{m-1} - \cdots - a_2 s - a_1$, and let $T \in \mathbb{R}_m$ be a nonsingular matrix such that $T^{-1}AT = A_c, T^{-1}b = b_c$, where $(A_c, \ b_c)$ is the associated controllable canonical form of (A, b) and $b_c = [0 \ \cdots \ 0 \ 1]^{\mathrm{T}} \in \mathbb{R}^m$.

Based on the protocol (8), we can derive the following agreement protocol for the system (17):

$$v_i = g^{\mathrm{T}} T^{-1} x_i - \sum_{j \in \mathcal{N}_i} \sum_{k=0}^{m-1} c_k e_{k+1}^{\mathrm{T}} T^{-1}(x_i - x_j), \quad (18)$$

where $g = [\ -a_1 \ \ -a_2 \ \ \cdots \ \ -a_m \ ]^{\mathrm{T}} \in \mathbb{R}^m, c_0, \ldots, c_{m-1}$ are defined as in (8) and $e_{k+1} \in \mathbb{R}^m, k = 0, \ldots, m-1$. While based on the protocol (9), we can establish another agreement protocol for the system (17) as

$$v_i = f^{\mathrm{T}} T^{-1} x_i - \sum_{j \in \mathcal{N}_i} e_1^{\mathrm{T}} T^{-1}(x_i - x_j), \quad (19)$$

where $f = [-a_1 \ \ -(c_1 + a_2) \ \ \cdots \ \ -(c_{m-1} + a_m)]^{\mathrm{T}} \in \mathbb{R}^m$, $c_1, \ \ldots, \ c_{m-1}$ are defined as in (9) and $e_1 \in \mathbb{R}^m$.

For the network (17) and (18) with a graph $\mathcal{G}$, assume there are N followers and n_l leaders, and the associated Laplacian matrix is given in (10). Label the followers 1 through N, and the leaders $N+1$ through $N+n_l$. The movement of each leader is dominated by an exogenous control input $z_i \in \mathbb{R}$

besides v_i, or say, the dynamics of the leaders is in the form

$$\dot{x}_i = Ax_i + b(v_i + z_i).$$

Let $x_f = [x_1^{\mathrm{T}} \ \cdots \ x_N^{\mathrm{T}}]^{\mathrm{T}}$ and $x_l = [x_{N+1}^{\mathrm{T}} \ \cdots \ x_{N+n_l}^{\mathrm{T}}]^{\mathrm{T}}$. Then

$$\begin{cases} \dot{x}_f = (I_N \otimes J_m - \mathcal{L}_f \otimes K_m)x_f - (l_{fl} \otimes K_m)x_l, \\ \dot{x}_l = (I_{n_l} \otimes J_m - \mathcal{L}_l \otimes K_m)x_l - (l_{lf} \otimes K_m)x_f \\ \qquad +(I_{n_l} \otimes b)z, \end{cases}$$

where $J_m = A + bg^{\mathrm{T}}T^{-1}$, $K_m = b\sum_{k=0}^{m-1} c_k e_{k+1}^{\mathrm{T}} T^{-1}$ and $z = [z_{N+1} \ \cdots z_{N+n_l}]^{\mathrm{T}}$. The dynamics of x_f is the controlled agreement system of the network (17) and (18). Let $x_f = (I_N \otimes T)\bar{x}_f$. We have

$$\dot{\bar{x}}_f = (I_N \otimes E_m - \mathcal{L}_f \otimes F_m)\bar{x}_f - (l_{fl} \otimes T^{-1}K_m)x_l,$$

where E_m and F_m are given in (12). Denote $\bar{x}_f = [\bar{x}_{11} \ \cdots \ \bar{x}_{1m}; \ \cdots; \ \bar{x}_{N1} \ \cdots \ \bar{x}_{Nm}]^{\mathrm{T}}$. Let $\bar{x}_f = P_1\bar{\bar{x}}_f$, where P_1 is a permutation matrix and $\bar{\bar{x}}_f = [\bar{x}_{11} \ \cdots \ \bar{x}_{N1}; \ \cdots; \ \bar{x}_{1m} \ \cdots \ \bar{x}_{Nm}]^{\mathrm{T}}$. It follows that

$$\dot{\bar{\bar{x}}}_f = (E_m \otimes I_N - F_m \otimes \mathcal{L}_f)\bar{\bar{x}}_f - P_1^{-1}(l_{fl} \otimes T^{-1}K_m)x_l.$$

Define $\bar{x}_l = [\bar{x}_{(N+1)1} \ \cdots \ \bar{x}_{(N+1)m}; \ \cdots; \ \bar{x}_{(N+n_l)1} \ \cdots \ \bar{x}_{(N+n_l)m}]^{\mathrm{T}}$ which satisfies $\bar{x}_l = (I_{n_l} \otimes T^{-1})x_l$; $\bar{\bar{x}}_l = [\bar{x}_{(N+1)1} \ \cdots \ \bar{x}_{(N+n_l)1}; \ \cdots; \ \bar{x}_{(N+1)m} \ \cdots \ \bar{x}_{(N+n_l)m}]^{\mathrm{T}}$; P_2 is a permutation matrix such that $\bar{x}_l = P_2\bar{\bar{x}}_l$. Then the system

$$\begin{aligned} &(E_m \otimes I_N - F_m \otimes \mathcal{L}_f, \ -P_1^{-1}(l_{fl} \otimes T^{-1}K_m)) \\ &\quad = (E_m \otimes I_N - F_m \otimes \mathcal{L}_f, \ -P_1^{-1}(l_{fl} \otimes T^{-1}K_m) \\ &\qquad \times (I_{n_l} \otimes T)P_2P_2^{-1}(I_{n_l} \otimes T^{-1})) \\ &\quad = (E_m \otimes I_N - F_m \otimes \mathcal{L}_f, \ -(F_m \otimes l_{fl}) \\ &\qquad \times P_2^{-1}(I_{n_l} \otimes T^{-1})). \end{aligned}$$

Let Q_1 and Q_2 be the controllability matrices of the systems $(E_m \otimes I_N - F_m \otimes \mathcal{L}_f, \ -P_1^{-1}(l_{fl} \otimes T^{-1}K_m))$ and $(E_m \otimes I_N - F_m \otimes \mathcal{L}_f, \ -(F_m \otimes l_{fl}))$, respectively. Then $Q_1 = Q_2P_2^{-1}(I_{n_l} \otimes T^{-1})$. Hence $\mathrm{rank}(Q_1) = \mathrm{rank}(Q_2)$. This indicates that the controllability of the network (17) and (18) is equivalent to that of the system (12). (The details of the observation above are just some simple computations, and hence are omitted.)

Based on the observation and Theorem 2, the following result holds.

Corollary 1. For a given network $\mathcal{G}_x$ with the dynamics (17)–(18), suppose there are n_l leaders and N followers, then the network is completely controllable if and only if the system $(-\mathcal{L}_f, \ -l_{fl})$ is completely controllable.

This result is valid for the network (17) and (19) as well. Until now, we can make a conclusion that the controllability of networks of agents with dynamics modeled as a completely controllable single-input LTI system is only determined by their topologies.

5 Influencing factors of controllability

Throughout this section, a network means the network of single-integrator agents (1), unless otherwise stated.

5.1 Effect of the selection of leaders on controllability

One influencing factor of the controllability of networks is the selection of the position and the number of leaders. In ref. [24], a necessary condition for the controllability of networks with leader-following structure has been established. We next extend the result to the case of networks with general graph.

Theorem 4. For a given network $\mathcal{G}_x$ with the dynamics (1)–(2) and a graph, suppose there are N followers and n_l leaders. Denote the induced subgraph on the followers as $\mathcal{G}_f$ and the induced subgraph on the leaders as $\mathcal{G}_l$. If the network is completely controllable, then for any strong component of $\mathcal{G}_f$, there exists at least one leader such that there is a path from the leader to the followers of the strong component.

Proof. Let $\mathcal{G}_f^1, \ldots, \mathcal{G}_f^s$ be the strong components of the subgraph $\mathcal{G}_f$. Introduce a new graph denoted by $\overline{\mathcal{G}}$, where the vertex set consists of the components of $\mathcal{G}_f$ (for convenience of expression, we also denote the vertices of $\overline{\mathcal{G}}$ by $\mathcal{G}_f^1, \ldots, \mathcal{G}_f^s$), and there exists an arc between a pair of distinct vertices $\mathcal{G}_f^i$ and $\mathcal{G}_f^j$ in $\overline{\mathcal{G}}$ if there exist arcs between the agents of the two components. Then there are no cyclic paths in the graph $\overline{\mathcal{G}}$ due to the definition of strong component. This also means that if there exists an arc from vertex $\mathcal{G}_f^i$ to $\mathcal{G}_f^j$, there must be no arcs from $\mathcal{G}_f^j$ to $\mathcal{G}_f^i$. Consequently, by rearranging the indices of the N agents of $\mathcal{G}_f$, the matrix $\mathcal{L}_f$ is taken in the form

$$\mathcal{L}_f = \begin{bmatrix} \mathcal{L}_{f1} & & & & \mathbf{0} \\ & \ddots & & & \\ \mathbf{0} & & \mathcal{L}_{fk} & & \\ * & & * & \mathcal{L}_{f(k+1)} & \\ & \ddots & & & \ddots & \\ * & & * & * & & \mathcal{L}_{fs} \end{bmatrix}, \quad (20)$$

where "$*$" represents zero or nonzero block, the row indices of $\mathcal{L}_{fi} \in \mathbb{R}^{n_i \times n_i}$ is associated with the indices of agents in the strong component $\mathcal{G}_f^i$, and for any $j = k+1, \ldots, s$, there exists nonzero block among the off-diagonal blocks of the jth row. For the block matrix (20), if we let the diagonal blocks and the zero off-diagonal blocks be scalar 0, and the nonzero off-diagonal blocks be scalar 1, then the resulting matrix becomes the adjacency matrix of the graph $\overline{\mathcal{G}}$. From the property of (20), it follows that for any vertex $\mathcal{G}_f^j, j = k+1, \ldots, s$ of $\overline{\mathcal{G}}$, there must exist a vertex $\mathcal{G}_f^i,\ i = 1, \ldots, k$ such that there is a path from $\mathcal{G}_f^i$ to $\mathcal{G}_f^j$ (this conclusion can be proved via induction). Furthermore, this yields that in the follower subgraph $\mathcal{G}_f$, for any strong component $\mathcal{G}_f^j,\ j = k+1, \ldots, s$ there must exist a strong component $\mathcal{G}_f^i,\ i = 1, \ldots, k$ such that there is a path from any vertex of $\mathcal{G}_f^i$ to any vertex of $\mathcal{G}_f^j$.

According to (20), the matrix l_{fl} can be rewritten as $l_{fl} = [B_1 \ \cdots \ B_s]^{\mathrm{T}}$, the block entry $B_i,\ i = 1,\ \ldots,\ s$ of which has the same row indices as $\mathcal{L}_f^i$. Then the controllability matrix $\mathcal{C}$ of the system $(-\mathcal{L}_f,\ -l_{fl})$ has the form

$$\mathcal{C} = \begin{bmatrix} -B_1 & \mathcal{L}_{f1}B_1 & \cdots & (-1)^N \mathcal{L}_{f1}^{N-1} B_1 \\ \vdots & \vdots & & \vdots \\ -B_k & \mathcal{L}_{fk}B_k & \cdots & (-1)^N \mathcal{L}_{fk}^{N-1} B_k \\ * & * & \cdots & * \end{bmatrix}.$$

If the conclusion does not hold, then there must exist a strong component $\mathcal{G}_f^i,\ i = 1,\ \ldots,\ k$ such that there is no path from any leader to the followers of the component $\mathcal{G}_f^i$ due to the property of (20). Thus $B_i = \mathbf{0}$, which results in the ith block row of $\mathcal{C}$ is zero. This causes a contradiction to the assumption that the network is completely controllable. The proof is completed.

Remark 5. Theorem 4 provides a necessary condition for the controllability of the network (1)-(2) and indicates that the positions of the leaders have important influence on the controllability. Just as done in the proof of Theorem 4, we introduce another new graph, denoted by $\overline{\overline{\mathcal{G}}}$, where the vertex set consists of the induced subgraph $\mathcal{G}_l$ and the strong components (or subgraphs) $\mathcal{G}_f^1, \ldots, \mathcal{G}_f^s$ of $\mathcal{G}_f$ (similarly, we denote the vertices of $\overline{\overline{\mathcal{G}}}$ by $\mathcal{G}_l,\ \mathcal{G}_f^1, \ldots, \mathcal{G}_f^s$, i.e., the vertex of $\overline{\overline{\mathcal{G}}}$ corresponds to a subgraph of $\mathcal{G}$); there exists an arc between any two distinct vertices of $\overline{\overline{\mathcal{G}}}$ if there exist arcs between the agents of their corresponding subgraphs of $\mathcal{G}$. Then Theorem 4 implies that the network $\mathcal{G}_x$ is completely controllable only if the graph $\overline{\overline{\mathcal{G}}}$ has a spanning tree with the root $\mathcal{G}_l$.

In addition, Proposition 1 implies that the number of the leaders has great influence on the controllability as well. Specifically, for a given network of N agents with $\mathcal{G} \in \mathfrak{G}$, let $\mathcal{G}^u$ be the underlying undirected graph of $\mathcal{G}$ and $\mathcal{L}^u$ be the associated Laplacian matrix of $\mathcal{G}^u$. Take arbitrarily $k < N$ agents as the leaders of the network. Denote the principle sub-matrix of $\mathcal{L}^u$ as $\mathcal{L}_f \in \mathbb{R}^{(N-k)\times(N-k)}$, which is obtained by deleting the rows and columns indicated by the indices of the leaders. If a principle sub-matrix $\mathcal{L}_f$ has no common eigenvalue with $\mathcal{L}^u$, then the network with k leaders is completely controllable according to Proposition 1. If any principle sub-matrix with order $N-k$ of $\mathcal{L}^u$ has common eigenvalue with it, then the network is uncontrollable under any k leaders, i.e., any k agents of the network are not able to control the remainder agents completely.

5.2 Effect of weights on controllability

It is known that the controllability of an LTI system depends upon the structures and parameters of its coefficient matrices, and is a structural property of the system. In sections 3 and 4, we have discussed the controlled agreement problem for networks with un-weighted graph. However, the agreement protocols are given in the forms of weighted ones in most literature, i.e., there exist link weights of the network distinct to 1 (see refs. [13, 14, 18–20]). Ignoring link weights of the network means ignoring the parameters of the system.

As a matter of fact, for a given un-weighted graph (or rather, for a given topological structure) and some prescribed leaders, we can turn an uncontrollable network with the topological structure into a completely controllable network by selecting appropriate weights for the communication links. As a trivial example, consider the un-weighted complete graph with three vertices (note that it was proved that an un-weighted complete graph is uncontrollable in ref. [21]). If we put the following weighted adjacency matrix on the graph

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \end{bmatrix},$$

then by appointing any one agent as leader, the resulting controlled agreement system $(-\mathcal{L}_f,\ -l_{fl})$ is completely controllable. This example shows that the link weights have important effect on the controllability of networks. Notice that the results in sections 3 and 4 are valid for networks with weighted graph as well. Therefore, for a given topological structure and some prescribed leaders, we wonder how to select appropriate link weights such that the corresponding weighted network is completely controllable, and whether or not there are some relations between the controllability of multi-agent systems and the structural controllability of linear control systems (see refs. [32, 34]). In what follows, we will introduce the concept of the structural controllability of networks and present a necessary and sufficient condition for this property. To do this, we recall some definitions and results concerning the structural controllability of linear control systems.

Consider a linear time-invariant system

$$\dot{x}(t) = A_0 x(t) + B_0 v(t)$$

with $x(t) \in \mathbb{R}^n$ and $v(t) \in \mathbb{R}^r$. Assume that the entries of the matrices are either fixed zeros or independent free parameters (such matrices are called to be structured matrices in some literature).

Definition 3[32,34]**.** A linear system (A_0, B_0) is structurally controllable if there exists a completely controllable system (A, B), which satisfies that there is a one-to-one correspondence between the locations of the fixed zeros and the free entries in the corresponding matrices of the two systems.

Lemma 2[34]**.** For a given linear system $(A_0,\ B_0)$, the system is structurally controllable if and only if the matrix $[A_0\ B_0]$ is irreducible and the generic rank g-rank$([A_0\ B_0]) = n$.

Note that the generic rank of a structured matrix A is defined as the maximum rank which A can attain as a function of the free parameters in A, denoted by g-rank(A).

We next introduce the definition of the structural controllability of linear control systems to multi-agent systems (or networks). Given a network $\mathcal{G}_x$ with the dynamics (1)–(2) and a topological structure, we can obtain its corresponding controlled agreement system (4) according to the partition of leaders and followers given in section 3. If we consider the zero off-diagonal entries of $-\mathcal{L}_f$ and the zero entries of $-l_{fl}$ to be fixed zeros, and the diagonal entries and the nonzero off-diagonal entries of $-\mathcal{L}_f$ (or 1 entries) and the nonzero entries of $-l_{fl}$ (or 1 entries) to be free parameters, then the system $(-\mathcal{L}_f,\ -l_{fl})$ has the structured property given in the system $(A_0,\ B_0)$. Under this scheme, we give the definition of the structural controllability of networks as follows.

Definition 4. For a given network $\mathcal{G}_x$ with the dynamics (1)–(2) and some prescribed leaders, the network is said to be structurally controllable if the system $(-\mathcal{L}_f,\ -l_{fl})$ is structurally controllable.

The following theorem states the main result on the structural controllability of networks.

Theorem 5. For a given network $\mathcal{G}_x$ with the dynamics (1)–(2) and a topological structure, suppose there are N followers and n_l leaders. Denote the induced subgraph on the followers as $\mathcal{G}_f$ and the induced subgraph on the leaders as $\mathcal{G}_l$. Then the network is structurally controllable if and only if for any strong component of $\mathcal{G}_f$, there exists at least one leader such that there is a path from the leader to the followers of the strong component.

Proof. Let $\mathcal{G}_f^1, \ldots, \mathcal{G}_f^s$ be the strong components of $\mathcal{G}_f$. Through rearranging the indices of the N agents in $\mathcal{G}_f$, the matrix pair $(-\mathcal{L}_f,\ -l_{fl})$ has the form

$$[-\mathcal{L}_f\ \ -l_{fl}] = \begin{bmatrix} -\mathcal{L}_{f1} & & & & \mathbf{0} & -B_1 \\ & \ddots & & & & \vdots \\ \mathbf{0} & & -\mathcal{L}_{fk} & & & -B_k \\ * & & * & -\mathcal{L}_{f(k+1)} & & -B_{k+1} \\ & \ddots & & & \ddots & \vdots \\ * & & * & * & -\mathcal{L}_{fs} & -B_s \end{bmatrix}, \quad (21)$$

where $*$ and $\mathcal{L}_{fi} \in \mathbb{R}^{n_i \times n_i},\ i = 1, \ldots, s$ are given in (20), $B_i,\ i = 1, \ldots, s$ are defined as in the proof of Theorem 4, and for any $j = k+1, \ldots, s$, there exists nonzero block among the off-diagonal blocks in the jth row.

Necessity. Suppose the network is structurally controllable, then the matrix $[-\mathcal{L}_f\ \ -l_{fl}]$ is irreducible. If the conclusion does not hold, the proof of Theorem 4 indicates that there must exist a strong component $\mathcal{G}_f^i,\ i = 1,\ \ldots,\ k$ such that there is no path from any leader to it. This results in $B_i = \mathbf{0}$. Therefore, there is a permutation matrix P such that

$$P[-\mathcal{L}_f\ \ -l_{fl}]P^{-1} = \begin{bmatrix} -\mathcal{L}_{fi} & \mathbf{0} & \mathbf{0} \\ * & * & \bar{B} \end{bmatrix},$$

which contradicts the fact that $[-\mathcal{L}_f\ \ -l_{fl}]$ is irreducible.

Sufficiency. Suppose for any strong component of $\mathcal{G}_f$, there exists at least one leader such that there is a path from the leader to it. Then for any $B_i,\ i = 1,\ \ldots,\ k$ in (21), $B_i \neq \mathbf{0}$. Denote the Laplacian matrix of the component $\mathcal{G}_f^j,\ j = 1,\ \ldots,\ s$ as $\mathcal{L}(j)$. Then $\mathcal{L}(j)$ has zero row sum and $\mathcal{L}_{fj} = \mathcal{L}(j) + \mathcal{D}_j$, where $\mathcal{D}_j$ is a diagonal matrix with rows and columns indexed by the vertices of $\mathcal{G}_f^j$, and each diagonal entry of $\mathcal{D}_j$ is the number of the neighbors, which come from $\mathcal{G}_l$ and other strong components, of the corresponding vertex. For $j = 1,\ \ldots,\ k$, $B_j \neq \mathbf{0}$ implies that $\mathcal{D}_j \neq \mathbf{0}$. Following the similar line of reasoning as the proof of Lemma 4 in ref. [35], we have $\mathcal{L}_{fj},\ j \in \{1,\ \ldots,\ k\}$ are invertible. Moreover, for $j = k+1,\ \ldots,\ s$, we also have $\mathcal{D}_j \neq \mathbf{0}$ because there exists nonzero block among the off-diagonal blocks of the jth row in (21). Hence, for $j = k+1,\ \ldots,\ s$, $\mathcal{L}_{fj}$ is invertible as well. In a word, we have $\text{rank}(-\mathcal{L}_f) = N$, which results in $\text{rank}([-\mathcal{L}_f\ \ -l_{fl}]) = N$. This proves that g-rank$([-\mathcal{L}_f\ \ -l_{fl}]) = N$. Next, if $[-\mathcal{L}_f\ \ -l_{fl}]$ is reducible, then there exists a permutation matrix P such that

$$-P\mathcal{L}_f P^{-1} = \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$-Pl_{fl} = \begin{bmatrix} \mathbf{0} \\ B_2 \end{bmatrix}.$$

This means there is no path from any leader to the followers corresponding to the indices of A_{11}, which contradicts the assumption. Hence $[-\mathcal{L}_f\ \ -l_{fl}]$ is irreducible. It follows from Lemma 2 that the controlled agreement system $(-\mathcal{L}_f,\ -l_{fl})$ is structurally controllable.

In summary, for a network with a given topological structure, the link weights, the position and the number of leaders have important influence on the controllability of the network. The research on the controllability of multi-agent systems based on agreement protocols needs more exploration of properties of graphs. In addition, there are some prospectives for further research, for example, selecting appropriate link weights such that the network is completely controllable; choosing proper positions and suitable number of leaders to make the network completely controllable.

6 Numerical examples

In this section, we give some simple numerical examples to illustrate the previous theoretical results. Consider the underlying undirected graph depicted in Figure 1. In the case of un-weighted graph, if we take any single agent as the leader of the network, then each resulting controlled agreement system is uncontrollable. This can be verified through Proposition 1. However, we have two methods to make the network completely controllable based on one selected leader.

One method is adding an agent to the leader group. As a simple example, add agent 4 to the leader group composed of agent 5. Then the resulting controlled agreement system is completely controllable according to Proposition 1. In this scenario, Figure 2 shows the motion trajectories of the followers 1, 2 and 3 when the dynamics of agents is described by single integrator. The three followers start from the initial positions in the plane represented by "$\circ$", and are steered to the positions in the plane represented by "$*$" (forming a straight line) in finite time. This indicates that the selection of leaders has great influence on the controllability of networks. Moreover, this example shows that if the network is completely controllable, the followers over the network can be steered to form any particular configuration in finite time by the leaders.

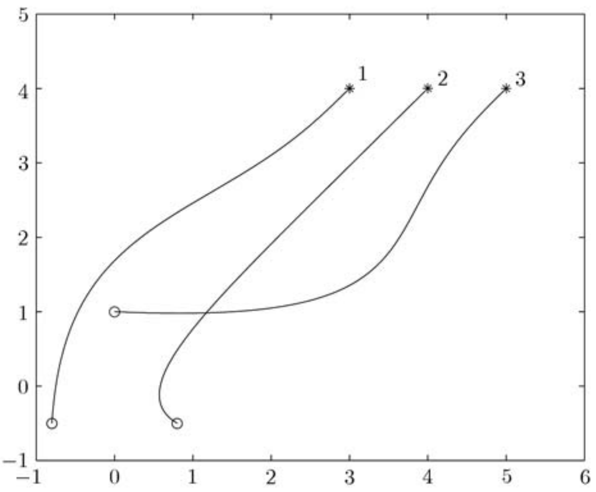

Figure 2 Motion trajectories of three followers 1, 2 and 3 with single-integrator dynamics.

Another method of improving the controllability of the network in Figure 1 is adding weights to the interaction links. For example, take the weighted adjacent matrix to be

$$\mathcal{A}=\begin{bmatrix}0&2&0&1&0\\2&0&3&0&0\\0&3&0&0&1\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix}.$$

Proposition 1 shows that the resulting controlled agreement system is completely controllable if any one agent is appointed the leader. Figure 3 displays the motion trajectories of the followers for the network of double-integrator (i.e., $m=2$ in the system (7)) agents with the protocol (9) and agent 5 being the leader. This method implies that the weights of interaction links affect greatly the controllability of networks as well.

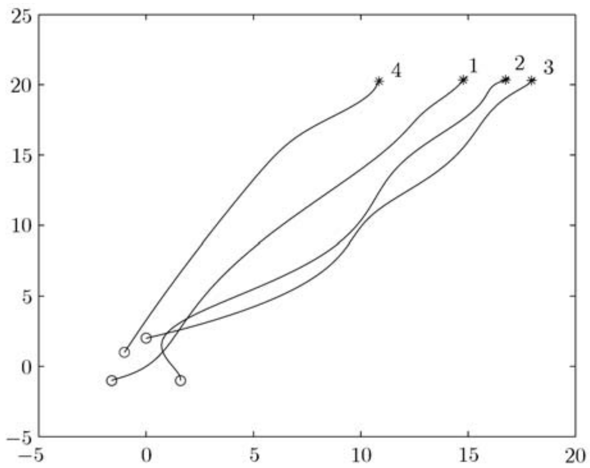

Figure 3 Motion trajectories of four followers 1, 2, 3 and 4 with double-integrator dynamics.

7 Conclusions

The necessary/sufficient conditions have been established for the controllability of networks of single-integrator agents. For networks with leader-following structure and networks with undirected graph, we have given an equivalence of the controllability based on model transformation from the agreement systems to their corresponding controlled agreement systems. For networks of high-order dynamic agents, we have proved that, under the same topology and the same prescribed leaders, its controllability is equivalent to that of networks with single-integrator agents, namely, the controllability of networks of high-order agents is independent of the dynamics of agents and is only determined by their topologies. Two important influencing factors of the controllability, that is, the selection of leaders and the coupling weights of graphs, have been analyzed from the viewpoint of graphical characterization.

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